Incoherence is Sufficient for Statistical RIP of Unit Norm Tight Frames: Constructions and Properties

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***Abstract*—Incoherent matrices, especially, incoherent unit norm tight frames (IUNTFs), are important in many present day applications. In addition to incoherence, the statistical restricted isometry property (StRIP) and statistical incoherence property (SInCoP) are important criteria for a matrix to provide theoretical guarantees for uniquely recovering sparse signals using computationally efficient algorithms.**

In this work, we show that the incoherence alone is sufficient to establish StRIP and SInCoP for unit norm tight frames (UNTFs). Further, we derive three simple properties that binary matrices need to satisfy, in order to produce IUNTFs with low coherence and high redundancy (ratio of the number of columns to the number of rows) via an existing embedding operation. We show that biadjacency matrices corresponding to biregular graphs satisfy the required properties. Thereby, we provide a connection between graph theory and the construction of IUNTFs. We also provide a bouquet of constructions of IUNTFs from finite fields and combinatorial designs. These can be used to produce IUNTFs with very general sizes.

Another important aspect of our construction is that the sparse recovery guarantees for the embedded IUNTFs can in fact be translated to the constituent binary matrix. We show that if the constituent $m \times M$ binary matrix has constant row and column weight, it can support sparse recovery through ℓ_1-minimization for all but an $\epsilon-$fraction of $t-$sparse signals chosen from a random signal model, provided $m = O(t(\log(\frac{M}{\epsilon}))^3)$, which is a significant improvement over the existing $m = O(t^2)$ bound, where m denotes the number of measurements. Also, the StRIP and SInCoP based approach results in matrices whose column size is exponential in the fourth root of the row size. To the best of our knowledge, this is the first construction of deterministic matrices satisfying StRIP and SInCoP with such high redundancy.

***Index Terms*—** Unit norm tight frame, biregular graph, sparse signal recovery, statistical restricted isometry property.

I. Introduction

The coherence of a matrix is defined as the maximum absolute value of inner products between any two distinct normalized columns. A matrix with small coherence is termed as an incoherent matrix. Incoherent matrices, especially, incoherent unit norm tight frames (IUNTFs), play a key role in a variety of applications including, but not limited to, compressed sensing, communications, and coding theory [1]. The statistical restricted isometry property (StRIP) and the statistical incoherence property (SInCoP) of a matrix, explored in [2], [3], are sufficient to provide guarantees for the unique recovery of sparse signals using the basis pursuit (BP) algorithm. The StRIP and SInCoP of a matrix are a function of its coherence, mean square coherence and spectral norm [2], [3]. As a result, the construction of IUNTFs [4]–[7] and matrices with StRIP has gained momentum in recent years [3], [8]–[10]. Our first objective in this paper is to show that, in case of UNTFs, the conditions on mean square coherence and spectral norm are redundant, and incoherence alone is sufficient to establish the StRIP and SInCoP.

IUNTFs with additional properties are of special interest. For example, matrices with high *redundancy* (defined as the ratio of the number of columns to the number of rows) are useful in sensing sparse vectors using as few measurements as possible. Similarly, matrices with low *density* (defined as the ratio of the number of nonzero elements to the total number of elements) are desirable as they can lead to low complexity algorithms for sparse signal recovery.[1] The existing IUNTFs with low redundancy, e.g., equiangular tight frames [12], mutually unbiased bases [13] and chirp matrices [14] exhibit StRIP and SInCoP, but are of very restricted sizes. Our second objective is to provide a simple construction of IUNTFs with large redundancy, low density and of arbitrary size using an existing embedding operation.

An embedding operation is proposed in [15] to obtain matrices with larger column size from an incoherent smaller dimensional binary matrix possessing constant column weight. The embedding is done by replacing each $1-$valued entry of each column of the binary matrix with a distinct row of a Hadamard or DFT matrix and replacing each zero by a row of zeros of size equal to the column size of the Hadamard or DFT matrix. A matrix is said to satisfy the $(\ell_1, t)-$recovery property if every $t-$sparse vector can be uniquely recovered via basis pursuit (BP). In [16], DFT matrices are embedded in binary matrices obtained from pairwise balanced designs to obtain tight frames with $(\ell_1, O(\sqrt{m}))-$recovery property, where m is the row size. In [12], the authors embed Hadamard or DFT matrices in binary matrices obtained from $(2, k, v)-$Steiner systems to obtain equiangular tight frames (ETFs). In [17], Hadamard or DFT matrices are embedded in binary matrices generated from a finite geometry to obtain low coherence matrices. It is also shown through numerical simulation that the embedded matrices exhibit superior sparse signal recovery performance compared to random Gaussian matrices. The authors in [18] use binary matrices obtained from vector spaces over finite fields in the embedding operation to obtain incoherent matrices with large column size.

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[1]For a survey of sparse recovery algorithms for sparse matrices, see [11].

From the above discussion, we see that embedding is a versatile approach that can be used to fulfill various objectives. For example, in [15], [18], it is used to obtain matrices with large column size, whereas in [12], it is used to construct ETFs. However, the existing analysis of the sparse recovery properties of embedded matrices is based on the $(\ell_1, t)-$property, which cannot guarantee the recovery of sparse signals with greater than $O(\sqrt{m})$ non-zero entries. Also, the ETFs constructed via the embedding operation suffers from low redundancy [12]. Overall, despite relying on the same underlying embedding operation, there is no unified framework that connects these diverse constructions. Therefore, it is of interest to develop a unified methodology for constructing highly redundant matrices with good sparse recovery properties using the embedding operation, which is our third objective in this paper.

In the context of the above, our contributions in this paper are as follows:

1) We show that, for UNTFs, incoherence alone suffices to establish StRIP and SInCoP. We note that previous results required the matrices to also satisfy certain conditions on the mean square coherence and spectral norm in order for the StRIP and SInCoP to hold; these conditions are redundant for IUNTFs.
2) We determine the underlying property of a binary matrix that enables one to produce IUNTFs by embedding an incoherent tight frame such as the Hadamard or DFT matrix. To elaborate, we show that binary matrices with the following three properties:
(i) each column contains the same number of ones,
(ii) each row contains the same number of ones,
(iii) the overlap between any two columns is sufficiently small (to be specified later),
are suitable candidates to produce IUNTFs through the embedding operation.
3) We show that biadjacency matrices associated with biregular graphs possess the required properties. This enables us to leverage the available literature on graph construction to design IUNTFs. We also present constructions of binary matrices with the required properties via (i) combinatorial designs and (ii) polynomials over finite fields. We present several constructions of binary matrices with the three required properties for very general sizes and with large redundancy. As the preview of the results to follow, we summarize the constructions presented in this work in Table I.
4) We also show that IUNTFs constructed via the embedding operation support sparse recovery through ℓ_1-minimization for all but $\epsilon-$fraction of $t-$sparse signals chosen from a random signal model provided $m = O(t(\log(\frac{M}{\epsilon}))^3)$, which is significantly better than $m = O(t^2)$ bound for recovering of $t-$sparse signals obtained via the $(\ell_1, t)-$property considered in [16], [19]. Here, t, m and M represent the sparsity, row size and column size, respectively.
5) The column size of most existing constructions (e.g., [12], [13], [14]) satisfying StRIP and SInCoP are polynomial (typically, the square) in the row size. In contrast, for a prime p, our IUNTFs are of size $p^2 \times p^{\sqrt{p}}$ with coherence $\frac{1}{\sqrt{p}}$ and density $\frac{1}{p}$. Further, they satisfy StRIP and SInCoP, and thereby support sparse recovery as mentioned in the previous point. Therefore, the coherence of the constructed IUNTFs is at most m^{α} and the column size is m^{β}, where $m = p^2$ is the row size, $\alpha = -\frac{1}{4}$ and $\beta = \frac{1}{2}m^{\frac{1}{4}}$. To the best of our knowledge, these are the first constructions of matrices satisfying StRIP and SInCoP with the column size being *exponential* in (the fourth root of) the row size.
6) We show that the sparse recovery guarantees for the embedded IUNTFs can be directly translated to the constituent binary matrices. In turn, this allows us to provide new and improved theoretical bounds for sparse recovery from such binary measurement matrices.

The rest of the paper is organized as follows. In Section II, we establish the StRIP and SInCoP of IUNTFs. In Section III, we discuss the general principle behind construction of IUNTFs from binary matrices, and identify the key properties that need to be satisfied to obtain IUNTFs via the embedding operation. In Sections IV, V and VI, we discuss the construction of IUNTFs from biregular graphs, finite field theory and combinatorial designs, respectively. The sparse recovery properties of constructed IUNTFs and their connection with existing constructions are discussed in Section VII. We present numerical results to validate the theoretical results and compare the sparse recovery properties of the proposed constructions with random Gaussian matrices in Sec. VIII. We end the paper with a few concluding remarks pointing to promising directions for future work.

