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Combined Base Station Association and Power Control in Multichannel Cellular Networks

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***Abstract*—A combined base station association and power control problem is studied for the uplink of multichannel multicell cellular networks, in which each channel is used by exactly one cell (i.e., base station). A distributed association and power update algorithm is proposed and shown to converge to a Nash equilibrium of a noncooperative game. We consider network models with discrete mobiles (yielding an atomic congestion game), as well as a continuum of mobiles (yielding a population game). We find that the equilibria need not be Pareto efficient, nor need they be system optimal. To address the lack of system optimality, we propose pricing mechanisms. It is shown that these mechanisms can be implemented in a distributed fashion.**

I. Introduction

Wireless communication systems have experienced tremendous growth over the last decade, and this growth continues unabated worldwide. Efficient management of resources is essential for the success of wireless cellular systems. In a mobile cellular system, mobiles adapt to time varying radio channels by adjusting base station (BS) associations and by controlling transmitter powers. Doing so, they not only maintain their quality of service (QoS) but also enhance their transmitters' battery lives. In addition, such controls reduce the network interference, thus maximizing spatial spectrum reuse. Distributed control is of special interest, since the alternative of centrally orchestrated control involves added infrastructure, the need for distribution of measurements, and hence system complexity.

Distributed control algorithms for single channel multicell networks have been extensively studied (Foschini & Miljanic [1], Yates [2], Hanly [3]). The monograph by Chiang et al. [4] and references therein provide an excellent survey of the area. Noncooperative games have been a natural tool for analysis and design of distributed power control algorithms. Scutari et al. [5] and Heikkinen [6] model distributed power control problems as potential games, while Altman & Altman [7] show that many of the cellular power control algorithms can be modeled as submodular games. In contrast, uplink resource allocation for *multichannel* multicell networks poses several challenges as observed in Yates [2] and Jiang et al. [8].

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We address the resource allocation problem in the uplink of a *multichannel multicell network*. Such a problem arises when a CDMA operator chooses to lease and utilize multiple frequency bands (channels) in order to reduce in-network interference, or multiple operators who lease different bands decide to cooperate. Many newer mobile devices are capable of operating over multiple CDMA bands, and thus have the option to choose from one of these distinct bands. We address a simplified version of this multichannel multicell problem where each BS operates on a separate frequency band, and so, there is no intercell interference.

A preview of our results is as follows. We propose a distributed algorithm for the combined base station association and power control problem, and subsequently model the problem as a player-specific congestion game. The equilibrium states of such algorithms, which are Nash equilibria of the corresponding games, may be far from system optimum. We resort to pricing mechanisms to induce mobiles to behave in a way that optimizes system cost. We also show that such a mechanism can be employed in a distributed fashion. Towards this end, we model the network as having a continuum of (nonatomic) mobiles, each offering infinitesimal load, which leads to a population game formulation. We then provide a marginal pricing mechanism that motivates a pricing strategy for the discrete mobiles case. Note that, unlike the case of transportation networks, mobiles are not really priced in cellular networks. The pricing is simply a part of the decision making routine built into each mobile in order bring about a *distributed* control mechanism that drives the system towards optimality.

The paper is organized as follows. In Section II we briefly discuss concepts of *finite noncooperative games* and *population games*. We study a network model with discrete mobiles in Section III. We propose a combined association and power control algorithm, model it as a noncooperative game, and analyze its performance. We extend this analysis to a network with a continuum of mobiles in Section IV. To address the inefficiency of the proposed algorithms, we design toll mechanisms in Section V. Finally, we conclude the paper with some remarks in Section VII. In Appendix A, we provide bounds on the *price of anarchy* [9] for the case of a continuum of mobiles. We omit a few of the proofs for lack of space; these can be found in our technical report [10].

Optimal power allocation and BS association in uplinks of multichannel multicell cellular networks have not been explored before. Ours is an attempt at a detailed coverage on what is possible in general, with more specific results in some special cases.

II. Game Preliminaries

A. Finite Noncooperative Games

A noncooperative strategic form game $(\mathcal{M}, (\mathcal{A}_i, i \in \mathcal{M}), (c_i, i \in \mathcal{M}))$ consists of a set of players $\mathcal{M} = \{1, \dots, M\}$. Each player i is accompanied by an action set $\mathcal{A}_i$ and a cost function $c_i : \times_{i=1}^M \mathcal{A}_i \to \mathbb{R}$. In this work, we assume all action sets to be finite. An action profile $\mathbf{a} = (a_i, i = 1, \dots, M)$ prescribes an action a_i for every player $i \in \mathcal{M}$. For $\mathbf{a} = (a_i, i = 1, \dots, M)$, denote $\mathbf{a}_{-i} := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_M)$ and $(b_i, \mathbf{a}_{-i}) := (a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_M)$.

Definition 2.1: Nash Equilibrium (NE): For an action profile $\mathbf{a}$, a mobile i's *best response*, $\mathcal{B}_i(\mathbf{a}) \subseteq \mathcal{A}_i$, is defined as $\mathcal{B}_i(\mathbf{a}) := \mathrm{argmin}_{b_i \in \mathcal{A}_i} c_i(b_i, a_{-i})$. $\mathbf{a}$ is said to be a Nash Equilibrium for the game if $a_i \in \mathcal{B}_i(\mathbf{a})$ for all $i \in \mathcal{M}$.

Definition 2.2: Potential Game: A game $(\mathcal{M}, (\mathcal{A}_i, i \in \mathcal{M}), (c_i, i \in \mathcal{M}))$ is said to be an ordinal potential game if there exists a function $V : \times_{i=1}^M \mathcal{A}_i \to \mathbb{R}$, known as an *ordinal potential function*, that satisfies

$$c_i(b_i, \mathbf{a}_{-i}) < c_i(\mathbf{a}) \Leftrightarrow V(b_i, \mathbf{a}_{-i}) < V(\mathbf{a})$$

for all $i \in \mathcal{M}, b_i \in \mathcal{A}_i, \mathbf{a} \in \times_{i=1}^M \mathcal{A}_i$. The game is called generalized ordinal potential game if there exists a potential function $V : \times_{i=1}^M \mathcal{A}_i \to \mathbb{R}$ satisfying

$$c_i(b_i, \mathbf{a}_{-i}) < c_i(\mathbf{a}) \Rightarrow V(b_i, \mathbf{a}_{-i}) < V(\mathbf{a})$$

for all $i \in \mathcal{M}, b_i \in \mathcal{A}_i, \mathbf{a} \in \times_{i=1}^M \mathcal{A}_i$.

Clearly, generalized ordinal potential games subsume ordinal potential games. Further, all minimizers of a potential function V are Nash equilibria of the game. Thus all generalized ordinal potential games $(\mathcal{M}, (\mathcal{A}_i, i \in \mathcal{M}), (c_i, i \in \mathcal{M}))$ admit at least one Nash equilibrium. Since the games are finite (finite number of players and finite action sets), they also have the finite improvement path (FIP) property, i.e., do not contain improvement cycles (Monderer & Shapley [11, Lemma 2.3]). Thus, in a finite generalized ordinal potential game, when players update as per the *better response strategy*, *round-robin* or *random* update processes converge to a Nash equilibrium in a finite number of steps. With the same strategies, an *asynchronous* update process also converges (Neel [12, Chapter 5]).

Remark 2.1: The strategic form games that have the FIP property also admit the finite best-response path (FBRP) property, i.e., they do not contain best response cycles (Milchtaich [13, Section 5]).[1] Thus, if players update as per the *best response* strategy, then also the above update processes converge to a Nash equilibrium in a finite number of steps. The reverse implication is not true in general - the FBRP property need not imply the FIP property.

Definition 2.3: Congestion Game: A game $(\mathcal{M}, (\mathcal{A}_i, i \in \mathcal{M}), (c_i, i \in \mathcal{M}))$ is said to be a player-specific weighted singleton congestion game if

1) there exists a set $\mathcal{N}$ such that $\mathcal{A}_i = \mathcal{N}$ for all $i \in \mathcal{M}$, and
2) there exist constants $(\beta_i, i \in \mathcal{M})$ and nonincreasing functions $f_{ij}, i \in \mathcal{M}, j \in \mathcal{N}$ such that

$$c_i(\mathbf{a}) = f_{ia_i}\Big(\sum_{\substack{l \in \mathcal{M}: \\ a_l = a_i}} \beta_l \Big) \text{ for all } \mathbf{a} \in \times_{i=1}^M \mathcal{A}_i, i \in \mathcal{M}.$$

In the above definition, we interpret $\mathcal{N}$ as a set of facilities and β_l as the load offered by player i. Then, $\sum_{l \in \mathcal{M}: a_l = a_i} \beta_l$ denotes the total load on facility a_i, under an action profile $\mathbf{a}$. The game is a *singleton* congestion game because each action picks exactly one facility. It is *weighted* because players offer different loads, and it is *player-specific* because the cost functions $c_i(\cdot)$ are player-specific.

Rosenthal [15] defined congestion games with unweighted players and player-independent cost functions, but more general action sets. The above generalization is due to Milchtaich [13] who showed that singleton weighted congestion games with player independent costs admit the FIP property but singleton player specific unweighted congestion games need not. Gairing et al. [16] studied these games in the special case of affine cost functions. Harks et al. [17] showed that a weighted congestion game admits a weighted potential function if and only if either all the cost functions are affine or they all are certain exponential functions. Mavronicolas et al. [18] considered a subclass of these games where each player-specific cost function is composed (by means of an abelian group operation) of a player-specific constant and a facility-specific nondecreasing function. Sbabou [19] considered another subclass for which all the Nash equilibria can be obtained without invoking the potential function or the finite best-reply property.

B. Population Games

A population game (Sandholm [20]) $(\mathcal{M}, (\mathcal{A}_l, l \in \mathcal{L}), (c_{lj}, l \in \mathcal{L}, j \in \mathcal{A}_l))$ consists of $\mathcal{L} = \{1, \dots, L\}$ classes of nonatomic populations of players. $\mathcal{M} = \cup_{l \in \mathcal{L}} \mathcal{M}_l$, and $M_l := |\mathcal{M}_l|$ denotes the total mass of the class l population. By a nonatomic population, we mean that the mass of each member of the population is infinitesimal. Players of class l are associated with an action set $\mathcal{A}_l$. Actions of these (class l) players lead to an action distribution $\mathbf{m}^l = (m_{lj}, j \in \mathcal{A}_l)$, where $\sum_{j \in \mathcal{A}_l} m_{lj} = M_l$. All the players within a class are alike. Thus the action distributions completely specify the play; we can characterize the states and dynamics of play solely in terms of action distributions. Let $\mathbf{m} = (\mathbf{m}^l, l \in \mathcal{L})$ denote the action distribution profile across the entire population, and let $\mathcal{M}^*$ denote the set of all such profiles. A population l is also accompanied by continuous cost density functions $c_{lj} : \mathcal{M}^* \to \mathbb{R}$.

Definition 2.4: Nash Equilibrium (NE): An action distribution profile $\mathbf{m}$ is a pure strategy Nash equilibrium for the game $(\mathcal{M}, (\mathcal{A}_i, i \in \mathcal{M}), (c_{lj}, l \in \mathcal{L}, j \in (\mathcal{A}_l))$ if and only if for all $l \in \mathcal{L}$ and $j \in \mathcal{A}_l$, a positive mass $m_{lj} > 0$ implies $c_{lj}(\mathbf{m}) \leq c_{lk}(\mathbf{m})$ for all $k \in \mathcal{A}_l$.

[1] A best response cycle is a finite best response path $\mathbf{a^1}, \cdots, \mathbf{a^k}$ such that $\mathbf{a^1} = \mathbf{a^k}$, and for some $j \in \{1, \cdots, k-1\}$, the deviating player in iteration j strictly benefits [14].

Remark 2.2: Definition 2.4 implies that, at a Nash equilibrium $\mathbf{m}$, for a class l, if j and k are any two facilities in $\mathcal{A}_l$ such that $m_{lj} > 0, m_{lk} > 0$, then $c_{lj}(\mathbf{m}) = c_{lk}(\mathbf{m})$.

Definition 2.5: Potential Game: A population game $(\mathcal{M}, (\mathcal{A}_l, l \in \mathcal{L}), (c_{lj}, l \in \mathcal{L}, j \in \mathcal{A}_l))$ is said to be a potential game if there exists a $\mathbf{C}^1$ function $V : \mathcal{M}^* \to \mathbb{R}$, known as a *potential function*, that satisfies

$$\frac{\partial V(\mathbf{m})}{\partial m_{lj}} = c_{lj}(\mathbf{m})$$

for all $l \in \mathcal{L}, j \in \mathcal{A}_l, \mathbf{m} \in \mathcal{M}^*$.