II. StRIP and SInCoP of IUNTFs

In this section, we show that IUNTFs satisfy StRIP and SInCoP. As a result, the basis pursuit algorithm can recover sparse signals uniquely whenever the signal arises from a generic random model [3]. Before presenting the main results of this section, we discuss the basics of frame theory and their properties.

A family of vectors $\{\phi_i\}_{i=1}^M$ in $\mathbb{C}^m$ is called a *frame* for $\mathbb{C}^m$, if there exist constants $0 < A \leq B < \infty$ such that

$$A \|z\|_2^2 \leq \sum_{i=1}^{M} |\langle z, \phi_i \rangle|^2 \leq B \|z\|_2^2, \forall z \in \mathbb{C}^m,$$

where A, B are called the lower and upper frame bounds, respectively. The redundancy of a frame is defined as the ratio of number of frame elements to the dimension of frame elements, i.e., $\frac{M}{m}$. Frames are divided into the following classes:

- If $A = B$, then $\{\phi_i\}_{i=1}^M$ is called an $A-$tight frame or simply a tight frame.
- If there exists a constant c such that $\|\phi_i\|_2 = c$ for all $i = 1, 2, \ldots, n$, then $\{\phi_i\}_{i=1}^M$ is an equal norm frame. If $c = 1$, then it is called a unit norm frame.
- If a frame is both unit norm and tight, it is called a unit norm tight frame (UNTF).

TABLE I. CONSTRUCTIONS PRESENTED IN THIS WORK.
HERE, p IS A PRIME POWER, AND n AND k ARE CHOSEN SUCH THAT AN EULER SQUARE OF INDEX (n, k) EXISTS

Biregular graph	Dimension $(m \times M)$	Coherence	Redundancy	Requirement for StRIP: $m = O(\cdot)$	Section
$p-$regular graph $D(2, p)$	$p^2 \times p^3$	$\leq \frac{2}{p}$	p	t	IV-A1
$(d, \frac{md}{M})-$regular LDPC code	$m \times M$	$\leq \frac{2}{d}$	$\frac{M}{m}$	t, for $d = O(\sqrt{m})$	IV-A2
$(d_r, d_\ell)-$regular QC LDPC code	$pd_r \times pd_r d_\ell$, $p \geq d_\ell + 1$	$\leq \frac{2}{d_r}$	d_ℓ	t, for $p = O(d_r)$	IV-A3
$(p^2, p^{r+1}, p, p^r)-$ biregular graph, $r \geq 1$	$p^2 \times p^{r+2}$	$\leq \frac{2r}{p}$	p^r	t, for fixed r	V-A
$(n^2, nk, n, k)-$biregular graph	$n^2 \times n^2 k$	$\leq \frac{2}{n}$	k	t	VI-A
$(2, p, p^2)-$Steiner system	$p^2 \times p^2(p+1)$	$\leq \frac{2}{p}$	$p+1$	t	VI-B
$(2, p+1, p^2+p+1)-$Steiner system	$(p^2+p+1) \times (p+1)(p^2+p+1)$	$\leq \frac{2}{p+1}$	$p+1$	t	VI-B
$(3, p+1, p^2+1)-$ Steiner system	$(p^2+1) \times p(p+1)(p^2+1)$	$\leq \frac{4}{p+1}$	$p(p+1)$	t	VI-B

- If a frame is unit norm and there exists a constant d such that $|\langle \phi_i, \phi_j \rangle| = d$, for $1 \leq i < j \leq M$, then $\{\phi_i\}_{i=1}^M$ is an equiangular frame.
- If a frame is both UNTF and equiangular, it is called an equiangular tight frame (ETF).

UNTFs are known to be well conditioned and provide stable representations [20]. It can be shown that a UNTF exists only when $A = \frac{M}{m}$. By considering the frame vectors as columns, one may obtain a full row rank matrix. In this sequel, we do not make any distinction between a frame and its associated matrix and use the two terms interchangeably.

Definition 1: (Coherence) The coherence μ_Φ of a frame $\Phi \in \mathbb{C}^{m \times M}$ is defined by

$$\mu_\Phi = \max_{1 \leq i, j \leq M, i \neq j} \frac{|\langle \phi_i, \phi_j \rangle|}{\|\phi_i\|_2 \|\phi_j\|_2},$$

where ϕ_i is the ith column of Φ.

Definition 2: (Mean square coherence) The mean square coherence $\overline{\mu}_\Phi^2$ of a frame $\Phi \in \mathbb{C}^{m \times M}$ is defined as

$$\overline{\mu}_\Phi^2 = \frac{1}{M-1} \max_j \sum_{i=1, i \neq j}^{M} |\langle \phi_i, \phi_j \rangle|^2. \tag{1}$$

Both μ_Φ and $\bar{\mu}_\Phi^2$ are important factors in determining StRIP of a given frame [3]. Our first result is an upper bound on the mean square coherence of any UNTF.

Theorem 1: For any UNTF $\Phi \in \mathbb{C}^{m \times M}$,

$$\overline{\mu}_\Phi^2 = \frac{M-m}{m(M-1)} < \frac{1}{m}.$$

Proof: See Sec. X-B1. ∎

A trivial upper bound for $\overline{\mu}_\Phi^2$ is μ_Φ^2, where μ_Φ is the coherence of Φ. Theorem 1 shows that the mean square coherence of any UNTF of a given size is *fixed*. This property will be crucial in showing that low coherence is enough to establish StRIP and SInCoP of IUNTFs.

Let $J_t(M)$ denote the set of all $t-$sized subsets of $\{1, \ldots, M\}$ and let P_{J_t} denote the uniform probability distribution on $J_t(M)$. We now formally define StRIP and SInCoP.

Definition 3: [3], [21], [22] Let $\Phi_{m \times M}$ be a frame for $\mathbb{R}^m$ or $\mathbb{C}^m$ containing M elements as its columns. Then, the frame Φ is said to satisfy the $(t, \delta, \epsilon)-$StRIP if

$$P_{J_t}\left(\|\Phi_J^T \Phi_J - I_{t \times t}\|_2 \geq \delta \right) < \epsilon,$$

where the random set $J \in J_t(M)$ is distributed according to P_{J_t} and Φ_J is the submatrix of Φ formed by taking the columns indexed by J.

Definition 4: [2], [3] An $m \times M$ matrix Φ is said to satisfy the $(t, \alpha, \epsilon)-$statistical incoherence property (SInCoP), if

$$P_{J_t}(\{I \in J_t(M) : \max_{i \notin I} \|\Phi_I^T \phi_i\|_2^2 \leq \alpha\}) \geq 1 - \epsilon.$$

Definition 5: (Random signal model S_t [2], [3]): A vector $x \in \mathbb{R}^M$ is said to be drawn from the random signal model S_t if it satisfies the following two properties:

1) Let $I \subset \{1, \cdots, M\}$ be a set containing the t indices corresponding to the entries of x that have the t largest absolute values. Then, the subset I is uniformly distributed over $J_t(M)$.
2) Conditional on I, the signs of the coordinates x_i, $i \in I$ are independent and identically distributed (i.i.d.) Rademacher random variables taking values in the set $\{1, -1\}$ with equal probability.

In [2], [3], it is shown that if a matrix satisfies StRIP and SInCoP, a sparse signal arising from the random signal model can be recovered with high probability using the basis pursuit algorithm, which solves the convex optimization problem

$$P_1(\Phi, y) : \min_x \|x\|_1 \text{ subject to } \quad \Phi x = y.$$

The next theorem states that IUNTFs satisfy StRIP.

Theorem 2: A UNTF $\Phi_{m \times M}$ with coherence $O(\frac{1}{\sqrt{m}})$ has the following properties:

1) $\Phi_{m \times M}$ satisfies the $(t, \delta, \frac{1}{t})-$StRIP for $m = O(t \log t)$.
2) $\Phi_{m \times M}$ satisfies the $(t, \delta, \epsilon)-$StRIP with $t < \frac{1}{\epsilon}$ for $m = O_\epsilon(t)$.[2]

Proof: See Sec. X-B2. ∎

The above theorem shows that matrices with coherence $O(m^{-\frac{1}{2}})$ satisfy StRIP. Next, we show that an IUNTF satisfies the SInCoP.

Theorem 3: A UNTF $\Phi_{m \times M}$ with coherence $\frac{1}{\sqrt{m}}$ satisfies $(t, \alpha, \epsilon)-$SInCoP for $m = O(t(\log(\frac{M}{\epsilon}))^{3/2})$ and $\alpha = O((\log(\frac{2M}{\epsilon}))^{-1})$.

Proof: See Sec. X-B3. ∎

The above theorem asserts that along with StRIP, matrices with coherence $O(m^{-\frac{1}{2}})$ also satisfy the SInCoP. Based on the above, the following theorem summarizes the sparse recovery guarantees for a UNTF $\Phi_{m \times M}$.

[2]Here, O_ϵ denotes the order function, whose constants may depend on ϵ.

Theorem 4: A UNTF $\Phi_{m\times M}$ with coherence $\frac{1}{\sqrt{m}}$ supports sparse recovery under the basis pursuit algorithm for all but a proportion $\epsilon (< \frac{1}{t})$ of $t-$sparse signals chosen from the random signal model S_t provided $m = O(t(\log(\frac{M}{\epsilon}))^{3/2})$.

Proof: The result follows directly from the [3, Theorem 15] (see Theorem 16 in Appendix X-A) as $\Phi_{m\times M}$ satisfies $(t,\delta,\epsilon)-$StRIP and $(t,\alpha,\epsilon)-$SInCoP for $m = O(t(\log(\frac{M}{\epsilon}))^{3/2})$. ■

Thus, we have shown that incoherence alone is sufficient for a UNTF to satisfy StRIP and SInCoP, and thereby guaranteeing successful recovery of sparse signals arising from the random signal model. In the next section, we provide various constructions of IUNTFs by using an existing embedding operation starting from constant column and row weight binary matrices.

III. From Binary Matrices to IUNTFs

In this section, we show that an embedding operator applied to a specially structured binary matrix can produce a UNTF with small coherence. The embedding operation proposed in [15] combines binary matrices with other low-coherence matrices, and is defined as follows:

Definition 6: [Embedding Operation] Suppose $A_{m\times M}$ is a binary matrix such that each column has k ones, and let $B_{k\times K}$ be a matrix. Define a new matrix $C = A \circledast B$ of size $m \times MK$ by replacing each 1-valued entry of each column of A with a distinct row of B and replacing each zero by a row of K zeros.

The following lemma relates the coherence μ_C of $C = A \circledast B$ with the coherence μ_A of A and the coherence μ_B of B.

Lemma 1: [15] Suppose $C = A \circledast B$, then $\mu_C = \max\{\mu_A, \mu_B\}$, provided all entries of B have same modulus.

From Lemma 1, we need the coherence of A and B to be small in order to ensure that the coherence of C is small. Our objective is to determine the properties of the binary matrix A and the matrix B which results in C being a UNTF with low coherence. First, we define a class of binary matrices as follows:

Definition 7: An $m \times M$ binary matrix is said to be a $(k, \frac{Mk}{m})$*−binary matrix of size* $m \times M$ if it satisfies the following two properties:

1) Each column contains k ones.
2) Each row contains $\frac{Mk}{m}$ ones.

Note that the density of $(k, \frac{Mk}{m})-$binary matrices of size $m \times M$ is $\frac{k}{m}$.

The following theorem states that UNTFs can be constructed from $(k, \frac{Mk}{m})-$binary matrices of size $m \times M$ via the embedding operation.

Theorem 5: Let Ψ be a $(k, \frac{Mk}{m})-$binary matrix of size $m \times M$ and $T_{k\times K}$ be a UNTF. Then $(\Psi \circledast T)_{m\times MK}$ is a UNTF.

Proof: See Sec. X-B4. ■

From the above theorem, we see that binary matrices with constant column and row weights are suitable candidates to produce UNTFs. However, in addition to being UNTFs, we seek matrices with low coherence and large redundancy. From Lemma 1, this implies that, one needs the matrices Ψ and T to have low coherence and large redundancy. For simplicity, in literature, T is usually chosen as an orthonormal basis, i.e., Hadamard or DFT matrix. In that case, the coherence and redundancy of $(\Psi \circledast T)$ depend solely on the coherence and redundancy of Ψ. We have the following definition:

Definition 8: A $(k, \frac{Mk}{m})-$binary matrix of size $m \times M$ is called a $(k, \frac{Mk}{m}, r)-$binary matrix of size $m \times M$ if the overlap between any two distinct columns is at most r.

The following theorem shows that by embedding an orthonormal basis inside a $(k, \frac{Mk}{m}, r)-$binary matrix of size $m \times M$, one can obtain an IUNTF.

Theorem 6: Let Ψ be a $(k, \frac{Mk}{m}, r)-$binary matrix of size $m \times M$ and T be a UNTF of size $k \times K$ with coherence μ_T. Let $\alpha = \max_{i,j} |T(i,j)|$, where $T(i,j)$ denotes the (i,j)th entry of T. Then, $(\Psi \circledast T)$ is a UNTF of size $m \times MK$ with coherence at most $\max\{\frac{r\alpha^2}{k}, \mu_T\}$.

In particular, if T be an orthonormal basis of size $k \times k$, then the coherence of the UNTF $(\Psi \circledast T)$ is at most $\frac{r\alpha^2}{k}$.

Proof: See Sec. X-B5. ■

Note that, one may choose $T_{k\times k}$ as the discrete cosine transform (DCT) matrix (or the discrete Fourier transform (DFT) matrix) to have $(\Psi \circledast T)$ as a real matrix (or a complex matrix). For the DCT matrix (or the DFT matrix) $\alpha \le \sqrt{\frac{2}{k}}$ (or $\alpha \le \frac{1}{\sqrt{k}}$). Then, if Ψ is a $(k, \frac{Mk}{m}, r)-$binary matrix of size $m \times M$, the coherence of the UNTF $(\Psi \circledast T)$ is at most $\frac{2r}{k}$ (or $\frac{r}{k}$). Instead of the DCT matrix, if we use T as the Hadamard matrix, a real IUNTF can be constructed with coherence $\frac{r}{k}$. Note that, Hadamard matrices do not exist for all orders. The Hadamard conjecture states that a Hadamard matrix of order $4k$ exists for every positive integer k. The construction of Hadamard matrices is known for order 2^n for every positive integer n.

Based on the above theorem, we desire to construct $(k, \frac{Mk}{m}, r)-$binary matrices which possess the following properties:

$(p1)$ Each column has k ones.

$(p2)$ Each row has an equal number of ones.

$(p3)$ The maximum overlap between any two columns is at most r $(< k)$.

We provide explicit constructions of matrices with the above properties in Sections IV, V and VI.

In the next section, we show that the biadjacency matrix associated with a biregular graph satisfies the desired properties.

IV. From Biregular Graphs to UNTFs

In this section, we provide a connection between biregular graphs and the construction of UNTFs via embedding.

A graph is called $d-$*regular* if all vertices have degree d. A bipartite graph is said to be *biregular* if all vertices on the same side of the bipartition have the same degree.

Definition 9: A bipartite graph with m left vertices and M right vertices is said to be an $(m, M, d_r, d_l)-$biregular graph if the m left vertices have degree d_l each and the M right vertices have degree d_r each. Note that a biregular graph satisfies $md_l = Md_r$.

The girth of a graph is defined as the length of its shortest cycle. The girth of a bipartite graph is always an even number, and in "simple" graphs (not more than one edge between any pair of vertices), the girth is at least four.

We define a special type of biregular graph for the purpose of obtaining binary matrices with the desired properties.

Definition 10: An (m, M, d_r, d_l) biregular graph is called an $(m, M, d_r, d_l, r)-$biregular graph if any two right vertices have at most r left vertices in common.

Definition 11: The biadjacency matrix associated with a bipartite graph with m left vertices and M right vertices is a binary matrix Φ of size $m \times M$ whose (i, j)th element ϕ_{ij} is one if there exists an edge between the ith left vertex and the jth right vertex, and is zero otherwise.

Theorem 7: Let Φ be the biadjacency matrix associated with an $(m, M, d_l, d_r, r)-$biregular graph and T be a UNTF of size $d_r \times K$ with coherence μ_T and $\alpha = \max_{i,j} |T(i, j)|$, where $T(i, j)$ denotes the (i, j)th entry of T. Then, $(\Phi \circledast T)$ is a UNTF of size $m \times MK$ with coherence at most $\max\{\frac{r\alpha^2}{d_r}, \mu_T\}$.

Proof: See Sec. X-B6. ■

Thus, if we choose T to be an orthogonal matrix of size $d_r \times d_r$ with unit-modulus entries (e.g., DFT or Hadamard matrix), then the UNTF constructed from a $(m, M, d_l, d_r, r)-$biregular graph has coherence at most $\frac{r}{d_r}$.

A. IUNTFs from Biregular Graphs with Large Girth

By Theorem 7, it is clear that to have an IUNTF from an $(m, M, d_r, d_l, r)-$biregular graph, we need d_r to be large and r to be small. In [23], it is shown that the overlap between any two columns of the biadjacency matrix of any bipartite graph with girth greater than 4 is at most 1. Hence, an $(m, M, d_r, d_l)-$biregular graph with girth greater than 4 is an $(m, M, d_r, d_l, 1)-$biregular graph. Consequently, from Theorem 7, one obtains a UNTF of size $m \times Md_r$ with coherence at most $\frac{1}{d_r}$ from the biadjacency matrix associated with an $(m, M, d_r, d_l, 1)-$biregular graph.