It is well known that Nash equilibria are the profiles which satisfy the Kuhn-Tucker first order conditions for a minimizer of the potential function (Sandholm [20, Proposition 3.1]). Any dynamics with *positive correlation* and *noncomplacency* (for e.g., the best response dynamics) approaches a Nash equilibrium [20].

We are interested in *nonatomic congestion games* (Sandholm [20]) in which $\mathcal{A}_l = \mathcal{N}, \forall l \in \mathcal{L}$, for a given set $\mathcal{N}$. As before, we interpret $\mathcal{N}$ as a set of facilities. Moreover, each class l has an associated offered load density $\gamma_l > 0$. An action distribution profile $\mathbf{m}$ leads to a congestion profile $(m_j, j \in \mathcal{N})$, where $m_j = \sum_{l \in \mathcal{L}} m_{lj}\gamma_l$. The cost density functions c_{lj} depend on $\mathbf{m}$ only through m_j, and are increasing in m_j.

C. Pricing

Levying of tolls is a conventional way to enforce system optimality in nonatomic networks. Beckman [21] and Dafermos & Sparrow [22] studied optimal tolls in transportation networks with a single class of users. Later Dafermos [23] and Smith [24] extended the analysis to multiclass networks. Roughgarden & Tardos [25] applied these ideas in computer networks and analyzed tolls for optimal routing.

In the atomic (discrete) setting, Caragiannis et al. [26] proposed tolls for two-terminal parallel-edge networks with unweighted users and linear latency functions. Subsequently, they considered the cases of heterogeneous users (with different sensitivities to taxes) and of asymmetric games where each client has at most two permissible choices [27]. Fotakis & Spirakis [28] studied *cost balancing tolls* for generic two-terminal networks with unweighted users and arbitrary increasing latency functions. Fotakis et al. [29] broadened this study to incorporate heterogeneous users and single-source multiple-sink networks. More recently, Jelinek et al. [30] analyzed the scenario where tolls have to respect some given upper bound restrictions on the links. They also focused on parallel-edge networks and unweighted users (either homogeneous or heterogeneous), and allowed arbitrary increasing latency functions. We propose an alternative toll mechanism, and demonstrate that the proposed tolls can be computed in a distributed fashion.

III. Discrete Mobiles

A. System Model

We now describe the model adopted in this work. We consider the uplink of a cellular network consisting of several BSs and mobiles. *Each BS operates in a distinct frequency band.* Let $\mathcal{N} = \{1, \ldots, N\}$ and $\mathcal{M} = \{1, \ldots, M\}$ denote the set of BSs and the set of mobiles, respectively.

A mobile must be associated with one BS at any time, and is free to choose the BS with which it associates. Let h_{ij} denote the power gain from mobile i to BS j. Let the receiver noise at all BSs have the average power σ^2. Let p_i denote the power transmitted by mobile i, and let a_i be the BS to which it is associated. Under an association profile $\mathbf{a} = (a_i, i = 1, \ldots, M)$, let $\mathcal{M}_j(\mathbf{a})$ be the set of mobiles associated with BS j. Under an association profile $\mathbf{a}$ and a power vector $\mathbf{p} = (p_i, i = 1, \ldots, M)$, the signal to interference plus noise ratio (SINR) of mobile i at BS a_i is

$$\frac{h_{ia_i}p_i}{\sum_{l \in \mathcal{M}_{a_i}(\mathbf{a}) \setminus \{i\}} h_{la_l}p_l + \sigma^2}$$

Mobile i has a target SINR requirement γ_i.

Remark 3.1: Assume a scenario where the channels are close together relative to their centre frequencies. Then the channel gains for various mobile-BS pairs can be taken to be functions of distances between them. In particular, if all the mobiles (respectively, the BSs) are collocated, then the channel gains will depend on BSs'(respectively, the mobiles') indices (see Sections III-C2 and III-C3).

B. The MAPC Algorithm

Yates [2] and Hanly [3] proposed an algorithm for distributed association and power control in single channel cellular networks. Convergence results for the algorithm are based on the concept of a *standard interference function*. The technique is based on a mobile reassociating itself with a BS with which it needs to use the least power; this fails to work in the case of a multichannel network and analogous convergence results for this algorithm may not hold (see Yates [2, Section VI]). Even in instances where the algorithm converges, it may get stuck at an association profile that is not Pareto efficient (see Definition 3.2).

We propose an alternative distributed algorithm for combined BS association and power control in multichannel multicell cellular networks. We also show its convergence. We make use of the following simple fact (see, for example, Kumar et al. [31, Chapter 5]). Consider the subproblem of power control with a fixed association $\mathbf{a}$. Define $\beta_l = \frac{\gamma_l}{1+\gamma_l}$, a measure of the "load" offered by mobile l.

Proposition 3.1: For a fixed association $\mathbf{a}$,
(i) The power control subproblem of BS j is feasible iff $\sum_{l \in \mathcal{M}_j(\mathbf{a})} \beta_l < 1$;
(ii) If the power control subproblem of BS j is feasible, there exists a unique efficient[2] power vector $\mathbf{p}(\mathbf{a})$ given by

$$p_i(\mathbf{a}) = \frac{\sigma^2}{h_{ij}} \frac{\beta_i}{1 - \sum_{l \in \mathcal{M}_j(\mathbf{a})} \beta_l}.$$

Throughout we assume that there exists at least one feasible association and power vector. Proposition 3.1 motivates the following algorithm.

[2]Efficiency is in the sense of minimizing the aggregate transmit power of all the mobiles.

Multichannel Association and Power Control (MAPC): Mobiles switch associations in a round-robin fashion in accordance with the the optimal power consumptions (given by Proposition 3.1(ii)) at the BSs with which these associate. More precisely, a switching mobile associates with a BS where it would require the least power. As the load at a BS changes, it immediately broadcasts the new load, and the associated mobiles update their powers to the optimal required powers as per the new loads. Mathematically, define

$$c_i(\mathbf{a}) = \frac{\sigma^2}{h_{ia_i}} \frac{\beta_i}{[1-\sum_{l\in\mathcal{M}_j(\mathbf{a})}\beta_l]^+}, \tag{1}$$

where $[x]^+ = \max(x,0)$. For $t = 0,1,2,\ldots$, mobile i where $i = 1+(t \mod M)$ updates its association and power at $t+1$ if $a_i(t) \notin \operatorname{argmin}_{j\in\mathcal{N}} c_i((j,\mathbf{a}(t)_{-i}))$. In this case,

$$a_i(t+1) \in \operatorname*{argmin}_{j\in\mathcal{N}} c_i((j,\mathbf{a}(t)_{-i})), \tag{2a}$$

and with $\mathbf{a}(t+1) = (a_i(t+1),\mathbf{a}(t)_{-i})$,

$$p_l(t+1) = c_l(\mathbf{a}(t+1)), \quad \forall l \in \mathcal{M}_{a_i(t)}(\mathbf{a}(t)) \cup \mathcal{M}_{a_i(t+1)}(\mathbf{a}(t+1)). \tag{2b}$$

Remark 3.2: Observe that while only one mobile updates its association at a time, all mobiles that perceive a change in load at their BSs update their powers to optimal values based on the new loads. If the power requirements of a mobile are identical at two or more BSs, one of those is chosen at random.

Remark 3.3: Consider the special case where the mobiles have a common target SINR requirement. In this case, even if the algorithm starts with an infeasible association, selfish moves of players eventually lead to a feasible one, and updates remain feasible thereafter.

This algorithm is also distributed in nature as the one proposed in [2]. BS j broadcasts its total congestion $\sum_{l\in\mathcal{M}_j(\mathbf{a})}\beta_l$ on a common control channel so that even non-associated mobiles receive this information. In addition, each mobile i is told its scaled gains $\frac{h_{ij}}{\sigma^2}$ by each BS $j \in \mathcal{N}$.

C. A Congestion Game Formulation

To show the convergence properties of the proposed algorithm, we model the system as a strategic form game. Let the mobiles be the players and the action set for each player be the possible associations, i.e, $\mathcal{A}_i = \mathcal{N}$ for all $i \in \mathcal{M}$. Define the cost functions of the players to be $c_i(\mathbf{a})$ for all $i \in \mathcal{M}$. It can be seen that above is a player-specific singleton weighted congestion game, and belongs to the subclass of congestion games with multiplicative player-specific constants described in [18]. In the following we refer to it as the strategic form game $(\mathcal{M},\mathcal{N},(c_i, i\in\mathcal{M}))$.

Before analyzing the general game, we consider the following special cases.

1) Single Class Traffic: This is the case where all the mobiles have a common target SINR requirement γ; $\beta := \frac{\gamma}{1+\gamma}$. In this case,

$$c_i(\mathbf{a}) = \frac{\sigma^2}{h_{ia_i}} \frac{\beta}{[1-|\mathcal{M}_{a_i}(\mathbf{a})|\beta]^+}$$

and we have a player specific unweighted singleton congestion game.

2) Collocated Mobiles: In this case, all mobiles are situated close together in a group. Thus $h_{ij} = h_j$ for all $i \in \mathcal{M}, j \in \mathcal{N}$, and

$$c_i(\mathbf{a}) = \frac{\sigma^2}{h_{a_i}} \frac{\beta_i}{[1-\sum_{l\in\mathcal{M}_{a_i}(\mathbf{a})}\beta_l]^+}.$$

This yields a player independent weighted singleton congestion game.

3) Collocated BSs: Here all BSs are assumed to be situated close together. Thus $h_{ij} = h_i$ for all $i \in \mathcal{M}, j \in \mathcal{N}$, and

$$c_i(\mathbf{a}) = \frac{\sigma^2}{h_i} \frac{\beta_i}{[1-\sum_{l\in\mathcal{M}_{a_i}(\mathbf{a})}\beta_l]^+}.$$

Now, we get a player specific weighted singleton congestion game.

The following result ensures that MAPC converges in each of these special cases.

Proposition 3.2: The finite strategic form game $(\mathcal{M},\mathcal{N},(c_i, i \in \mathcal{M}))$ is a generalized ordinal potential game and thus admits the FIP property in each of the following cases.

1) $\beta_i = \beta$ for all $i \in \mathcal{M}$,
2) $h_{ij} = h_j$ for all $i \in \mathcal{M}, j \in \mathcal{N}$,
3) $h_{ij} = h_i$ for all $i \in \mathcal{M}, j \in \mathcal{N}$.

Proof: In each case, we show that the game $(\mathcal{M},\mathcal{N},(c_i, i\in\mathcal{M}))$ is better response equivalent (Neel [12, Chapter 5]) to a generalized ordinal potential game (by demonstrating a potential function for the latter). This implies that, in each case, $(\mathcal{M},\mathcal{N},(c_i, i\in\mathcal{M}))$ itself is a generalized ordinal potential game. It is also finite which implies that the FIP property holds.