In the following, we discuss different existing biregular graphs with large girth and the construction of IUNTFs from their biadjacency matrices.

1) Construction from regular bipartite graphs: Let $n \geq 2$ be an integer and p be a power of a prime. In [24], a family of $p-$regular bipartite graphs, denoted by $D(n, p)$, is constructed with p^n left vertices and p^n right vertices. When n is odd, the girth of $D(n, p)$ is at least $n + 5$ [24] and when n is even, the girth of $D(n, p)$ is at least $n + 4$ [25]. Hence, for all n, the girth of $D(n, p)$ is at least $2\lceil \frac{n}{2} \rceil + 4$. In particular, $D(2, p)$ and $D(3, p)$ have girth 6 and 8, respectively. Now, the biadjacency matrix Φ of $D(n, p)$ is a $p^n \times p^n$ binary matrix with the following properties:

1) Each column has p ones.
2) Each row has p ones.
3) As the girth is greater than 4, the overlap between any two columns (rows) is at most one. Hence, the coherence is at most $\frac{1}{p}$.

Now, if $T_{p\times K}$ is a UNTF, if we define $\alpha = \max_{i,j} |T(i, j)|$, $(\Phi \circledast T)_{p^n\times Kp^n}$ is a UNTF with coherence $\max\{\frac{\alpha^2}{p}, \mu_T\}$. If $T_{p\times p}$ is a DCT or a DFT matrix, then $(\Phi \circledast \frac{1}{\sqrt{p}}T)_{p^n\times p^{n+1}}$ is a UNTF with coherence at most $\frac{2}{p} = 2m^{-\frac{1}{n}}$ or $\frac{1}{p} = m^{-\frac{1}{n}}$, respectively, where $m = p^n$ is the row size. If Φ comes from $D(2, p)$, then $(\Phi \circledast \frac{1}{\sqrt{p}}T)_{p^2\times p^3}$ is a UNTF with coherence $\frac{2}{p}$ (in real case) or $\frac{1}{p}$ (in complex case). In the complex case, when $m = p^2$, the coherence is at most $\frac{1}{\sqrt{m}}$, which is close to the Welch bound [26].

Apart from $D(n, p)$, there are also $d-$regular bipartite expander graphs with m left and right vertices [27], [28]. The $m\times m$ binary matrix Φ associated with a regular expander has d ones in each row and column. Now, if the girth of the graph is greater than 4, then Φ has at most one overlap between any two distinct columns. As a result, the coherence of Φ is $\frac{1}{d}$. If $T_{d\times K}$ is a UNTF, and letting $\alpha \triangleq \max_{i,j} |T(i, j)|$, we have that $(\Phi \circledast T)_{m\times Km}$ is a UNTF with coherence $\max\{\frac{\alpha^2}{d}, \mu_T\}$. If $T_{d\times d}$ is a DCT or a DFT matrix, then $(\Phi \circledast \frac{1}{\sqrt{d}}T)_{m\times md}$ is a UNTF with coherence at most $\frac{2}{d}$. Thus, expander graphs can also be used to construct UNTFs.

2) Construction from low density parity check (LDPC) codes: Bipartite graphs with girth greater than 4 have been used as the parity check matrix of LDPC codes [29]. A $(d_r, d_l)-$regular LDPC code is represented by a parity check matrix Φ in which each column has d_r ones and each row has d_l ones [29]. A general method is presented in [30] to construct LDPC codes having a large girth via the progressive edge-growth (PEG) algorithm. A PEG constructed $(d, \frac{Md}{m})-$regular LDPC code has girth greater than 4, if $\frac{\log(Md-M-m+1)}{\log[(d-1)(\frac{Md}{m}-1)]} \geq 2$. This condition approximately requires $d < (\frac{m^2}{M})^{\frac{1}{3}}$ [23], [29], which implies (i) if $d \approx m^{1/2}$, then $M \leq \sqrt{m}$, (ii) if $d \approx m^{1/3}$, then $M \leq m$ and (iii) if $d \approx m^{1/4}$, then $M \leq m^{5/4}$. This bound on d is somewhat loose, as, in practice the PEG algorithm can produce LDPC codes with much larger d. Now, the parity check matrix $\Phi_{m\times M}$ constructed from the PEG algorithm has the following properties :

1) Each column has d ones, where $d < (\frac{m^2}{M})^{\frac{1}{3}}$.
2) Each row has Md/m ones.
3) As the girth is greater than 4, the overlap between any two columns (rows) is at most one. Hence, the coherence is at most $\frac{1}{d}$.

Suppose $T_{d\times K}$ is a UNTF and $\alpha = \max_{i,j} |T(i, j)|$, , then $(\Phi \circledast T)_{m\times KM}$ becomes a UNTF with coherence at most $\max\{\frac{\alpha^2}{d}, \mu_T\}$. In particular, if $T_{d\times d}$ is a DCT or a DFT matrix, then $(\Phi \circledast \frac{1}{\sqrt{d}}T)_{m\times Md}$ is a UNTF with coherence at most $\frac{2}{d}$ or $\frac{1}{d}$, respectively.

3) Construction from quasi-cyclic LDPC Codes: Another well known class of LDPC codes is the quasi-cyclic LPDC (QC-LDPC) code [31]. Let Φ be the parity check matrix of a $(d_r, d_\ell)-$regular QC-LDPC code of length $M = pd_\ell$, where p is a positive integer. It is shown in [31] that a necessary condition for Φ to have girth greater than or equal to 6 in the Tanner graph representation of a $(d_r, d_\ell)-$regular QC-LDPC code is $p \geq d_\ell$ (i.e., $M \geq d_\ell^2$) if d_ℓ is odd, and $p \geq d_\ell + 1$ (i.e., $M \geq d_\ell(d_\ell + 1)$) if d_ℓ is even. Therefore, the matrix $\Phi_{pd_r\times pd_\ell}$ whose corresponding graph has girth greater than or equal to 6 has the following properties:

1) Each column contains d_r ones.
2) Each row contains d_ℓ ones.

3) The overlap between any two columns is at most one. Hence, the coherence of Φ is at most $\frac{1}{d_r}$.

Suppose $T_{d_r \times K}$ is a UNTF and $\alpha = \max_{i,j} |T(i,j)|$, then $(\Phi \circledast T)_{pd_r \times Kpd_\ell}$ becomes a UNTF with coherence at most $\max\{\frac{\alpha^2}{d_r}, \mu_T\}$. In particular, if $T_{d_r \times d_r}$ is a DCT or DFT matrix, $(\Phi \circledast \frac{1}{\sqrt{d_r}} T)_{pd_r \times pd_\ell d_r}$ is a UNTF with coherence at most $\frac{2}{d_r}$ or $\frac{1}{d_r}$, respectively.

In this section, we used biregular graphs with large girth to construct IUNTFs via the embedding operation. In the next section, we obtain biadjacency matrices based on polynomials over finite fields such that the intersection between two distinct columns of the biadjacency matrix can be greater than one. The matrices we obtain in this manner have much larger redundancy than the constructions presented above, at the cost of a slightly higher coherence.

V. IUNTFs from Biadjacency Matrices Obtained from Finite Fields

In this section, we show that binary matrices with the required properties can be obtained by evaluating polynomials over finite fields as in [32]. In order to obtain small coherence, most of the existing constructions focus on binary matrices with a large number of ones in each column and as little overlap between pairs of columns as possible. However, as row weights have no direct impact on the coherence, these constructions do not examine the row structure. But, for constructing IUNTFs using the embedding procedure, we need binary matrices with the constant row weight. We extend and refine the approach in [32] to produce structured binary matrices with flexible row size and large redundancy. Then, by applying the embedding operation on such structured binary matrices, we obtain IUNTFs with large redundancy.

A. Construction for Prime and Prime Power Dimension

In this section, we provide a method for constructing binary matrices with the required properties $(p1)$, $(p2)$ and $(p3)$ for the case when row size is a prime or a prime power.

Theorem 8: For p being a prime or a prime power and $1 \leq r < k \leq p$, a $(k, p^r, r)-$binary matrix of size $pk \times p^{r+1}$ with coherence at most $\frac{r}{k}$ can be constructed using polynomials of degree at most r over the finite field $\mathbb{F}_p$.

Proof: See Sec. X-B7. ■

Thus, one can construct incoherent binary matrices with constant row and column weights via Theorem 8 using polynomials over finite fields.

Relationship with the biregular graph based construction: The constructed binary matrix V^p is the biadjacency matrix of a $(pk, p^{r+1}, p^r, k, r)-$biregular graph.

We note that, for $k = p$ and $r > 1$, the above construction produces the binary matrices reported in [32]. The above result shows that the rows of the binary matrix have constant row weight and therefore the construction can be used to produce IUNTFs via the embedding operation.