1) Let us first observe that, in MAPC algorithm, mobiles do not switch to a BS if the new aggregate load of the BS exceeds (or equals) 1. Therefore, in any improvement path, if a BS's aggregate load becomes permissible (i.e., < 1), it continues to be below 1. After finitely many steps we get a partition of the set of BSs in two sets such that BSs in the first set have permissible loads while those in the second set do not, and mobiles do not switch across these sets (the latter set may be empty). Hence, to investigate the FIP property, we focus on the set of BSs with permissible load and on the mobiles associated with them. Alternatively, we assume that, after finitely many steps, all the BSs have permissible loads. Now note that the strategic form game $(\mathcal{M},\mathcal{N},(c_i, i\in\mathcal{M}))$ is better response equivalent to $(\mathcal{M},\mathcal{N},(-\frac{1}{c_i}, i\in\mathcal{M}))$. Also note that

$$-\frac{1}{c_i(\mathbf{a})} = -\frac{h_{ia_i}}{\sigma^2}\frac{[1-|\mathcal{M}_{a_i}(\mathbf{a})|\beta]^+}{\beta}.$$

The function $V_1 : \mathcal{N}^{\mathcal{M}} \to \mathbb{R}$ given by

$$V_1(\mathbf{a}) = -\frac{1}{\sigma^2\beta}\prod_{l\in\mathcal{M}} h_{la_l} \prod_{k\in\mathcal{N}} \left(\prod_{t=1}^{|\mathcal{M}_k(\mathbf{a})|} [1-t\beta]^+\right)$$

satisfies

$$V_1(j,\mathbf{a}_{-i}) - V_1(\mathbf{a}) = -\left(\frac{1}{c_i(j,\mathbf{a}_{-i})} - \frac{1}{c_i(\mathbf{a})}\right)\prod_{l\in\mathcal{M}\setminus\{i\}} h_{la_l} \times \prod_{k\in\mathcal{N}}\left(\prod_{t=1}^{|\mathcal{M}_k(\mathbf{a})\setminus\{i\}|}[1-t\beta]^+\right)$$

for all $i \in \mathcal{M}, j \in \mathcal{N}, \mathbf{a} \in \mathcal{N}^{\mathcal{M}}$. Notice that all the product terms in the right hand side are strictly positive because all the BSs have permissible load. Thus the game $(\mathcal{M},\mathcal{N},(-\frac{1}{c_i}, i \in \mathcal{M}))$ is a generalized ordinal potential game with a potential function V_1.[3]

2) The strategic form game $(\mathcal{M},\mathcal{N},(c_i, i \in \mathcal{M}))$ is better response equivalent to $(\mathcal{M},\mathcal{N},(-\frac{\beta_i}{c_i}, i \in \mathcal{M}))$. Also note that

$$-\frac{\beta_i}{c_i(\mathbf{a})} = -\frac{h_{a_i}[1-\sum_{l\in\mathcal{M}_{a_i}(\mathbf{a})}\beta_l]^+}{\sigma^2}.$$

For the function $V_2 : \mathcal{N}^{\mathcal{M}} \to \mathbb{R}$ given by

$$V_2(\mathbf{a}) = -\sum_{i\in\mathcal{M}} \frac{h_{a_i}\beta_i\left(\left[1-\sum_{l\in\mathcal{M}_{a_i}(\mathbf{a})}\beta_l\right] + (1-\beta_i)\right)}{\sigma^2},$$

$$V_2(j,\mathbf{a}_{-i}) - V_2(\mathbf{a}) = -2\beta_i\left(\frac{h_j[1-\sum_{l\in\mathcal{M}_j(\mathbf{a})}\beta_l - \beta_i]}{\sigma^2} - \frac{h_{a_i}[1-\sum_{l\in\mathcal{M}_{a_i}(\mathbf{a})}\beta_l]}{\sigma^2}\right)$$

for all $i \in \mathcal{M}, j \in \mathcal{N}, \mathbf{a} \in \mathcal{N}^{\mathcal{M}}$. Therefore

$$-\frac{\beta_i}{c_i(j,\mathbf{a}_{-i})} < -\frac{\beta_i}{c_i(\mathbf{a})} \Rightarrow V_2(j,\mathbf{a}_{-i}) < V_2(\mathbf{a}).$$

So V_2 is a potential function for the game $(\mathcal{M},\mathcal{N},(-\frac{\beta_i}{c_i}, i \in \mathcal{M}))$, and so the latter is a generalized ordinal potential game.

3) The strategic form game $(\mathcal{M},\mathcal{N},(c_i, i \in \mathcal{M}))$ is better response equivalent to $(\mathcal{M},\mathcal{N},(-\frac{\beta_i}{h_i c_i}, i \in \mathcal{M}))$. Also note that

$$-\frac{\beta_i}{h_i c_i(\mathbf{a})} = -\frac{[1-\sum_{l\in\mathcal{M}_{a_i}(\mathbf{a})}\beta_l]^+}{\sigma^2}.$$

The function $V_3 : \mathcal{N}^{\mathcal{M}} \to \mathbb{R}$ defined as

$$V_3(\mathbf{a}) = -\sum_{i\in\mathcal{M}} \frac{\beta_i[1-\sum_{l\in\mathcal{M}_{a_i}(\mathbf{a})}\beta_l]}{\sigma^2}$$

satisfies

$$V_3(j,\mathbf{a}_{-i}) - V_3(\mathbf{a}) = -2\beta_i\left(\frac{[1-\sum_{l\in\mathcal{M}_j(\mathbf{a})}\beta_l - \beta_i]}{\sigma^2} - \frac{[1-\sum_{l\in\mathcal{M}_{a_i}(\mathbf{a})}\beta_l]}{\sigma^2}\right)$$

for all $i \in \mathcal{M}, j \in \mathcal{N}, \mathbf{a} \in \mathcal{N}^{\mathcal{M}}$. Therefore

$$-\frac{\beta_i}{h_i c_i(j,\mathbf{a}_{-i})} < -\frac{\beta_i}{h_i c_i(\mathbf{a})} \Rightarrow V_3(j,\mathbf{a}_{-i}) < V_3(\mathbf{a}).$$

So the game $(\mathcal{M},\mathcal{N},(-\frac{\beta_i}{h_i c_i}, i \in \mathcal{M}))$ is an generalized ordinal potential game with V_3 as a potential function. ■

[3]This potential function is similar to those proposed in [16] for linear cost functions, and in [18] for cost functions composed of player-specific constants and facility-specific functions.

Now, we focus on the general case. Gairing et al. [16] show (via a counter-example with 3 players) that player-specific weighted singleton congestion games with affine cost functions are not necessarily generalized ordinal potential games, and so, need not possess the FIP property. This negative result applies to our game also, and convergence proofs based on potential functions cannot be used. However, it follows from [13] that the strategic form game $(\mathcal{M},\mathcal{N},(c_i, i \in \mathcal{M}))$ admits (i) FIP property if $|\mathcal{N}| = 2$, (ii) FBRP property if $|\mathcal{M}| = 2$.

Georgiou et al. [32] establish that player-specific weighted singleton congestion games with 3 players and linear cost functions possess FBRP property. Mavronicolas et al. [18] broaden this result to generic cost functions with player-specific constants.[4] Specifically, they show in an exhaustive manner that such games do not possess any *best response cycles*.[5] Their result and proof technique extend to the game $(\mathcal{M},\mathcal{N},(c_i, i \in \mathcal{M}))$ even though the cost functions c_i are not linear. Thus, the game $(\mathcal{M},\mathcal{N},(c_i, i \in \mathcal{M}))$ can be shown to possess the FBRP property if $|\mathcal{M}| = 3$.

In the case of more than 3 players, convergence of the best response dynamics in weighted singleton congestion games with linear cost functions is an open problem [32], [34]. Georgiou et al. [32] conjecture that such games always admit at least one NE. Though functions c_i are not linear, the game $(\mathcal{M},\mathcal{N},(c_i, i \in \mathcal{M}))$ is best response equivalent to another game in which costs are composed of multiplicative player-specific constants and affine nondecreasing functions. Also, simulations run on numerous instances of the game suggest that players' updates as per the best response strategy always converge in a finite number of steps. We therefore conjecture that

Conjecture 3.1: The finite strategic form game $(\mathcal{M},\mathcal{N},(c_i, i \in \mathcal{M}))$ admits the FBRP property and thus possesses at least one pure strategy Nash equilibrium.

The FBRP property ensures that **MAPC** converges in a finite number of steps (see Remark 2.1). However, the round-robin update process requires some coordination to ensure that the designated mobile updates its association in a slot. Let us consider the following variants of **MAPC**.

1) *Random update process:* At each t, one mobile is randomly chosen to update its association, while ensuring that all the mobiles have strictly positive probabilities of being chosen. In a framework with no synchronizing agent and with an arbitrarily fine time-scale, it is unlikely that two mobiles update simultaneously. Random update process is a natural candidate in this setup.
2) *Asynchronous update process:* At each t, each mobile i updates its association with probability $\epsilon_i \in (0,1)$. There is thus a strictly positive probability that any subset of

[4]On the other hand, they also demonstrated a best response cycle in a game with 3 players and costs composed of additive player-specific constants and facility-specific nondecreasing functions. More recently, Gairing and Klimm [33] demonstrated lack of a NE in a 4 player singleton weighted congestion game with concave cost functions that differ by player-specific additive constants only.

[5]There does not seem to be any reason why this technique cannot be extended to more than 3 players; though the number of possibilities in the exhaustive search may become enormous.

mobiles may update their associations simultaneously. As before, all mobiles update their powers based on the new loads. This algorithm does not require any coordination among mobiles (to ensure one by one updates), and is thus fully distributed.

The FBRP property of the game $(\mathcal{M},\mathcal{N},(c_i, i \in \mathcal{M}))$ implies that these two algorithms also converge to a NE in finite number of steps with probability 1 (see Section II-A).

D. System Optimality

A system optimal power allocation should bring about the lowest interference environment. This motivates the following definition of system optimality.

Definition 3.1: For an association profile $\mathbf{a}$, define a system performance measure $C(\mathbf{a}) = \sum_{i=1}^{M} c_i(\mathbf{a})$ with $c_i(\mathbf{a})$ defined in (1). We define an association profile $\mathbf{a^o}$ to be system optimal if it minimizes $C(\mathbf{a})$ over all possible associations $\mathbf{a} \in \times_{i=1}^{M} \mathcal{A}_i$.

Let us now recall the following notion of Pareto efficiency [31, Chapter 5].

Definition 3.2: An association profile $\mathbf{a}$ is said to be Pareto dominated by another association profile $\mathbf{a}'$ if $c_i(\mathbf{a}') \le c_i(\mathbf{a})$ for all $i \in \mathcal{M}$ with $c_i(\mathbf{a}') < c_i(\mathbf{a})$ for some i. An association profile $\mathbf{a}$ is said to be Pareto efficient if it is not Pareto dominated by any other association profile in $\times_{i=1}^{M} \mathcal{A}_i$.

Clearly any association profile that is system optimal is also Pareto efficient. Thus, if there is a unique Pareto efficient association profile, it is also the unique system optimal one. However, unlike the case of single channel networks, joint association and power control problems in multichannel networks do not in general admit a unique Pareto efficient association profile. In particular, when $|\mathcal{M}| > |\mathcal{N}|$, there cannot be unique Pareto efficient association profile.[6] To see this, define Θ_i for any mobile i as the set of best match BSs as follows

$$\Theta_i := \operatorname*{argmin}_{j \in \mathcal{N}} \frac{\sigma^2 \gamma_i}{h_{ij}}$$

The system optimal association profile $\mathbf{a^o}$ is clearly Pareto efficient. Next, two cases are possible.

1) For all i, $a_i^o \in \Theta_i$. Since $|\mathcal{M}| > |\mathcal{N}|$, there exist two mobiles i and l such that $a_i^o = a_l^o$.
2) There exists a mobile i such that $a_i^o \notin \Theta_i$.

Consider a mobile i as in Case 1, or as in Case 2. Let $\mathbf{a}'$ be another profile which is system optimal subject to i being associated with any of its best match BSs and no other mobile being associated with that BS. It can be easily checked that $\mathbf{a}'$ is also Pareto efficient.

As the following example illustrates, **MAPC** may settle at a Pareto inefficient association profile, and hence may not be system optimal.

Example 3.1: Consider a network with two BSs, two mobiles, and a common SINR requirement γ. The two BSs operate in disjoint bands. Assume

$$h_{12} < h_{11} < \frac{h_{12}}{(1-\gamma)}$$
$$\text{and} \qquad h_{21} < h_{22} < \frac{h_{21}}{(1-\gamma)}.$$

The unique Pareto efficient association is $(a_1 = 1, a_2 = 2)$ with power allocation $(\frac{\sigma^2}{h_{11}}\gamma, \frac{\sigma^2}{h_{22}}\gamma)$. However, if we start with initial association $(a_1 = 2, a_2 = 1)$, **MAPC** will not move forward, because a unilateral switch requires larger power to meet the target SINR. Neither mobile will switch to the BS with which it has a better channel. Hence, $(\frac{\sigma^2}{h_{12}}\gamma, \frac{\sigma^2}{h_{21}}\gamma)$ is a steady state power vector at which the algorithm settles; thus $(a_1 = 2, a_2 = 1)$ is Pareto inefficient.

In the following we consider special cases, and investigate whether the proposed algorithm leads to a system optimal association profile.

1) Collocated Mobiles and Single Class Traffic: Even in this special case, **MAPC** may settle at a Pareto inefficient NE as shown in the following example.

Example 3.2: Consider a 2-cell network with 4 collocated mobiles and $\beta_i = \beta, i = 1,2,3,4$. Assume that h_1 and h_2 satisfy

$$\begin{aligned} h_1(1-3\beta) &= h_2(1-2\beta), \\ h_1(1-2\beta) &> h_2(1-\beta). \end{aligned}$$

The following facts are easily verified. Both the inequalities can be met simultaneously. The association $(a_1 = a_2 = a_3 = 1, a_4 = 2)$ is a NE from which the algorithm does not move. This association is Pareto dominated by $(a_3 = a_4 = 1, a_1 = a_2 = 2)$ which is another NE. Thus **MAPC** may settle at a Pareto inefficient NE.

Consider now a variant of **MAPC** in which mobile $i = 1+(t \mod M)$ updates its association at $t+1$ if[7]

$$a_i(t) \notin \operatorname*{argmin}_{j \in \mathcal{N}} \left(c_i(j, \mathbf{a}(t)_{-i}), \sum_{l \in \mathcal{M}_j(j,\mathbf{a}(t)_{-i})} \beta_l \right).$$

In this case,

$$a_i(t+1) \in \operatorname*{argmin}_{j \in \mathcal{N}} \left(c_i(j, \mathbf{a}(t)_{-i}), \sum_{l \in \mathcal{M}_j(j,\mathbf{a}(t)_{-i})} \beta_l \right).$$

In words, a mobile selects a least loaded BS (after taking its own load into account) among the ones which require transmission with the least power. We name this variant **MAPC***.

Proposition 3.3: Consider the case where $\beta_i = \beta$ and $h_{ij} = h_j$ for all $i \in \mathcal{M}, j \in \mathcal{N}$. If the strategic form game $(\mathcal{M},\mathcal{N},(c_i, i \in \mathcal{M}))$ contains no best response cycles, then **MAPC*** converges in a finite number of steps.

Proof: See [10]. ∎

The FBRP property (Conjecture 3.1) ensures that **MAPC*** converges in a finite number of steps. We now show that **MAPC*** converges to a Pareto efficient NE in the special case of collocated mobiles and single class traffic.

[6] However, different Pareto efficient association profiles may be identical up to a permutation, e.g., if two mobiles are indifferent with respect to their SINR requirements and channel gains to all the BSs.

[7] The minimization is with respect to the lexicographical ordering.

Proposition 3.4: For the noncooperative game $(\mathcal{M}, \mathcal{N}, (c_i, i \in \mathcal{M}))$, when $h_{ij} = h_j$ and $\beta_i = \beta$ for all $i \in \mathcal{M}, j \in \mathcal{N}$, the steady states of **MAPC*** are the Pareto efficient NEs of the game.

Proof: Propositions 3.2 and 3.3 imply that MAPC* converges in a finite number of steps in this special case. For any association profile $\mathbf{a}$, let $m_j(\mathbf{a})$ be the number of mobiles associated with BS j. Let $\mathbf{a}$ be a NE, and $\mathbf{a}'$ be another profile dominating $\mathbf{a}$. We show that the proposed variant of MAPC does not settle at $\mathbf{a}$.

We first argue that congestion vectors $\mathbf{m}(\mathbf{a}) = (m_N(\mathbf{a}), \cdots, m_N(\mathbf{a}))$ and $\mathbf{m}(\mathbf{a}')$ cannot be identical. Indeed if this is the case, $\mathbf{a}'$ is obtained by permuting the mobiles' associations in $\mathbf{a}$ in some way. But then their payoffs undergo the same permutation, which makes it impossible for all of them to gain.

We define $g_j = \frac{\sigma^2}{h_j}$ and $f(m) = \frac{\beta}{[1-m\beta]^+}$. Then a mobile associated with BS j incurs a cost $g_j f(m_j(\mathbf{a}))$. Further, $\mathbf{a}$ being a NE,

$$g_j f(m_j(\mathbf{a})) \le g_k f(m_k(\mathbf{a}) + 1)$$

for all $j, k \in \mathcal{N}$. In particular,

$$\begin{aligned} & m_j(\mathbf{a}) \le m_k(\mathbf{a}) + 1 \quad \text{if } g_k \le g_j, && \text{(3a)} \\ \text{and} \quad & m_j(\mathbf{a}) < m_k(\mathbf{a}) + 1 \quad \text{if } g_k < g_j. && \text{(3b)} \end{aligned}$$

Next, we define

$$\begin{aligned} \bar{c} \quad &:= \max_{j \in \mathcal{N}: m_j(\mathbf{a}) > 0} g_j f(m_j(\mathbf{a})), \\ \text{and } \mathcal{N}_1 \quad &:= \operatorname*{argmax}_{j \in \mathcal{N}: m_j(\mathbf{a}) > 0} g_j f(m_j(\mathbf{a})) \end{aligned}$$

Under $\mathbf{a}'$ none of the mobiles incurs a cost more than $\bar{c}$. In particular, those associated with a BS $j \in \mathcal{N}_1$ under a' must have cost less than $\bar{c}$. This implies $m_j(\mathbf{a}') \le m_j(\mathbf{a})$ for all $j \in \mathcal{N}_1$. Now suppose that $m_j(\mathbf{a}') = m_j(\mathbf{a})$ for all $j \in \mathcal{N}_1$, and $m_k(\mathbf{a}') > m_k(\mathbf{a})$ for a $k \in \mathcal{N} \setminus \mathcal{N}_1$. Then,

$$g_k f(m_k(\mathbf{a}')) \ge g_k f(m_k(\mathbf{a}) + 1) \ge g_j f(m_j(\mathbf{a}))$$

for any $j \in \mathcal{N}_1$. The last inequality holds because $\mathbf{a}$ is an NE. Thus we have that

$$g_k f(m_k(\mathbf{a}')) \ge \bar{c},$$

and hence there are more mobiles incurring costs greater than or equal to $\bar{c}$ under $\mathbf{a}'$ than under $\mathbf{a}$. This contradicts the hypothesis that $\mathbf{a}'$ Pareto dominates $\mathbf{a}$. Thus there must be BSs $j \in \mathcal{N}_1, k \in \mathcal{N} \setminus \mathcal{N}_1$ with $m_j(\mathbf{a}') < m_j(\mathbf{a})$ and $m_k(\mathbf{a}') > m_k(\mathbf{a})$ which is same as $m_k(\mathbf{a}') \ge m_k(\mathbf{a}) + 1$. Again, $\mathbf{a}$ being an NE,

$$g_k f(m_k(\mathbf{a}')) \ge \bar{c}.$$

But the hypothesis that $\mathbf{a}'$ Pareto dominates $\mathbf{a}$ implies that

$$g_k f(m_k(\mathbf{a}')) \le \bar{c}.$$

Thus k must belong to the set

$$\mathcal{N}_2 := \{k \in \mathcal{N} \setminus \mathcal{N}_1 : g_k f(m_k(\mathbf{a}) + 1) = \bar{c}\}$$

Moreover, $m_k(\mathbf{a}') = m_k(\mathbf{a}) + 1$.

Now, we claim that there exist BSs $j \in \mathcal{N}_1$ and $k \in \mathcal{N}_2$ such that $g_j < g_k$. Assume this claim holds. Then,

$$g_k f(m_k(\mathbf{a}) + 1) = \bar{c} = g_j f(m_j(\mathbf{a}))$$

implies that $m_j(\mathbf{a}) > m_k(\mathbf{a}) + 1$. Thus, under the proposed algorithm, one of the mobiles associated with BS j moves to BS k, i.e., the algorithm does not settle at $\mathbf{a}$.

We prove the claim via contradiction. Suppose $g_j \ge g_k$ for all $j \in \mathcal{N}_1, k \in \mathcal{N}_2$. Obtaining $\mathbf{a}'$ from $\mathbf{a}$ may involve three types of load transfers.

1) One mobile moves from a BS $j \in \mathcal{N}_1$ to a BS $k \in \mathcal{N}_2$ such that $g_j = g_k$. By the definition of $\mathcal{N}_2$, such moves only permute the overall cost profile, and by themselves cannot lead to $\mathbf{a}'$.
2) One mobile moves from a BS $j \in \mathcal{N}_1$ to a BS $k \in \mathcal{N}_2$ such that $g_j > g_k$. Then, the cost reduces for $m_j(\mathbf{a}) - 1$ mobiles that are still with BS j, but increases to $\bar{c}$ for $m_k(\mathbf{a}) > m_j(\mathbf{a}) - 1$ mobiles (see (3a)). Such moves also cannot lead to the association profile $\mathbf{a}'$.
3) $n > 1$ mobiles move from a BS $j \in \mathcal{N}_1$ to BSs $k_1, \ldots, k_n \in \mathcal{N}_2$ (they have to move to different BSs, again by the definition of $\mathcal{N}_2$). Now, the cost reduces for $m_j(\mathbf{a}) - n$ mobiles, but increases to $\bar{c}$ for

$$\sum_{l=1}^{n} m_{k_l}(\mathbf{a}) \ge n(m_j(\mathbf{a}) - 1) > m_j(\mathbf{a}) - n \quad \text{(see (3b))}$$

mobiles. Such moves also cannot lead to the association profile $\mathbf{a}'$.

Thus there must be BSs $j \in \mathcal{N}_1$ and $k \in \mathcal{N}_2$ such that $g_j < g_k$ as claimed. This completes the proof of the proposition. ■

However, the obtained Pareto efficient association profile need not be system optimal. This is demonstrated by Example 4.2 for the case of a continuum of mobiles.

2) Collocated BSs and Single Class Traffic: Next, we consider the case where the mobiles have identical target SINR requirements and the BSs are collocated, so that $h_{ij} = h_i$ for all $i \in \mathcal{M}, j \in \mathcal{N}$. For any association profile $\mathbf{a}$, define its *support* $\mathcal{S}_{\mathbf{a}}$ to be the set $\{j \in \mathcal{N} : a_l = j \text{ for some } l \in \mathcal{M}\}$. We say that $\mathbf{a}$ has *full support* if $\mathcal{S}_{\mathbf{a}} = \mathcal{N}$.

Lemma 3.1: In the game $(\mathcal{M}, \mathcal{N}, (c_i, i \in \mathcal{M}))$, when $\beta_i = \beta$ and $h_{ij} = h_i$ for all $i \in \mathcal{M}, j \in \mathcal{N}$, any association profile with full support is Pareto efficient.

Proof: See [10]. ■

Proposition 3.5: All the Nash equilibria in the game $(\mathcal{M}, \mathcal{N}, (c_i, i \in \mathcal{M}))$, when $\beta_i = \beta$ and $h_{ij} = h_i$ for all $i \in \mathcal{M}, j \in \mathcal{N}$, are Pareto efficient.

Proof: Let $\mathbf{a}^*$ be a Nash equilibrium. The following are the two possible scenarios.

1) $\mathbf{a}^*$ *does not have full support:* We must have $|\mathcal{M}| \le |\mathcal{N}|$. Indeed, if $|\mathcal{M}| > |\mathcal{N}|$ and $\mathbf{a}^*$ does not have full support, then there must be mobiles i and l with $a_i^* = a_l^*$ and a BS j with $\mathcal{M}_j(\mathbf{a}^*) = \emptyset$. Clearly, mobile i benefits by moving to BS j. This contradicts the fact that $\mathbf{a}^*$ is a NE. Next, $i \ne l$ implies $a_i^* \ne a_l^*$ for the same reason as explained above. Since all BSs have the same channel gain to a mobile, $\mathbf{a}^*$ is Pareto efficient.

2) $\mathbf{a}^*$ *has full support:* Lemma 3.1 implies that $\mathbf{a}^*$ is Pareto efficient

■

However, a NE need not be system optimal if the mobiles are not collocated as shown in the following example.

Example 3.3: Consider a 2-cell network with 5 mobiles. The 2 BSs are collocated. Further, $h_{ij} = ih$ and $\beta_i = \beta, i = 1, 2, 3, 4, 5$ where $\frac{1}{4} < \beta < \frac{1}{3}$. Any profile in which two mobiles associate with one BS, and the remaining three with another is a NE. On the other hand, $(a_1 = a_2 = 1, a_3 = a_4 = a_5 = 2)$ and $(a_1 = a_2 = 2, a_3 = a_4 = a_5 = 1)$ are the only socially optimal NEs.

3) Collocated BSs and Symmetrically Placed Mobiles: Now, we consider the case where $h_{ij} = h$ for all $i \in \mathcal{M}, j \in \mathcal{N}$. Mobiles may have different target SINR requirements. In this case also MAPC may settle at a Pareto inefficient NE as shown in the following example.

Example 3.4: Consider a 2-cell network with 6 mobiles. The BSs are collocated and the mobiles are symmetrically located around them. Assume $\beta_1 = \beta_2 = 0.3, \beta_3 = 0.4, \beta_4 = \beta_5 = 0.5$ and $\beta_6 = 0.6$. It can be seen that $(a_1 = a_6 = 1, a_2 = a_5 = 2, a_3 = a_4 = 3)$ is a feasible association profile, so we have a feasible problem at hand. But $(a_1 = a_2 = 1, a_3 = a_6 = 2, a_4 = a_5 = 3)$ is a NE which is infeasible and also Pareto inefficient because it is dominated by $(a_1 = a_2 = 1, a_3 = 2, a_4 = a_5 = a_6 = 3)$.

4) Collocated Mobiles, Symmetrically Placed BSs, and Single Class Traffic: In the special case when all the mobiles are collocated and all the BSs are symmetrically placed with respect to the collocated mobiles, we have the following result.

Proposition 3.6: All the NEs in the game $(\mathcal{M}, \mathcal{N}, (c_i, i \in \mathcal{M}))$, with $\beta_i = \beta$ and $h_{ij} = h$ for all $i \in \mathcal{M}, j \in \mathcal{N}$, are system optimal.