B. Construction for Composite Dimension

For the composite case, we use the following composition rule given in [33] for combining binary matrices. The following result has been proved there.

Lemma 2: ([33, Lemma 4]) For $i = 1, 2$, let Ψ_i be a binary matrix of size $m_i \times M_i$ consisting of k_i row blocks of size n_i each (i.e., $m_i = k_i n_i$). In each column, every row block contains a single 1, and the intersection between any two columns is at most r_i. Assume that $r = \max\{r_1, r_2\} < k \leq \min\{k_1, k_2\} \leq \min\{n_1, n_2\}$. Then, the composition rule, denoted by $*$, produces a matrix $\Psi = \Psi_1 * \Psi_2$ of size $n_1 n_2 k \times M_1 M_2$ containing k row blocks of size $n_1 n_2$ each. In each column of Ψ, every row block contains a single 1, and the intersection between any two columns is at most r and the density of Ψ is $\frac{1}{n_1 n_2}$.

Using the above result recursively, we have the following result for general m.

Theorem 9: Let $m = p_1 \cdots p_t$, where $p_1, \ldots, p_t$ are distinct primes or prime powers and $r < k \leq \min\{p_1, \ldots, p_t\}$. Then, there exists a $(k, m^r, r)-$binary matrix of size $mk \times m^{r+1}$ with coherence at most $\frac{r}{k}$.

Proof: See Sec. X-B8. ■

The above theorem can be used to construct IUNTFs by embedding V_r^m with other low coherence tight frames. Specifically, embedding the binary matrix V_r^m with other tight frames with low coherence yields IUNTFs. Suppose $T_{k \times K}$ is a UNTF with $\alpha = \max_{i,j} |T(i,j)|$ and let V_r^m be the $mk \times m^{r+1}$ binary matrix constructed using Theorem 9. Then, $(V_r^m \circledast T)$ is a UNTF of size $mk \times Km^{r+1}$ with coherence $\mu_{V \circledast T}$ at most $\max\{\frac{r\alpha^2}{k}, \mu_T\}$.

So far, we have discussed the construction of IUNTFs via the embedding operation using structured binary matrices obtained from biregular graphs and finite field theory. In next section, we construct IUNTFs using binary matrices obtained from combinatorial designs.

VI. IUNTFs from Combinatorial Designs

In this section, we show that incidence matrices of some combinatorial designs produce binary matrices with the required properties.

A. IUNTFs from Euler Square Matrices

It is known that when an Euler square of index (n, k) exists with $k < n$,[3] a binary Euler square matrix Φ of size $nk \times n^2$ can be constructed such that each column of Φ contains k ones and the overlap between any two columns is at most one [35]. As a result, the coherence of Φ is at most $\frac{1}{k}$. A close look at the construction of Euler square matrix Φ reveals that each row contains n ones. Therefore, Φ is the biadjacency matrix associated with an $(nk, n^2, k, n)-$biregular graph. In [36], it is shown that the girth of this biregular graph is 6, which also shows that the overlap between any two columns or any two rows of Φ is at most one. For $T_{k \times K}$ being a UNTF with $\alpha =$

[3] The construction of Euler squares for several values of n and k can be found in [34].

TABLE II. IUNTFS FROM STEINER SYSTEMS ($\mu = O(m^{\alpha})$, $M = O(m^{\beta})$)

Steiner system	m	M	$\mu \leq$	α	β
$(2, p, p^2)$	p^2	$p^2(p+1)$	$\frac{2}{p}$	$-\frac{1}{2}$	$\frac{3}{2}$
$(2, p+1, p^2+p+1)$	p^2+p+1	$(p+1)(p^2+p+1)$	$\frac{2}{p+1}$	$-\frac{1}{2}$	$\frac{3}{2}$
$(2, p+1, p^3+p^2+p+1)$	p^3+p^2+p+1	$(p+1)(p^2+1)(p^2+p+1)$	$\frac{2}{p+1}$	$-\frac{1}{3}$	$\frac{5}{3}$
$(2, p, p^3)$	p^3	$p^3(p^2+p+1)$	$\frac{2}{p}$	$-\frac{1}{3}$	$\frac{5}{3}$
$(2, p+1, p^3)$	p^3+1	$(p+1)p^2(p^2-p+1)$	$\frac{2}{p}$	$-\frac{1}{3}$	$\frac{5}{3}$
$(3, p+1, p^2+1)$	p^2+1	$p(p+1)(p^2+1)$	$\frac{4}{p+1}$	$-\frac{1}{2}$	$\frac{3}{2}$

$\max_{i,j}|T(i,j)|$, one can construct a UNTF $(\Phi \circledast T)_{nk\times Kn^2}$ with coherence $\max\{\frac{\alpha^2}{k}, \mu_T\}$.

One interesting fact is that Φ^T is a binary matrix of size $n^2 \times nk$ with n ones in each column, k ones in each row, and the overlap between any two distinct columns is at most one. Therefore, Φ^T is an $(n,k,1)-$binary matrix of size $n^2 \times nk$. As a result, the coherence of Φ^T is at most $\frac{1}{n}$. Notice that Φ^T is an over-determined matrix, that is, its row size is greater than its column size. Hence, it is ordinarily not useful as a compressed sensing matrix. For $T_{n\times K}$ being a UNTF with $\alpha = \max_{i,j}|T(i,j)|$, one can construct a UNTF $(\Phi^T \circledast T)_{n^2\times Knk}$ with coherence $\max\{\frac{\alpha^2}{n}, \mu_T\}$. This UNTF is underdetermined because $K \geq n$, and hence is useful for compressed sensing. In particular, if we choose T as a DCT or DFT matrix of size $n \times n$, then $(\Phi^T \circledast \frac{1}{\sqrt{n}}T)_{n^2\times n^2 k}$ becomes a UNTF with coherence at most $\frac{2}{n} = O(\frac{1}{\sqrt{m}})$, where $m = n^2$ is the row size of the UNTF.

For p being prime or prime power, an Euler square of index $(p, p-1)$ exists. For T being a DCT or DFT matrix of size $(p-1)\times(p-1)$, we get $(\Phi \circledast \frac{1}{\sqrt{p-1}}T)_{p(p-1)\times(p-1)p^2}$, with $\mu_{(\Phi\circledast T)}$ at most $\frac{2}{p-1} = O(\frac{1}{\sqrt{m}})$, where $m = p(p-1)$ is the row size of $(\Phi \circledast \frac{1}{\sqrt{p-1}}T)$.

B. IUNTFs from Steiner Systems

The incidence matrix Φ of a $(t,k,n)-$Steiner system is the biadjacency matrix of an $\left(n, \frac{\binom{n}{t}}{\binom{k}{t}}, \frac{\binom{n-1}{t-1}}{\binom{k-1}{t-1}}, k\right)-$biregular graph with coherence $\frac{t-1}{k}$. For $T_{k\times K}$ being a UNTF with $\alpha = \max_{i,j}|T(i,j)|$, one can construct a UNTF $(\Phi \circledast T)$ of size $n\times K\frac{\binom{n}{t}}{\binom{k}{t}}$ with coherence $\max\{\frac{(t-1)\alpha^2}{k}, \mu_T\}$. A necessary divisibility condition [37] for the existence of $(t,k,n)-$Steiner systems is as follows:

$$\binom{k-i}{t-i} \text{ divides } \binom{n-i}{t-i} \, \forall\, 0 \leq i \leq t-1.$$

If, for a fixed t, there exists a $(t,k,n)-$Steiner system with $k = O(\frac{1}{\sqrt{n}})$, then by taking T as a $k\times k$ DCT or DFT matrix, we get the UNTF $(\Phi \circledast \frac{1}{\sqrt{k}}T)$ of size $n \times K\frac{\binom{n}{t}}{\binom{k}{t}}$ with coherence at most $\frac{2(t-1)}{k} = \frac{2(t-1)}{\sqrt{n}} = O(\frac{1}{\sqrt{m}})$, where $m = n$ is the row size. There are several known constructions of $(t,k,n)-$Steiner systems, and the UNTFs obtained from some of the known constructions is listed in the Table II.

C. IUNTFs from Balanced Incomplete Block Designs (BIBDs)

In combinatorial mathematics, among all block designs, the most intensely studied are the BIBDs, which were historically developed to address statistical issues in the design of experiments. Binary matrices with the required properties can be obtained from the incidence matrices of BIBDs [38].

A BIBD is a set X of $m \geq 2$ elements called varieties or treatments and a collection of $M > 0$ subsets of X, called blocks, such that the following conditions are satisfied:

1) Each block consists of exactly k varieties, $m > k > 0$.
2) Each variety appears in exactly w blocks, $w > 0$.
3) Each pair of varieties appear simultaneously in exactly λ blocks, $\lambda > 0$.