Proof: The mobiles as well as BSs are indistinguishable in this game. At a NE, let m_j be the number of mobiles associated with BS j. We first prove that at any NE, the vector of mobiles' costs is unique up to permutations. To prove this, it suffices to prove that the vector $\mathbf{m} = (m_j, j \in \mathcal{N})$ for a NE is unique up to permutations. As $\mathbf{m}$ yields a NE, the following must hold for all $j, k \in \mathcal{N}$:

$$\begin{aligned} \frac{\sigma^2}{h} \frac{\beta}{1 - m_j \beta} &\leq \frac{\sigma^2}{h} \frac{\beta}{1 - m_k \beta - \beta} \\ \text{or } m_j &\leq m_k + 1. \end{aligned} \tag{4}$$

Define $n = \lfloor \frac{M}{N} \rfloor$ and $l = M - nN$. From (4) we see that $\mathbf{m}$ given by $m_j = n + 1, j = 1, \ldots, l, m_j = n, j = l + 1, \ldots, N$ characterizes one of the NEs; other NEs are permutations of this vector, and m is unique up to permutations. We now show that $\mathbf{m}$ is a system optimal congestion vector, and the system optimality of all other NEs follows. To do this observe that

$$C(\mathbf{a}) = \frac{\sigma^2}{h} \sum_{i \in \mathcal{M}} \frac{\beta}{1 - m_{a_i} \beta} = \frac{\sigma^2}{h} \sum_{j \in \mathcal{N}} \frac{m_j \beta}{1 - m_j \beta}$$

is a *Schur-convex* function in $(m_1, \ldots, m_N)$ because $\frac{x}{1-x}$ is a convex function. This implies that the minimum value is attained at a vector which is as close to uniform as possible, i.e., a vector that is *majorized* by any other vector (Marshall & Olkin [35]).[8] All such vectors are permutations of $\mathbf{m}$ (Alternatively, if there exist BSs j and k such that $m_j \geq m_k + 2$, moving a mobile from BS j to BS k results in a strictly lower cost). This concludes the proof. ■

IV. Continuum of Mobiles

In this section, we consider a nonatomic version of the system in Section III-A. Such a model is of interest for two reasons. First, for many of the fixed QoS traffic classes (e.g., voice), the target SINR requirements in CDMA cellular systems are very small. In a typical IS 95 CDMA system with system bandwidth 1.25 MHz, chip rate 1.2288 Mcps, data rate 9.6 Kbps, and target $\frac{E_b}{N_0} = 6$ dB, the target SINR turns out to be -15 dB, i.e., $\frac{1}{32}$ (Kumar et al. [31, Chapter 5]). If we assume that at any time the number of mobiles associated with a BS is large, it is reasonable to say that an incoming mobile or an outgoing mobile has a negligible effect on the congestion. Secondly, we have seen that our proposed algorithm may end up with inefficient associations. There is extensive work on toll mechanisms that induce system optimality in networks with a continuum of mobiles. The analysis of toll-mechanisms (or pricing) on a multichannel multicell network with a continuum of mobiles can be expected to shed light on the existence and properties of pricing mechanisms for networks with discrete mobiles.

A. System Model

Let $\mathcal{M} = \cup_{l=1}^{L} \mathcal{M}_l$ be an infinite set of $\mathcal{L} = \{1, \ldots, L\}$ classes of nonatomic mobiles. By nonatomic mobiles, we mean that the effect of a single mobile at a BS is infinitesimal. The population of class l mobiles has "mass" M_l. All the mobiles in a class are collocated and require equal minimum SINR. In particular, all such mobiles have the same power gains to any of the BSs (gains from a mobile to different BSs can be different). Assume $\mathcal{N}$ to be the finite set of BSs. As before, σ denotes the common standard deviation of receiver noise at all BSs. Let γ_l be the common minimum required SINR density for class l mobiles, and h_{lj} be the power gain between a class l mobile and BS j. An association profile a is a measurable function $a : \mathcal{M} \to \mathcal{N}$. Any association a leads to a congestion profile $\mathbf{m}(a) = (m_{lj}(a), l \in \mathcal{L}, j \in \mathcal{N})$, $m_{lj}(a)$ being the mass of class l mobiles associated with BS j. Let $\mathcal{M}^*$ denote the set of all such congestion profiles.

Under an association profile a and a power density allocation $p : \mathcal{M} \to \mathbb{R}_+$, the SINR density for $x \in \mathcal{M}_l, l \in \mathcal{L}$ is

$$\frac{h_{la(x)} p(x)}{\sum_{l=1}^{L} \int_{\mathcal{M}_l} 1_a(x, z) h_{la(z)} p(z) dz + \sigma^2},$$

$$\text{where} \qquad 1_a(x, z) = \begin{cases} 1, & \text{if } a(x) = a(z) \\ 0, & \text{otherwise} \end{cases}$$

Our definition of a "class" makes all the mobiles in a class alike, and so, congestion profiles are sufficient to characterize

[8]The condition $h_{ij} = h_i$ for all $j \in \mathcal{N}$ is used to deduce that NE profiles are majorized by any non NE profile; the condition $h_{ij} = h_j$ for all $i \in \mathcal{M}$ is used to deduce Schur-convexity of $C(a)$.

the system. In the sequel, we just use m_{lj} for $m_{lj}(a)$ for convenience. The dependence on a is understood.

Consider again the subproblem of power control with a fixed congestion profile $\mathbf{m}$. The following result is analogous to Proposition 3.1, and is shown in [10, Appendix A].

Proposition 4.1:
1) The power control subproblem of BS j is feasible iff $\sum_{l\in\mathcal{L}} m_{lj}\gamma_l < 1$.
2) If the power control subproblem of BS j is feasible, there exists a unique efficient[9] power density p given by

$$p(x) = \frac{\sigma^2}{h_{lj}} \frac{\gamma_l}{1 - \sum_{l\in\mathcal{L}} m_{lj}\gamma_l},$$

$\forall x \in \mathcal{M}_l$ such that $a(x) = j, l \in \mathcal{L}$, where a is the underlying association profile.

An evolutionary dynamics can be proposed to address the combined association and power control problem. To this end, we define functions $c_{lj} : \mathcal{M}^* \to \mathbb{R}_+$, where $c_{lj}(\mathbf{m})$ denotes the minimum power density for class l mobiles associated with BS j under congestion profile $\mathbf{m}$, as

$$c_{lj}(\mathbf{m}) = \frac{\gamma_l\sigma^2}{h_{lj}[1 - \sum_{l\in\mathcal{L}} m_{lj}\gamma_l]^+}.$$

For notational convenience, define

$$g_{lj} = \frac{\gamma_l\sigma^2}{h_{lj}},$$

$$m_j = \sum_{l=1}^{L} \gamma_l m_{lj}, \forall j \in \mathcal{N}$$

$$\text{and} \qquad c(z) = \begin{cases} \frac{1}{1-z}, & \text{if } z < 1 \\ \infty, & \text{if } z \geq 1. \end{cases}$$

We then have

$$c_{lj}(\mathbf{m}) = g_{lj}c(m_j). \tag{5}$$

Again we assume that the system is feasible, i.e., there exists a feasible assignment, as done in Section III-B. This boils down to assuming $\sum_{l\in\mathcal{L}} \gamma_l M_l < N$ in the case of nonatomic mobiles. Now, structures of the cost functions allow us to restrict attention to the region where $m_j < 1, \forall j \in \mathcal{N}$; if $m_j \geq 1$ for a $j \in \mathcal{N}$, all the mobiles associated with j incur infinite cost.

B. A Congestion Game Formulation

We model the problem as a nonatomic congestion game. The continuum of mobiles constitute the population, and $\mathcal{N}$ denotes the common action set for players of all the classes. Class l players are accompanied by cost functions $c_{lj}(\mathbf{m}), j \in \mathcal{N}$. In the following, we refer to it as the game $(\mathcal{M}, \mathcal{N}, (c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$.

Proposition 4.2: The nonatomic game $(\mathcal{M}, \mathcal{N}, (c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$ is a potential game. Furthermore, it admits at least one NE, and the set of NEs coincides with the set of minimizers of the potential function.

[9]Efficiency is in the sense that the aggregate transmit power across the continuum of mobiles is minimized.

Proof: In the region $\{\mathbf{m} : m_j < 1, \forall j \in \mathcal{N}\}$, the function $V : \mathcal{M}^* \to \mathbb{R} \cup \{+\infty\}$ defined as

$$V(\mathbf{m}) := \sum_{j\in\mathcal{N}} \left(\sum_{l\in\mathcal{L}} \gamma_l m_{lj} \log g_{lj} + \int_0^{m_j} \log c(x)dx \right), \tag{6}$$

is a $\mathbf{C}^1$ function with

$$\frac{\partial V(\mathbf{m})}{\partial m_{lj}} = \gamma_l \log g_{lj} + \gamma_l \log c(m_j) = \gamma_l \log c_{lj}(\mathbf{m})$$

for all $l \in \mathcal{L}, j \in \mathcal{N}, \mathbf{m} \in \mathcal{M}^*$. Thus the nonatomic game $(\mathcal{M}, \mathcal{N}, (\gamma_l \log c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$ is a potential game with $V(\mathbf{m})$ as a potential function (see Definition 2.5). Note that the strategic form game $(\mathcal{M}, \mathcal{N}, (c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$ is better response equivalent to $(\mathcal{M}, \mathcal{N}, (\gamma_l \log c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$. Thus the former is also a potential game with the same potential function $V(\mathbf{m})$.

Now consider the following optimization problem

$$\begin{aligned} \text{Minimize} \quad & V(\mathbf{m}) \\ \text{subject to} \quad & \sum_{j\in\mathcal{N}} m_{lj} = M_l, \; l \in \mathcal{L} && \text{(7a)} \\ & m_{lj} \geq 0, \; l \in \mathcal{L}, j \in \mathcal{N}. && \text{(7b)} \end{aligned}$$

All the conditions are self-explanatory. Observe that

$$\frac{\partial^2 V(\mathbf{m})}{\partial m_{ik} m_{lj}} = \begin{cases} \gamma_i\gamma_l c(m_j) & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, with an appropriate ordering of the components of m, the Hessian of $V(\mathbf{m})$ is given by

$$\nabla^2 V(\mathbf{m}) = \begin{bmatrix} c(m_1)D & 0 & \dots & 0 \\ 0 & c(m_2)D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c(m_N)D \end{bmatrix},$$

where

$$\begin{aligned} D \quad &:= \quad \Gamma\Gamma^T, && \text{(8a)} \\ \text{and } \Gamma \quad &:= \quad [\gamma_1, \cdots, \gamma_L]^T. && \text{(8b)} \end{aligned}$$

Clearly, D, and hence $\nabla^2 V(\mathbf{m})$ is a positive semi-definite matrix. Thus, $V(\mathbf{m})$ is a convex function of $\mathbf{m}$. Since we are minimizing a convex objective function subject to linear constraints, there exists at least one minimizer, and all minima are global minima. Also, Kuhn-Tucker first order conditions are necessary and sufficient [36, Section 5.5.3]. Combining this with the fact that NEs are the profiles which satisfy the Kuhn-Tucker first order conditions for a minimizer of the potential function (see Section II-B), we see that the set of NEs coincides with the set of minimizers of the potential function. ■

Remark 4.1: The assertion in the above proposition does not hold for general population games. While all local minimizers of potential function are equilibria, not all equilibria minimize potential (even locally) in general [20, Section 3]. This is unlike finite player potential games where only equilibria are the local minimizers of potential functions.

Furthermore, NEs have the following property (see [37, Proposition 3.3]).

Proposition 4.3: The congestion at a BS is constant across all the NEs of the game $(\mathcal{M}, \mathcal{N}, (c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$. Consequently, the cost density for a class is also constant across all the NEs.

Remark 4.2: At NEs, the congestions (at BSs) by class, m_{lj}, are not unique because the objective function (6) is not *strictly* convex with respect to this set of variables.

C. System Optimality

Analogous to the definition in Section III-D, we define the system performance measure

$$C(\mathbf{m}) := \sum_{j \in \mathcal{N}} \sum_{l=1}^{L} m_{lj} g_{lj} c(m_j). \tag{9}$$

A congestion profile $\mathbf{m}^* \in \mathcal{M}^*$ is said to be system optimal if it minimizes $C(\mathbf{m})$ over all possible profiles $\mathbf{m} \in \mathcal{M}^*$.

In contrast with the discrete mobiles case where equilibria need not be Pareto efficient (see Example 3.1), we have the following result for the nonatomic case.

Proposition 4.4: All NEs of the nonatomic game $(\mathcal{M}, \mathcal{N}, (c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$ are Pareto efficient.

Proof: Let $\mathbf{m}$ be a NE congestion profile. Under a NE, the cost densities for the mobiles of the same class are equal, irrespective of their associations (see Remark 2.2). Thus, it is sufficient to prove that there does not exist another congestion profile $\mathbf{m}'$ such that for every class l, and for all BSs j, k, with $m_{lj} > 0, m'_{lk} > 0$,

$$c_{lk}(\mathbf{m}') \leq c_{lj}(\mathbf{m}), \tag{10}$$

and strict inequality holds for some such l, j and k. Assume that such an $\mathbf{m}'$ exists. Then,

$$g_{lk} c(m'_k) < g_{lj} c(m_j) \leq g_{lk} c(m_k)$$

where the last inequality follows because $\mathbf{m}$ is a NE and $m_{lj} > 0$. This yields $m'_k < m_k$. This further implies that there is a BS s such that $m'_s > m_s$, and a class t such that $m'_{ts} > m_{ts}$. By the strictly increasing property of c, we have

$$g_{ts} c(m'_s) > g_{ts} c(m_s) \geq g_{tr} c(m_r)$$

for a BS r such that $m_{tr} > 0$. Such a BS of course exists and the latter inequality follows because $\mathbf{m}$ is a NE. The two inequalities imply $c_{ts}(\mathbf{m}') > c_{tr}(\mathbf{m})$, and so the tuple t, r, s violates (10). Thus the assumption that $\mathbf{m}'$ Pareto dominates $\mathbf{m}$ is incorrect. This completes the proof. ■

We show that the NEs are system optimal if all the mobiles are collocated, and all the BSs are symmetrically placed around them.

Proposition 4.5: All NEs in the nonatomic game $(\mathcal{M}, \mathcal{N}, (c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$, with $h_{lj} = h$ for all $l \in \mathcal{L}, j \in \mathcal{N}$, are system optimal.

Proof: In the case of collocated base stations

$$\begin{aligned} C(\mathbf{m}) &= \frac{\sigma^2}{h} \sum_{j \in \mathcal{N}} \sum_{l=1}^{L} \gamma_l m_{lj} c(m_j) \\ &= \frac{\sigma^2}{h} \sum_{j \in \mathcal{N}} m_j c(m_j) \end{aligned}$$

For the reason described earlier, we restrict attention to the region where $m_j < 1, \forall j \in \mathcal{N}$. In this region,

$$\frac{d}{dm_j} m_j c(m_j) = \frac{1}{(1 - m_j)^2},$$

and so $m_j c(m_j)$ is a convex function of m_j. Thus $C(\mathbf{m})$ is a Schur-convex function of $(m_j, 1 \leq j \leq N)$, and is minimized at any $\mathbf{m}^*$ with

$$m_j^* = \frac{1}{N} \sum_{l \in \mathcal{L}} \gamma_l M_l$$

for all $j \in \mathcal{N}$. When $h_{lj} = h$ for all $l \in \mathcal{L}, j \in \mathcal{N}$, any congestion profile with equal congestion at all the BSs is a NE. Thus, the system optimal profile $\mathbf{m}^*$ is a NE. Since all the NEs incur equal cost (see Proposition 4.3), all NEs are system optimal. ■

However, NEs need not be system optimal if BSs are not collocated, or mobile are not collocated. We illustrate these facts through the following examples.

Example 4.1: Consider an infinite set $\mathcal{M}$ of nonatomic mobiles belonging to two classes; class 1 and class 2 mobiles have masses M_1 and $3M_1$ respectively. Assume common minimum SINR density requirement γ, and let $3M_1\gamma < 1$. Let there be two collocated BSs. Let the power gain between a class l mobile and a BS be h_l, $h_1 < \frac{h_2}{3}$. A congestion profile is a NE if and only if it assigns equal load to both the BSs. Thus, the total cost incurred at NE

$$C^* = \frac{\gamma M_1 \sigma^2}{h_1(1 - 2\gamma M_1)} + \frac{3\gamma M_1 \sigma^2}{h_2(1 - 2\gamma M_1)}.$$

Next, consider a profile in which class 1 mobiles associate with BS 1 and class 2 mobiles associate with BS 2. The cost incurred now is

$$C = \frac{\gamma M_1 \sigma^2}{h_1(1 - \gamma M_1)} + \frac{3\gamma M_1 \sigma^2}{h_2(1 - 3\gamma M_1)}.$$

It can be easily checked that $C < C^*$ if

$$M_1 < \frac{1}{\gamma} \frac{h_2/3 - h_1}{h_2 - h_1}.$$

Example 4.2: Consider an infinite set $\mathcal{M}$ of nonatomic mobiles all belonging to same class; $M := |\mathcal{M}|$. Assume common minimum SINR density requirement γ, and let $M\gamma < 1$. Let there be two BSs with h_j the gain to BS j, $j = 1, 2$. An NE congestion profile $(\alpha^* M, (1 - \alpha^*)M)$ is given as

1) if $\frac{h_1}{h_2} \leq (1 - M\gamma)$, $\alpha^* = 0$,
2) if $\frac{h_2}{h_1} \leq (1 - M\gamma)$, $\alpha^* = 1$,
3) otherwise, α^* satisfies

$$\begin{aligned} \frac{\gamma\sigma^2}{h_1(1 - \alpha^*\gamma M)} &= \frac{\gamma\sigma^2}{h_2(1 - (1 - \alpha^*)\gamma M)} \\ \text{i.e., } \frac{1 - \alpha^*\gamma M}{1 - (1 - \alpha^*)\gamma M} &= \frac{h_2}{h_1}. \end{aligned} \tag{11}$$

On the other hand, a congestion profile $(\alpha^o M, (1 - \alpha^o M)$ will be system optimal if and only if α^o solves the following optimization problem:

$$\begin{aligned} &\text{Minimize} \quad \frac{\alpha\gamma M \sigma^2}{h_1(1 - \alpha\gamma M)} + \frac{(1 - \alpha)\gamma M \sigma^2}{h_2(1 - (1 - \alpha)\gamma M)} \\ &\text{subject to} \quad 0 \leq \alpha \leq 1. \end{aligned} \tag{12}$$

This is a convex optimization problem, and it is straightforward to show that

1) if $\sqrt{\frac{h_1}{h_2}} \leq (1 - M\gamma)$, $\alpha^o = 0$,
2) if $\sqrt{\frac{h_2}{h_1}} \leq (1 - M\gamma)$, $\alpha^o = 1$,
3) otherwise, α^o satisfies

$$\frac{1 - \alpha^o \gamma M}{1 - (1 - \alpha^o)\gamma M} = \sqrt{\frac{h_2}{h_1}} \tag{13}$$

Hence, if $\min\{\frac{h_1}{h_2}, \frac{h_2}{h_1}\} > 1 - M\gamma$, then $\min\{\sqrt{\frac{h_1}{h_2}}, \sqrt{\frac{h_2}{h_1}}\} > 1 - M\gamma$, and α^* and α^o must satisfy (11) and (13) respectively. In such a case, the NE will be system optimal if and only if $h_1 = h_2$.

V. Pricing for System Optimality

A. Continuum of Mobiles

In this section, we show that there is a toll mechanism that can induce system optimal associations and power allocations in a cellular network with multiple classes of mobiles. We also show that the mechanism can be employed in a distributed fashion.

Define

$$c'(z) := \begin{cases} \frac{d}{dz}c(z) = \frac{1}{(1-z)^2}, & \text{if } z < 1 \\ \infty, & \text{if } z \geq 1 \end{cases}$$

Consider a congestion profile $\mathbf{m} = (m_{lj}, l \in \mathcal{L}, j \in \mathcal{N})$. We propose that a class l mobile joining BS j be levied a toll

$$t_{lj}(\mathbf{m}) = \gamma_l \sum_{i=1}^{L} m_{ij} g_{ij} c'(m_j). \tag{14}$$

Now, define $\bar{c}_{lj}(\cdot) = c_{lj}(\cdot) + t_{lj}(\cdot), \forall l \in \mathcal{L}, j \in \mathcal{N}$, and consider the nonatomic game $(\mathcal{M}, \mathcal{N}, (\bar{c}_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$. Players may incur different power costs ($c_{lj}(\cdot)$) in different NEs of this game. Therefore, one has to distinguish between the following two cases (see Fotakis & Spirakis [38]).

1) A toll mechanism is said to *weakly enforce* system optimality if some NE of the game with tolls is an optimal profile.
2) It is said to *strongly enforce* system optimality if all the NEs of the game with tolls are optimal profiles.

We show that tolls $t_{lj}(\cdot)$ weakly enforce system optimality in all cases and strongly enforce it in a special setting.

Proposition 5.1: The nonatomic game $(\mathcal{M}, \mathcal{N}, (\bar{c}_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$ is a potential game. Furthermore, a congestion profile $\mathbf{m}$ is system optimal only if it is a NE of this game.

Proof: See [10]. ∎

If all the mobiles are collocated, the proposed tolls strongly enforce system optimality.

Proposition 5.2: All NEs in the nonatomic game $(\mathcal{M}, \mathcal{N}, (\bar{c}_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$, with $h_{lj} = h_j$ for all $l \in \mathcal{L}, j \in \mathcal{N}$, are system optimal.

Proof: It suffices to show that $C(\mathbf{m})$ is a convex function if $h_{lj} = h_j$ for all $l \in \mathcal{L}, j \in \mathcal{N}$. Then, any congestion profile satisfying Karush-Kuhn-Tucker conditions (i.e., any NE) is system optimal.

If $h_{lj} = h_j$ for all $l \in \mathcal{L}, j \in \mathcal{N}$,

$$C(\mathbf{m}) = \sum_{j \in \mathcal{N}} \frac{m_j c(m_j) \sigma^2}{h_j}.$$

Using the observation $c(x) + xc'(x) = c'(x)$, it is easy to see that

$$\frac{\partial^2 C(\mathbf{m})}{\partial m_{ik} m_{lj}} = \begin{cases} \frac{\gamma_i \gamma_l c''(m_j)\sigma^2}{h_j} & \text{if } k = j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nabla^2 C(\mathbf{m}) = \begin{bmatrix} \frac{c''(m_1)\sigma^2}{h_1} D & 0 & \dots & 0 \\ 0 & \frac{c''(m_2)\sigma^2}{h_2} D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{c''(m_N)\sigma^2}{h_N} D \end{bmatrix},$$

with D given by (8a)-(8b). It is now obvious that $\nabla^2 C(\mathbf{m})$ is a positive semi-definite matrix, and so $C(\mathbf{m})$ is a convex function of $\mathbf{m}$. ∎

However, the tolls $t_{lj}(\mathbf{m})$ may fail to strongly enforce a system optimal congestion profile even if all the BSs collocated, the mobiles require a constant SINR density γ, but they are not collocated. To see this, consider the congestion profile $\mathbf{m}^*$ with

$$m^*_{lj} = \frac{M_l}{N} \ \forall l \in \mathcal{L}, j \in \mathcal{N}.$$

It can be easily checked that, for all $l \in \mathcal{L}$,

$$\bar{c}_{lj}(\mathbf{m}^*) = \frac{\gamma\sigma^2}{h_l} c\left(\frac{\gamma M}{N}\right) + \sum_{i=1}^{L} \frac{\gamma^2\sigma^2}{h_i} \frac{M_i}{N} c'\left(\frac{\gamma M}{N}\right),$$

which is independent of $j \in \mathcal{N}$. Thus $\mathbf{m}^*$ is a NE of the game $(\mathcal{M}, \mathcal{N}, (\bar{c}_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$. But $\mathbf{m}^*$ may not be system optimal (see Example 4.1).

Remark 5.1: 1) $\bar{c}_{lj} = c_{lj} + t_{lj}$ can be interpreted as the marginal cost due to additional association of class l mobiles to BS j. The term c_{lj} is the power density incurred by these new mobiles, and t_{lj} is the increase in power consumption densities of the mobiles already associated with BS j, integrated over all such mobiles. Economists call them “private cost” and “social cost”, respectively. Selfish mobiles do not care for the social cost, while the social optimality criterion accounts for this marginal externality [39].

2) The cost functions for various classes have a certain structure in the settings of interest to us. Mobile classes that consider a BS pay tolls proportional to their required SINR densities. In particular, tolls are uniform across all the mobile classes that have equal SINR requirements. This is special to our setting; usually one does not see uniform tolls in the case of multiclass networks (see Dafermos [23], Smith [24]).

This toll mechanism can be implemented in a distributed fashion. All the BSs broadcast the tolls (normalized by SINR densities) along with their aggregate congestions as before.[10] All mobiles need to know their scaled gains $\frac{h_{lj}}{\sigma^2}$ to each BS $j \in \mathcal{N}$. A mobile then makes a choice taking both power density and toll into account.