BIBDs are referred to as $(m, M, w, k, \lambda)-$designs, and they obey the following relations:

$$Mk = mw \quad \text{and} \quad w(k-1) = \lambda(m-1).$$

Given an $(m, M, w, k, \lambda)-$design, one can represent it as an $m \times M$ matrix Φ_λ called the incidence matrix of the design. The rows are labeled with the varieties of the design and the columns are labeled with the blocks. We place a 1 in the (i,j)th cell of the matrix if variety i is contained in block j, and we place a 0 otherwise. Then,

1) Each column of Φ_λ has k ones.
2) Each row of Φ_λ has w ones.
3) Each pair of distinct columns has λ common ones. In particular, when $\lambda = 1$, the overlap between any two distinct columns becomes exactly one.

Remark 1: The definition 7 is a relaxation of the concept of BIBD in the sense that it requires all vertices to be contained in the same number of blocks, and requires all blocks to contain the same number of vertices, but discards the BIBD requirement that any two distinct vertices are contained in the same number of blocks.

As a consequence of the above, the coherence of Φ_1 is $\frac{1}{k}$. For $T_{k\times K}$ being a UNTF with $\alpha = \max_{i,j}|T(i,j)|$, one can construct a UNTF $(\Phi_1 \circledast T)_{m\times MK}$ with coherence $\max\{\frac{\alpha^2}{k}, \mu_T\}$. In particular, if we choose T to be a DCT or DFT matrix of size $k \times k$, then $\mu_{(\Phi_1 \circledast \frac{1}{\sqrt{k}}T)}$ becomes at most $\frac{2}{k}$. The construction of BIBDs for various dimensions and parameters has been explored in [38].

D. IUNTFs from Disjunct Matrices

One can also find structured binary matrices with the required properties from the construction of disjunct matrices in the non-adaptive group testing literature. For $k > d$, in [39], a binary matrix Φ of size $\binom{n}{d} \times \binom{n}{k}$ is constructed with the following properties:

1) Each column contains $\binom{k}{d}$ ones.
2) Each row has $\binom{n-d}{k-d}$ ones.
3) The overlap between any two distinct columns is at most $\binom{k-1}{d}$.
4) The coherence of Φ is at most $1 - \frac{d}{k}$.

Now, if T is a DFT matrix of size $\binom{k}{d} \times \binom{k}{d}$, then $(\Phi \circledast \frac{1}{\sqrt{\binom{k}{d}}} T)$ becomes a UNTF of size $\binom{n}{d} \times \binom{k}{d} \binom{n}{k}$ with coherence at most $1 - \frac{d}{k}$.

In the next section, we discuss the sparse recovery properties of IUNTFs constructed from biregular graphs.

VII. Sparse Recovery Properties and Discussion

In this section, we begin by showing that IUNTFs obtained from the embedding operation satisfy StRIP and SInCoP. We also connect the constructions presented thus far to existing constructions from the literature in Subsection VII-A.

The next theorem uncovers the sparse recovery properties of the matrix $(\Phi \circledast T)_{m \times M}$.

Theorem 10: $(\Phi \circledast T)_{m \times M}$ with coherence $O(\frac{1}{\sqrt{m}})$ supports sparse recovery under the basis pursuit algorithm for all but a proportion $\epsilon \, (< \frac{1}{t})$ of $t-$sparse signals chosen from the random signal model S_t provided $m = O(t(\log(\frac{M}{\epsilon}))^{3/2})$.

Proof: Follows from Theorem 4. ■

Theorem 11: $(\Phi \circledast T)_{m \times M}$ with coherence $O(m^{-1/3})$ (or $O(m^{-1/4})$) supports sparse recovery under the basis pursuit algorithm for all but a proportion $\epsilon \, (< \frac{1}{t})$ of $t-$sparse signals chosen from the random signal model S_t provided $m = O(t(\log(\frac{M}{\epsilon}))^{9/4})$ (or $m = O(t(\log(\frac{M}{\epsilon}))^{3})$).

Proof: From [3, Theorem 2] and [3, Theorem 4], it is easy to check that $(\Phi \circledast T)$ with coherence at most $O(m^{-1/3})$ (or $O(m^{-1/4})$) satisfies $(t, \delta, \epsilon)-$StRIP and $(t, \alpha, \epsilon)-$SInCoP for $m = O(t(\log(\frac{M}{\epsilon}))^{9/4})$ (or $m = O(t(\log(\frac{M}{\epsilon}))^{3}),$) where $t < \frac{1}{\epsilon}$ and $\alpha = O((\log(\frac{2M}{\epsilon}))^{-1})$. Consequently, the proof of the theorem follows directly from [3, Theorem 15]. ■

The next theorem shows that the sparse recovery guarantees of the IUNTF obtained by applying the embedding operation on a binary matrix translates to the binary matrix itself.

Theorem 12: Let Φ be a $(k, \frac{mk}{m})-$binary matrix with size $m \times M$ with coherence at most $O(m^{-1/2})$ (or $O(m^{-1/3})$ or $O(m^{-1/4})$). Then, Φ supports sparse recovery under the basis pursuit algorithm for all but a proportion $\epsilon \, (< \frac{1}{t})$ of $t-$sparse signals chosen from the random signal model S_t provided $m = O(t(\log(\frac{M}{\epsilon}))^{3/2})$ (or $m = O(t(\log(\frac{M}{\epsilon}))^{9/4})$ or $m = O(t(\log(\frac{M}{\epsilon}))^{3})$).

Proof: See Sec. X-B9. ■

For a $(k, \frac{mk}{m})-$binary matrix of size $m \times M$ with coherence at most $O(m^{-1/2})$ (or $O(m^{-1/3})$ or $O(m^{-1/4})$), the classical sparse recovery bound based on coherence can guarantee recovery of $t-$sparse signals whenever m is in the order of t^2 (or t^3 or t^4). Theorem 12 provides a significantly better recovery guarantee as compared to the coherence based bound.

A. Discussion

- Among all the constructions presented here, the constructions yielding UNTFs with coherence $O(\frac{1}{\sqrt{m}})$ are of special interest. For example, IUNTFs constructed from $D(2,p)$, Euler square of index $(p-1, p)$ and $(2, p, p^2)-$Steiner system possess coherence of near optimal order, where p is a prime or a prime power. Among them, IUNTFs with large redundancy are preferred in many applications. For p being a prime or prime power, IUNTFs constructed in Section V and IUNTFs constructed from the $(2, p, p^2)-$Steiner system are of row size $m = p^2$ and with coherence $O(\frac{1}{\sqrt{m}})$. But the redundancy of the former (i.e., the IUNTFs constructed in Section V) is p^r for any $r > 1$ and the redundancy of the latter is p. Hence, the former has better compression ratio compared to the latter.
- In [17], the embedding operation on binary matrices (constructed from Steiner systems) with coherence in the order of $m^{-\frac{1}{2}}$ or $m^{-\frac{1}{3}}$ has been used to obtain column extended matrices, where m is the row size. Through numerical experiments, it is shown that the column extended matrices exhibit better sparse recovery performance compared to random Gaussian matrices. The theoretical analysis in [17] based on coherence can guarantee recovery of $t-$sparse signals whenever m is in the order of t^2 or t^3. Using our construction methodology, we have shown that such column extended matrices obtained via the embedding operation are indeed UNTFs. Therefore, by Theorems 4 and 11, we are able to establish that such UNTFs can recover $t-$sparse signals whenever $m = O(t(\log(\frac{M}{\epsilon}))^{3/2})$ or $m = O(t(\log(\frac{M}{\epsilon}))^{9/4})$, which are significantly better than the theoretical bound given in [17]. Therefore, we are able to provide a mathematical justification for the superior recovery performance of embedded matrices through their StRIP and SInCoP.
- In [16], the authors construct $m \times M$ matrices with $(\ell_1, t)-$recovery property for $t < \frac{\sqrt{m}}{4}$, which is tight to within a multiplicative factor of $4\sqrt{2}$. Taking p to be a prime or prime power, IUNTFs of size $p^2 \times p^3$ constructed from $D(2,p)$, Euler Square and $(2, p, p^2)-$Steiner systems have coherence at most $\frac{1}{p}$. Consequently, they have $(\ell_1, t)-$recovery property for $t < \frac{\sqrt{m}}{2}$, which is tight to within a multiplicative factor of $2\sqrt{2}$ and $m = p^2$. Thus, by considering the coherence itself, we see that the constructions presented here offer a slight improvement over those in [16].
- In [16], [19], it is shown that the embedded matrices obtained from pairwise balanced design can recover all $t-$sparse signals obtained via the $(\ell_1, t)-$property provided $m = O(t^2)$. On the other hand, IUNTFs constructed via the embedding operation support sparse recovery through ℓ_1-minimization for all but an $\epsilon-$fraction of $t-$sparse signals chosen from a generic signal model provided $m = O(t(\log(\frac{M}{\epsilon}))^{\frac{3}{2}})$, which is significantly better than the results reported in [16], [19].
- In [3], it is shown that ETFs and chirp matrices have a redundancy of order m, the row size, and satisfy

$(t, \delta, \epsilon)-$StRIP provided $m = O(t)$. For a fixed r, IUNTFs obtained from finite field theory in Section V have redundancy $m^{\frac{r}{2}+1}$ and also satisfy $(t, \delta, \epsilon)-$StRIP provided $m = O(t)$. Thus, they possess better redundancy compared to ETFs and chirp matrices for $r > 2$.