[10]Normalized tolls $\frac{t_{lj}}{\gamma_l}$ are uniform across all mobile classes that consider a BS. A mobile can recover the exact toll from the normalized value.

B. Discrete Mobiles

Pricing mechanisms for networks with discrete mobiles are relatively difficult to design and analyze (Fotakis & Spirakis [38]). Again, we propose a toll mechanism that weakly enforces system optimality in all cases and strongly enforces it in a special setting. The mechanism is motivated by the toll mechanism for the nonatomic case (Theorem 5.1).

Consider the network model of Section III-A and an association profile $\mathbf{a}'$. Let mobile i evaluate BS j for association. Define $\mathbf{a} = (j, \mathbf{a}'_{-i})$. Analogous to the nonatomic case, define "private" and "social" costs as

$$\begin{aligned} c_i(\mathbf{a}) &= \frac{\sigma^2}{h_{ij}} \frac{\beta_i}{[1-\sum_{k\in\mathcal{M}_j(\mathbf{a})}\beta_k]^+}, \\ \text{and } t_i(\mathbf{a}) &= \sum_{l\in\mathcal{M}_j(\mathbf{a})\setminus\{i\}} \frac{\sigma^2}{h_{lj}} \left(\frac{\beta_l}{[1-\sum_{k\in\mathcal{M}_j(\mathbf{a})}\beta_k]^+} \right. \\ &\qquad \left. - \frac{\beta_l}{[1-\sum_{k\in\mathcal{M}_j(\mathbf{a})\setminus\{i\}}\beta_k]^+} \right), \quad (15) \end{aligned}$$

respectively.[11] Clearly, $c_i(\mathbf{a})$ is the required power of mobile i if it joins BS j, while $t_i(\mathbf{a})$ is the aggregate increase in power consumption of all other mobiles associated with BS j. We propose a toll mechanism with tolls $t_i : \mathcal{N}^M \to \mathbb{R}$ given by (15). This yields a new game $(\mathcal{M}, \mathcal{N}, (\bar{c}_i, i \in \mathcal{M}))$ with cost functions for an association profile $\mathbf{a}$ given by

$$\begin{aligned} \bar{c}_i(\mathbf{a}) &= c_i(\mathbf{a}) + t_i(\mathbf{a}) \\ &= \sum_{l\in\mathcal{M}_{a_i}(\mathbf{a})} \frac{\sigma^2}{h_{la_i}} \frac{\beta_l}{[1-\sum_{k\in\mathcal{M}_{a_i}(\mathbf{a})}\beta_k]^+} - \\ &\qquad \sum_{l\in\mathcal{M}_{a_i}(\mathbf{a})\setminus\{i\}} \frac{\sigma^2}{h_{la_i}} \frac{\beta_l}{[1-\sum_{k\in\mathcal{M}_{a_i}(\mathbf{a})\setminus\{i\}}\beta_k]^+}. \quad (16) \end{aligned}$$

Proposition 5.3: The finite strategic form game $(\mathcal{M}, \mathcal{N}, (\bar{c}_i, i \in \mathcal{M}))$ is an ordinal potential game and thus admits the FBRP property.

Proof: See [10]. ∎

It is shown in [10] that the potential function $V(\mathbf{a})$ equals the system performance measure $C(\mathbf{a})$ defined in Section III-D. Hence an association profile $\mathbf{a}^o$ that optimizes system performance is also a (global) minimizer of $V(\mathbf{a})$, and therefore a NE of the potential game with tolls. *So, we see that tolls $t_i(\mathbf{a})$ weakly enforce a system optimal association profile.* In general, tolls do not strongly enforce a system optimal association profile. For instance reconsider Example 3.1. The association profile $(a_1 = 2, a_2 = 1)$ is inefficient, but an NE for the game $(\mathcal{M}, \mathcal{N}, (\bar{c}_i, i \in \mathcal{M}))$.

In the following we consider special cases, and investigate the effect of the proposed tolls.

1) Collocated Mobiles with Single Class Traffic: Let us consider the special case when all the mobiles are collocated and have identical minimum SINR requirements. In other words, $h_{ij} = h_j$ and $\beta_i = \beta$ for all $i \in \mathcal{M}, j \in \mathcal{N}$. The potential function for this special case can be written as

$$V(\mathbf{a}) = \sum_{j\in\mathcal{N}} \frac{\sigma^2}{h_j} \frac{|\mathcal{M}_j(\mathbf{a})|\beta}{[1-|\mathcal{M}_j(\mathbf{a})|\beta]^+}$$

Define $g_j = \frac{\sigma^2}{h_j}, f(m) = \frac{m\beta}{[1-m\beta]^+}$ and $m_j(\mathbf{a}) = |\mathcal{M}_j(\mathbf{a})|$ for all $j \in \mathcal{N}$. Then $\mathbf{m}(\mathbf{a}) = (m_j(\mathbf{a}), j \in \mathcal{N})$ denotes the congestion profile under $\mathbf{a}$. Since mobiles are indistinguishable, any two association profiles that lead to identical congestion profiles are essentially indifferent from the point of view of analysis. Thus we talk solely in terms of congestion profiles. Abusing notation (the argument of $V(\cdot)$ was earlier defined to be the association profile $\mathbf{a}$), we write

$$V(\mathbf{m}) = \sum_{j\in\mathcal{N}} g_j f(m_j).$$

Since $(\mathcal{M}, \mathcal{N}, (\bar{c}_i, i \in \mathcal{M}))$ is a finite potential game, an association profile $\mathbf{m}^*$ will be a NE if and only if

$$g_j f(m_j^*) + g_k f(m_k^*) \le g_j f(m_j^* - 1) + g_k f(m_k^* + 1) \quad (17)$$

for all $k \neq j,\ j, k \in \mathcal{N}$. The following proposition shows that tolls $t_j(\mathbf{a})$ strongly enforce a system optimal association profile in case of collocated mobiles with single class traffic.

Proposition 5.4: All the NEs in the game $(\mathcal{M}, \mathcal{N}, (\bar{c}_i, i \in \mathcal{M}))$, with $h_{ij} = h_j$ and $\beta_i = \beta$ for all $i \in \mathcal{M}, j \in \mathcal{N}$, are system optimal. In other words, the tolls strongly enforce system optimality.

Proof: Let $\mathbf{m}^o$ be a system optimal congestion profile, and $\mathbf{m}^*$ any other profile such that $V(\mathbf{m}^*) > V(\mathbf{m}^o)$. Partition the set $\mathcal{N}$ as $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_+ \cup \mathcal{N}_-$ such that

$$\begin{aligned} j \in \mathcal{N}_0 &\iff m_j^* = m_j^o \\ j \in \mathcal{N}_+ &\iff m_j^* \ge m_j^o + 1 \\ j \in \mathcal{N}_- &\iff m_j^* \le m_j^o - 1 \end{aligned}$$

Start with the congestion profile $\mathbf{m}^*$, and move mobiles from BSs $\mathcal{N}_+$ to BSs $\mathcal{N}_-$ one mobile at a time, so that we end up with the congestion profile $\mathbf{m}^o$. In this process we get a succession of congestion profiles, each of which satisfies

$$\begin{aligned} m_j = m_j^* &\quad \forall \quad j \in \mathcal{N}_0 \\ m_j \le m_j^* &\quad \forall \quad j \in \mathcal{N}_+ \\ m_j \ge m_j^* &\quad \forall \quad j \in \mathcal{N}_- \end{aligned}$$

There must exist a pair of successive congestion profiles $\mathbf{m}'$ and $\mathbf{m}''$ such that $V(\mathbf{m}') > V(\mathbf{m}'')$, with $\mathbf{m}''$ possibly the ultimate congestion profile $\mathbf{m}^o$. Let $\mathbf{m}''$ be obtained from $\mathbf{m}'$ by the transfer of a mobile from BS $j \in \mathcal{N}_+$ to a BS $k \in \mathcal{N}_-$. We then have

$$g_j f(m_j') + g_k f(m_k') > g_j f(m_j' - 1) + g_k f(m_k' + 1)$$

which is same as

$$g_j(f(m_j') - f(m_j' - 1)) > g_k(f(m_k' + 1) - f(m_k')). \quad (18)$$

Recall that f is a convex function and $m_j' \le m_j^*, m_k' \ge m_k^*$. Using these in (18), we get

$$g_j(f(m_j^*) - f(m_j^* - 1)) > g_k(f(m_k^* + 1) - f(m_k^*)),$$

[11] In (15), when both terms within parentheses are ∞, the expression is taken to be ∞; we may think of driving β to the true values from below, and the first term always dominates the second. Same remark holds for other such expressions also.

i.e.,

$$g_j f(m_j^*) + g_k f(m_k^*) > g_j f(m_j^* - 1) + g_k f(m_k^* + 1)$$

which implies that $\mathbf{m}^*$ is not a NE (see (17)). This completes the proof. ∎

2) Collocated Mobiles and Symmetrically Placed BSs: Now we consider another special case when all the mobiles are collocated and all the BSs are symmetrically placed with respect to the collocated mobiles. In this case $h_{ij} = h$ for all $i \in \mathcal{M}, j \in \mathcal{N}$. We have the following result.

Proposition 5.5: With $h_{ij} = h$ for all $i \in \mathcal{M}, j \in \mathcal{N}$, the NEs in the game $(\mathcal{M}, \mathcal{N}, (\bar{c}_i, i \in \mathcal{M}))$ coincide with those in $(\mathcal{M}, \mathcal{N}, (c_i, i \in \mathcal{M}))$.

Proof: See [10]. ∎

Thus tolls may not strongly enforce a system optimal association profile in this case (see Example 3.4).

3) Collocated BSs with Single Class Traffic: Even in this special case tolls $t_j(\mathbf{a})$ may fail to strongly enforce a system optimal association profile. For an illustration reconsider Example 3.3. The association profile $(a_1 = a_3 = 1, a_2 = a_4 = a_5 = 2)$ is not system optimal, but an NE for the game $(\mathcal{M}, \mathcal{N}, (\bar{c}_i, i \in \mathcal{M}))$.

Remark 5.2: 1) While tolls at a BS are equal for all the mobiles not associated with it and having equal SINR requirements, they are mobile dependent for all associated ones (see (15)). This is unlike in nonatomic case where we saw uniform tolls at a BS for all the mobiles with equal SINR requirements.

2) The modified algorithm (the one accounting for tolls) can be implemented in distributed fashion. All the BSs broadcast quantities $t_j^o(\mathbf{a})$ given by

$$t_j^o(\mathbf{a}) = \sum_{l \in \mathcal{M}_j(\mathbf{a})} \frac{\sigma^2}{h_{lj}} \frac{\beta_l}{[1 - \sum_{k \in \mathcal{M}_j(\mathbf{a})} \beta_k]^+}$$

along with their aggregate congestions $\sum_{k \in \mathcal{M}_j(\mathbf{a})} \beta_k$. All the mobiles need to know the scaled gains $\frac{h_{ij}}{\sigma^2}$ of their own channels to all the BSs $j \in \mathcal{N}$. Mobiles use these broadcast information to calculate their powers and tolls, and choose a BS taking both into account.

Discussion: The proposed pricing technique can be used to induce a system optimal routing in atomic weighted network congestion games with arbitrary nondecreasing edge latency functions [40].[12] In this setting, the joint BS association and power control problems can be viewed as network congestion games over two-terminal parallel-edge networks: the edges are identified with BSs, and latencies are identified with minimum power requirements. It turns out that the proposed tolls weakly enforce a system optimal routing profile in general network congestion games. They strongly enforce a system optimal routing profile if

1) the network is two-terminal series parallel,
2) the mobiles are unweighted (i.e, have identical weights), and
3) the latency functions are standard.[13]

[12] Here, the system cost is weighted sum of the latencies of all the mobiles.

[13] A latency function $c(\cdot)$ is called *standard* if $mc(m)$ is convex [39], e.g., $c(m) = \frac{1}{1-m}$.

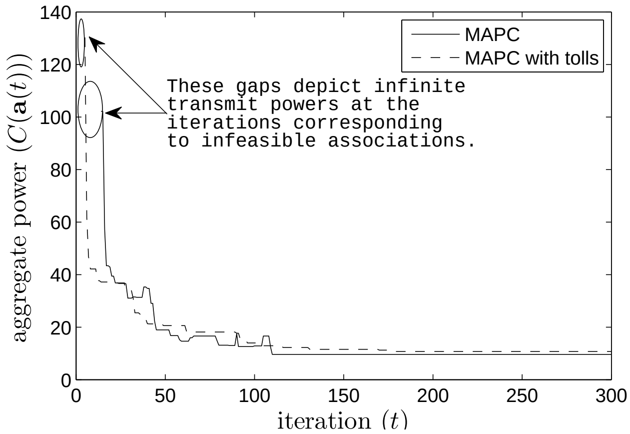

Fig. 1. Aggregate transmit powers over all the mobiles versus number of iterations.