- Delsarte-Goethals codes satisfy a result similar to Theorem 4 with $m = O(k(\log(\frac{M}{\epsilon}))^3)$ [3, Proposition 13]. The Delsarte-Goethals codes have $\mu = O(m^{-1/4})$. In comparison, the IUNTF $(\Phi \circledast T)$ has $\mu = O(m^{-1/2})$, and thus it possesses better reconstruction properties, with the same dependence of the row size on the sparsity.
- According to Theorem 9, we can construct a $(p, p^r, r)-$binary matrix of size $p^2 \times p^{r+1}$ with coherence at most $\frac{r}{p}$. Now, for $r = O(p^{1/3})$ (and $p^{1/2}$), one can have $m \times m^{O(m^{1/6})}$ (and $m \times m^{O(m^{1/4})+1}$) with coherence $O(m^{-1/3})$ (and $O(m^{-1/4})$). Now, we can construct IUNTFs from these binary matrices with coherence at most $O(m^{-1/3})$ (and $O(m^{-1/4})$) and with redundancy $m^{O(m^{1/6})}$ (and $O(m^{O(m^{1/4})+1})$). For example, UNTFs of sizes $9^2 \times 9^5$, $16^2 \times 16^6$, $25^2 \times 25^7$ and $49^2 \times 49^9$ can be constructed with coherences at most $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{7}$, respectively. Now, according to Theorem 11, they satisfy StRIP and SInCoP. Thus, we can construct highly redundant IUNTFs with StRIP and SInCoP. To the best of our knowledge, these are the *only known constructions of matrices with redundancy that is exponential in the fourth root of the number of rows.*
- In this work, we have considered x to be sparse in the canonical basis, and study the linear system $y = \Phi x$, where Φ is a UNTF. In practice, x may not be sparse in standard basis but sparse in a dictionary A, where $x = A\alpha$ and α is sparse. Typically, A is an orthonormal basis or a UNTF, then, ΦA is also an UNTF, which implies that $\bar{\mu}^2_{\Phi A}$ and $\|\Phi A\|_2^2$ are $\frac{M-m}{m(M-1)}$ and $\frac{M}{m}$, respectively. But, one cannot guarantee that ΦA possesses low coherence even when Φ has low coherence. Consequently, it is difficult to establish StRIP and SInCoP of ΦA. It is also known that randomly constructed Φ matrices that satisfy the restricted isometry property for ensuring sparse recovery continue to satisfy it as long as the distribution of Φ is invariant to right multiplication by a unitary matrix. However, this idea does not extend to deterministic constructions. Studying this aspect is beyond the scope of the present work.

In summary, the results presented thus far provide a unified approach to constructing matrices with low coherence, large redundancy, and attractive sparse recovery properties. We illustrate these via numerical results in the next section.

VIII. Numerical Results

In this section, we empirically validate the theoretical bounds derived above for IUNTFs and binary matrices. Specifically, we first show that using the IUNTFs constructed via the embedding operation, BP can recover sparse signals for which the sparsity level that is approximately linear in the row size (see Theorem 10). Next, we show that a similar linear recovery property holds for binary matrices constructed from biregular graphs via the OMP algorithm (see Theorem 12).

Table III. BP recovery performance using IUNTFs obtained from finite fields

Size $(m \times M)$	$k_{\textbf{max}}^{\textbf{BP-IUNTF}}$	$k_{\textbf{max}}^{\textbf{BP-IUNTF}}/m$
49×2401	6	0.1
121×14641	13	0.1
169×28561	18	0.1
289×83521	29	0.1

Table IV. BP recovery performance with binary matrices obtained from finite fields

Size $(m \times M)$	$k_{\textbf{max}}^{\textbf{BP-bin}}$	$k_{\textbf{max}}^{\textbf{BP-bin}}/m$
49×392	9	0.18
121×1452	22	0.18
169×2366	31	0.18
289×5202	50	0.19
361×7220	61	0.17

At each sparsity level, we generate 1000 sparse signals according to the model S_k in Definition 5. That is, the support of the sparse vector is chosen uniformly at random, but the nonzero values are chosen to be Gaussian distributed with zero mean and unit variance. We recover the sparse vector from its lower-dimensional projection obtained from the measurement matrices presented in this paper. We quantify the sparse recovery performance using average normalized mean square error (NMSE): if x is the the original sparse vector, and $\hat{x}$ is the recovered vector, the NMSE is defined as $\mathbb{E}(\|\hat{x}-x\|_2^2/\|x\|_2^2)$. The reconstruction is considered successful if the NMSE is below a threshold of 0.05. Codes for all the constructions discussed in this paper are available online at https://github.com/pradipspc/UNTF.git.

A. Binary Matrices and IUNTFs Obtained using Finite Fields

We construct $(p, p^2, 2)-$binary matrices of size $p^2 \times p^3$ with coherence at most $\frac{2}{p}$, density $\frac{1}{p}$ and redundancy p as described in Sec V. We consider four distinct values of p, that is, $p = 7, 11, 13, 17$. From these binary matrices, we construct real-valued IUNTFs by embedding DCT matrices. We thus obtain IUNTFs of size $p^2 \times p^4$ with coherence at most $\frac{4}{p}$, density $\frac{1}{p}$ and redundancy p^2. They also satisfy $O(p^2)$-StRIP according to Theorem 11. Hence, according to Theorems 11 and 12, one can expect BP to successfully recover sparse signals coming from the random signal model with $O(p^2)$-sparsity using IUNTFs and binary matrices, respectively. However, the $(p, p^2, 2)-$binary matrix of size $p^2 \times p^3$ is rank deficient, and BP algorithms such as ℓ_1 magic [40] cannot be used directly in this case. Hence, we convert it to a full rank matrix by concatenating an identity matrix of size $p^2 \times p^2$.

In Table III, we show the recovery performance of BP using these IUNTFs as sensing matrices. The maximum sparsity at which recovery is successful, denoted by $k_{\textbf{max}}^{\textbf{BP-IUNTF}}$, is approximately $\lfloor 0.1p^2 \rfloor$. Therefore, we see that the sparsity scales linearly with row size, as expected. Also, we show the

TABLE V. BP RECOVERY PERFORMANCE FOR IUNTFS OBTAINED USING EULER SQUARES

Size $(m \times M)$	$k_{\text{max}}^{\text{BP-IUNTF}}$	$k_{\text{max}}^{\text{BP-IUNTF}}/m$
121×242	54	0.45
169×338	76	0.44
289×578	133	0.46
361×722	165	0.45

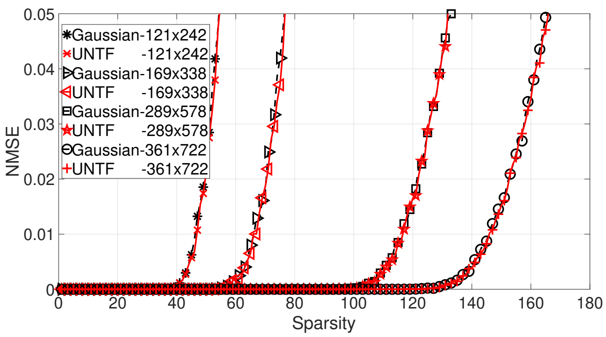

Figure 1. BP NMSE for UNTFs obtained using Euler Squares and Gaussian matrices

TABLE VI. OMP RECOVERY PERFORMANCE FOR BINARY MATRICES OBTAINED FROM EULER SQUARES

Size $(m \times M)$	$k_{\text{max}}^{\text{OMP-bin}}$	$k_{\text{max}}^{\text{OMP-bin}}/m$
136×289	57	0.42
171×361	75	0.44
253×529	117	0.46
406×841	195	0.48

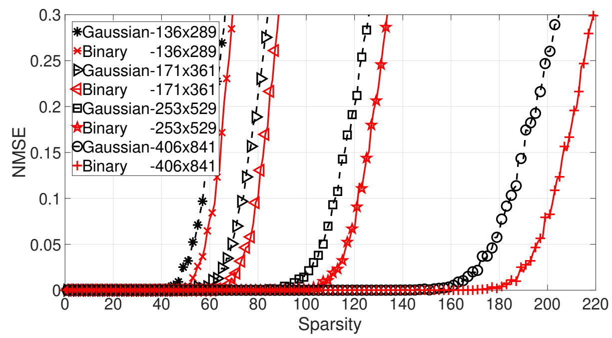

Figure 2. OMP NMSE for binary matrices obtained using Euler squares and Gaussian matrices

BP recovery performance using the binary matrices as sensing matrices in Table IV. The maximum sparsity at which recovery is successful, denoted by $k_{\text{max}}^{\text{BP-bin}}$, is approximately $\lfloor 0.18p^2 \rfloor$. Hence, once again, the maximum sparsity grows linearly with row size.