VI. Simulation

We now demonstrate the proposed joint BS association and power control algorithms via simulation. To illustrate, we consider a cellular network with 30 mobiles and 3 BSs (thus leading to 3^{30} possible association profiles). The BSs use mutually noninterfering channels (see Figure 2). The mobiles are scattered independently and uniformly. We take the channel gains to be equal to the path losses which are assumed to follow the inverse square law. More precisely, for any mobile i and BS j that are a distance d_{ij} apart,

$$h_{ij} = (\max\{1, d_{ij}\})^{-2}.$$

The receiver noise at any BS has the average power 0.1 mW. The target SINRs γ_is are independently and uniformly sampled from the interval $[0.05, 0.1]$ for all i. Thus $\beta_i < 0.1$ and $\sum \beta_i < 3$, which is necessary for feasibility of the joint BS association and power control problem at hand.

We start with an arbitrary association and consider the *random* update process, wherein at each iteration a randomly chosen mobile updates, with all mobiles equally likely to be chosen. We have implemented MAPC (proposed in Section III-B) and also its variant with tolls as described in Section V-B. We plot the aggregate power over all the mobiles in Figure 1. The initial BS association is infeasible and so are a few subsequent ones, resulting in infinite transmit powers in the first few iterations. But the proposed algorithms quickly lead to feasible associations and power allocations. This is evident from Figure 1 where we observe finite aggregate transmit powers after first few iterations. The proposed algorithms also quickly converge to equilibrium BS associations (NEs in the respective games). Notice that we have simulated the most general case for which MAPC's convergence has not been formally established. The demonstrated convergence of MAPC corroborates Conjecture 3.1. While the equilibrium associations of these algorithms need not be system optimal, they are seen to substantially reduce the aggregate power compared to an arbitrary association. Also recall that aggregate transmit power acts as a potential function for the game in Section V-B (Proposition 5.3). Therefore, the aggregate power

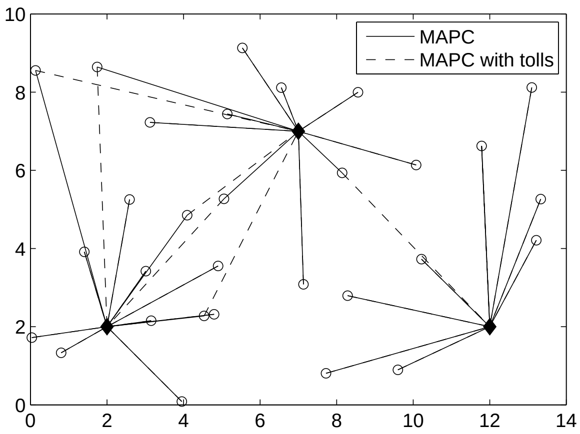

Fig. 2. A cellular network with 30 mobiles and 3 BSs. The steady state associations under the two algorithms are also shown.

for MAPC with tolls decreases after each iteration and settles at a local minimum (see Figure 1).

Finally, we show the steady state associations for both the algorithms in Figure 2. We observe that a few mobiles (6 mobiles in Figure 2) may associate with different BSs under the two algorithms.

VII. Conclusion

We studied the combined association and power control problem in multichannel multicell cellular networks in which a different channel is used by each cell, and so, there is no intercell interference. We studied the cases of discrete mobiles and a continuum of mobiles. We proposed several distributed mechanisms motivated by the techniques of game theory. We studied the inefficiency of the distributed algorithm in the case of a continuum of mobiles. It is an open question whether such inefficiency can be quantified in the case of discrete mobiles. To mitigate the inefficiency, we proposed toll mechanisms in both the settings.

Appendix A
Price of Anarchy: Continuum of Mobiles

Recall that a NE is not necessarily a system optimal congestion profile (see Example 4.2). *Price of anarchy* [9] (or, *Coordination ratio* [41]) characterizes the inefficiency caused by the selfish behavior of players; it is the ratio of the cost of the worst NE and the optimal cost. We observed in Proposition 4.3 that, in the nonatomic case, mobiles incur the same cost at all the NEs. We can then define price of anarchy as follows.

Definition A.1: Let $\mathbf{m}$ be a NE, and $\mathbf{m}^o$ be a system optimal congestion profile. Then the *price of anarchy* is

$$\mathrm{PoA} = \frac{C(\mathbf{m})}{C(\mathbf{m}^o)}.$$

We restrict our analysis to a single class population. We assume that all the mobiles have identical minimum required SINR density γ and identical power gain h_j to BS j, $j \in \mathcal{N}$.

A. Two BSs

First we consider a case with 2 BSs as in Example 4.2. Let $h_1 > h_2$.[14] Also, let $(\alpha^* M, (1-\alpha^*)M)$ and $(\alpha^o M, (1-\alpha^o)M)$ be the congestion profiles under a NE and a system optimal association, respectively. Recall from Example 4.2 that

1) if $\gamma M \leq 1 - \sqrt{\frac{h_2}{h_1}}$, then $\alpha^* = \alpha^o = 1$
2) if $1 - \sqrt{\frac{h_2}{h_1}} < \gamma M \leq 1 - \frac{h_2}{h_1}$, then $\alpha^* = 1$, and from (13)
$$\alpha^o = \frac{\sqrt{h_1} - \sqrt{h_2} + \gamma M \sqrt{h_2}}{(\sqrt{h_1} + \sqrt{h_2})\gamma M}$$
3) if $\gamma M > 1 - \frac{h_2}{h_1}$, then from (11)
$$\alpha^* = \frac{h_1 - h_2 + \gamma M h_2}{(h_1 + h_2)\gamma M},$$
and α^o is as above.

$C(\mathbf{m})$ and $C(\mathbf{m}^o)$ are obtained via substituting $\alpha = \alpha^*$ and $\alpha = \alpha^o$, respectively, in the objective function (12). Straightforward calculations give that

$$\mathrm{PoA}(M) = \begin{cases} 1 & \text{if } M \leq \frac{1-\sqrt{\lambda}}{\gamma}, \\ \frac{\lambda(2-\gamma M)\gamma M}{(1-\gamma M)(2\sqrt{\lambda}-(1-\gamma M)(1+\lambda))} & \text{if } \frac{1-\sqrt{\lambda}}{\gamma} \leq M \leq \frac{1-\lambda}{\gamma}, \\ \frac{\gamma M(1+\lambda)}{2\sqrt{\lambda}-(1-\gamma M)(1+\lambda)} & \text{if } M \geq \frac{1-\lambda}{\gamma} \end{cases}$$

where $\lambda := \frac{h_2}{h_1} < 1$. Further calculations also yield that $\mathrm{PoA}(M)$ is continuous at $M = \frac{(1-\lambda)}{\gamma}$, and

$$\frac{\mathrm{dPoA}(M)}{\mathrm{d}M} \begin{cases} \geq 0 \text{ if } M < \frac{1-\lambda}{\gamma}, \\ \leq 0 \text{ if } M > \frac{1-\lambda}{\gamma} \end{cases}$$

Thus, the price of anarchy is maximized when $M = \frac{1-\lambda}{\gamma}$. Moreover, the maximum price of anarchy is

$$\frac{1-\lambda^2}{2\sqrt{\lambda} - \lambda(1+\lambda)}.$$

Viewing this now as a function of $\lambda \in (0,1]$, we see that the maximum price of anarchy decreases with λ. We also observe that $\mathrm{PoA} \to \infty$ as $\lambda \to 0$, i.e., arbitrarily high PoAs can be realized in 2 BS networks.

B. N BSs

Again, without any loss of generality, we assume that $h_1 \geq h_2 \geq \cdots \geq h_N$. We also assume that the population's mass is Δ_j when it spills over BS j under NE. Clearly, $\Delta_2 \leq \Delta_3 \leq \cdots \leq \Delta_N$. In the case of 2 BSs we proved that price of anarchy is maximized when the population spills over BS 2 under NE. In the case of $N > 2$ BSs also, simulations suggest that the price of anarchy is maximized at one of the spill over points $\{\Delta_j, j = 2, \ldots, N\}$. We have however not been able to prove this observation. We illustrate this observation in [10, Figure 1].

[14]If $h_1 = h_2$ equal fraction of population join each of the BSs under the NE and the system optimal association, and the price of anarchy is 1.

However, we prove that the price of anarchy decreases with mass for $M \geq \Delta_N$. We define $e_N := \sum_{k \leq N} \frac{1}{h_k}$ and $e_N^* := \sum_{k \leq N} \frac{1}{\sqrt{h_k}}$. It can be easily checked that, for $M \geq \Delta_N$,

$$\begin{aligned}\mathrm{PoA}(M) &= \frac{e_N M\gamma}{e_N M\gamma - (e_N N - e_N^{*2})} \\ &= 1 + \frac{e_N N - e_N^{*2}}{e_N M\gamma - (e_N N - e_N^{*2})}\end{aligned}$$

from which the claim follows (see [10, Appendix B] for details). Thus, to obtain a bound on the price of anarchy, we only focus on $M \leq \Delta_N$. For $M \leq \Delta_N$, the load on BS j

$$m_j \leq \frac{1}{\gamma}\left(1 - \frac{h_N}{h_j}\right)$$

under NE. We use this observation in the next section.

C. A Bound on the Price of Anarchy

Now, we derive a sharp bound on the price of anarchy for single class networks with arbitrary number of BSs, and gains $h_j \in [h_{\min}, h_{\max}]$ for all the BSs. We follow Roughgarden [39, Chapter 3].

In the BS association game, a generic cost function is of the form

$$c_h(m) := \frac{\sigma^2}{h}\frac{\gamma}{1-\gamma M},$$

and

$$\mathcal{C} := \{c_h(\cdot) : h \in [h_{\min}, h_{\max}]\}$$

is the class of all feasible cost functions. Observe that the functions $c_h(\cdot)$ and the class $\mathcal{C}$ both are standard.[15] We define

$$\bar{c}_h(m) := \frac{\mathrm{d}(mc_h(m))}{\mathrm{d}m}.$$

We also assume that the load on a BS with gain h_j does not exceed

$$\theta_h := \frac{1}{\gamma}\left(1 - \frac{h_{\min}}{h}\right)$$

under NE. Thus, we redefine *anarchy value* for a cost function $c_h(\cdot)$ as[16]

$$\alpha(c_h) := \sup_{m \leq \theta_h} [\lambda\mu + (1-\lambda)]^{-1}$$

where $\lambda \in (0,1)$ satisfies $\bar{c}_h(\lambda m) = c_h(m)$ and $\mu := \frac{c_h(\lambda m)}{c_h(m)} \leq 1$. Both λ and μ are functions of m; we do not show this dependence explicitly. Straightforward calculations yield that

$$\begin{aligned}\lambda &= \frac{1-\sqrt{1-m\gamma}}{m\gamma}, \\ \mu &= \sqrt{1-m\gamma}, \\ \text{and } \alpha(c_h) &= \sup_{m \leq \theta_h} \frac{1}{2}\left[1 - \frac{1}{1+\sqrt{1-m\gamma}}\right]^{-1} \\ &= \frac{1}{2}\left(1 + \sqrt{\frac{h}{h_{\min}}}\right).\end{aligned}$$

[15]A cost function $c(\cdot)$ is called *standard* if $mc(m)$ is convex. A class $\mathcal{C}$ is *standard* if it contains a nonzero function and if each $c(\cdot) \in \mathcal{C}$ is standard [39].

[16]The original definition [39, Definition 3.3.2] considers supremum over $m \in (0, \infty)$.

The *anarchy value* for class $\mathcal{C}$ is (see [39, Definition 3.3.3])

$$\alpha(\mathcal{C}) = \sup_{c_h \in \mathcal{C}} \alpha(c_h) = \frac{1}{2}\left(1 + \sqrt{\frac{h_{\max}}{h_{\min}}}\right)$$

It can be easily checked that [39, Theorem 3.3.8] remains valid with our new definition of anarchy value. Thus, price of anarchy is bounded by $\alpha(\mathcal{C})$. For any $0 < \epsilon \leq \alpha(\mathcal{C}) - 1$, a price of anarchy $\geq \alpha(\mathcal{C}) - \epsilon$ is realized in a network in which (i) there is one BS with gain $h_{\max}$, (ii) there are several BSs with gain $h_{\min}$ (minimum number depending on ϵ), and (iii) the population has mass $\theta_{h_{\max}}$ (see the proof of [39, Lemma 3.4.3] for details).

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