B. Binary Matrices and IUNTFs Obtained from Euler Squares

We construct real-valued IUNTFs by embedding DCT matrices into the transpose of binary matrices obtained from Euler squares of index $(p, 2)$ for $p = 11, 13, 17, 19$ as discussed in Sec. VI-A. The resulting IUNTFs are of size $p^2 \times 2p^2$ with coherence at most $\frac{2}{p}$, density $\frac{1}{p}$ and redundancy 2. They also satisfy $O(p^2)$-StRIP as shown in Sec. VI-A. Therefore, one can expect BP can recover sparse signals with $O(p^2)$-sparsity.

We show the results in Table V and Figure 1. The maximum sparsity at which recovery is successful, denoted by $k_{\text{max}}^{\text{BP-IUNTF}}$, is approximately $\lfloor 0.45p^2 \rfloor$, thus showing the linear scaling of sparsity with the number or rows. Further, Figure 1 shows that the recovery performance of the IUNTFs is virtually identical to that of random Gaussian matrices (every entry is chosen to be i.i.d. with zero mean and unit variance). Among the random matrix based constructions, Gaussian matrices are known to be optimal in terms of the row size required for successful sparse recovery [41]. Thus, the deterministic constructions based on IUNTFs presented in this paper have similar sparse recovery performance as random Gaussian matrices.

Next, we show the recovery performance of binary matrices of size $p(\frac{p-1}{2}) \times p^2$ constructed from the Euler square of index $(p, \frac{p-1}{2})$ for $p = 17, 19, 23, 29$. These binary matrices are of size $p(\frac{p-1}{2}) \times p^2$ with coherence at most $\frac{2}{p-1}$, density $\frac{1}{p-1}$ and redundancy $\frac{2p}{p-1} \approx 2$.

It turns out, similar to the binary matrices constructed in Sec. VIII-A, the Gramian matrices of such binary matrices are not positive definite, and, as a result, standard convex optimization solvers such as ℓ_1 magic [40] are not applicable in this case. Hence, the solutions are computed using the orthogonal matching pursuit (OMP) algorithm. The results are summarized in Table VI and Figure 2. The table clearly shows the linear relationship between the maximum sparsity of successfully recovered sparse signals, denoted by $k_{\text{max}}^{\text{OMP-bin}}$, and the row size of the binary matrices. Also, Figure 2 shows that the NMSE performance of the binary matrices via OMP is better than that of the optimal random Gaussian matrices. Thus, the deterministic constructions presented in this paper are competitive with respect to their random Gaussian counterparts, especially under recovery via OMP.

IX. CONCLUSIONS

In this work, we identified the properties of a binary matrix so that one can obtain unit norm tight frames via an embedding operation. We showed that it is possible to obtain binary matrices satisfying the required properties from biregular graphs, combinatorial designs, finite field theory and non-adaptive group testing. We also showed that our constructed IUNTFs satisfy StRIP and SInCoP.

In the construction of IUNTFs via the embedding operation, the main ingredient is to have a binary matrix satisfying $(p1)$, $(p2)$ and $(p3)$. We hope that our present work will motivate other new constructions of structured binary matrices. Another interesting aspect of IUNTFs constructed via the embedding operation is that they are block orthonormal. Investigating the use of such block orthogonal sensing matrices for block-sparse vector recovery is another promising direction for future work.

X. APPENDIX

A. Statistical Restricted Isometry Property

In this subsection, we briefly review some of the sparse signal recovery guarantees associated with the statistical restricted isometry property (StRIP) (see Definition 3).

Theorem 13: [22] Let $\Phi_{m\times M}$ be a frame. Then, if

$$\sqrt{\mu_\Phi^2 t\log(t+1)} + \frac{t}{M}\|\Phi\|_2^2 \le c\delta, \text{ where } c \text{ is a constant,}$$

Φ satisfies $(t, \delta, \frac{1}{t})$-StRIP.

The following theorem provides sufficient conditions for a frame to satisfy StRIP.

Theorem 14: [22], [3] Let $\Phi_{m\times M}$ be a frame such that $\mu_\Phi = O_\epsilon(\frac{1}{\sqrt{m}})$, $\overline{\mu}_\Phi^2 = O_\epsilon(\frac{1}{m})$ and $\|\Phi\|_2^2 = O_\delta(\frac{M}{m})$. Then,

1) $m = O_\epsilon(t\log t)$ is sufficient for $\Phi_{m\times M}$ to satisfy $(t,\delta,\epsilon)-$StRIP, with $\epsilon < \frac{1}{t}$.
2) $m = O_\epsilon(t)$ suffices for $\Phi_{m\times M}$ to satisfy $(t,\delta,\epsilon)-$StRIP with $t < \frac{1}{\epsilon}$. Here, O_ϵ and O_δ denote functions whose constants may depend on ϵ and δ, respectively.

It is shown in [3] that ETFs and chirp matrices satisfy StRIP. Random sparse signals can be recovered with high probability using the basis pursuit algorithm if the matrix satisfies the SInCoP, defined in Section 4, along with StRIP [2], [3]. The following theorem relates the mutual coherence related properties of a matrix to the SInCoP.

Theorem 15: [3, Theorem 4] Let $\Phi_{m\times M}$ be a unit norm frame such that

$$\mu_\Phi^4 \le \frac{(1-a)^2\beta^2}{32t(\log(2M/\epsilon))^3} \quad \text{and} \quad \overline{\mu}_\Phi^2 \le \frac{a\beta}{t\log(2M/\epsilon)},$$

where $\beta > 0$ and $0 < a < 1$ are constants. Then Φ has the $(t,\alpha,\epsilon)-$SInCoP with $\alpha = \beta/\log(2M/\epsilon)$, $0 \le t \le M$ and $\epsilon > 0$ is a small number.

The following theorem relates the StRIP and SInCoP to the recovery of $t-$sparse signals via the basis pursuit algorithm.

Theorem 16: [3] Let x be a $t-$sparse signal from the model S_t in Section 5. Suppose that the matrix $\Phi_{m\times M}$ satisfies the following:

1) $(t,\delta,\epsilon)-$StRIP.
2) $(t, \frac{(1-\delta)^2}{8\log(2M/\epsilon)}, \epsilon)$-SInCoP.

Then, x can be recovered uniquely via basis pursuit with probability exceeding $1-3\epsilon$.

B. Proofs of Theorems

1) Proof of Theorem 1: Since $\Phi \in \mathbb{C}^{m\times M}$ is a UNTF, $\|\Phi x\|_2^2 = \frac{M}{m}\|x\|_2^2 \,\forall\, x \in \mathbb{C}^M$, and $\|\phi_i\|_2 = 1\,\forall i$. From (1),

$$\begin{aligned}(M-1)\overline{\mu}_\Phi^2 &= \max_j \sum_{i=1,i\neq j}^{M} |\langle \phi_i, \phi_j\rangle|^2 \\ &= \max_j(\|\Phi^T\phi_i\|_2^2 - \|\phi_i\|_2^2) \\ &= \max_j\left(\frac{M}{m}\|\phi_i\|_2^2 - \|\phi_i\|_2^2\right) = \frac{M-m}{m}. \quad (2)\end{aligned}$$

Thus,

$$\overline{\mu}_\Phi^2 = \frac{M-m}{m(M-1)} < \frac{1}{m}.$$

2) Proof of Theorem 2: Note that $\Phi_{m\times M}$ satisfies the following properties:

(p_1') $\mu_\Phi = O(\frac{1}{\sqrt{m}})$,
(p_2') $\overline{\mu}_\Phi^2 = \frac{M-m}{m(M-1)}$, according to Theorem 1,
(p_3') $\|\Phi\|_2^2 = \frac{M}{m}$, as it is a UNTF.

Now, from [22, Theorem B] (see Appendix X-A, Theorem 13) $\mu_\Phi = O((t\log t)^{-\frac{1}{2}})$ and $\|\Phi\|_2^2 = O(\frac{M}{t})$ are sufficient for a frame Φ to satisfy $(t,\delta,\frac{1}{t})-$StRIP. Since $\Phi_{m\times M}$ satisfies (p_1') and (p_3'), it has $(t,\delta,\frac{1}{t})$-StRIP for $m = O(t\log t)$.

For the second part, a simplified version of the main result in [3, Theorem 2] is sufficient, and it states that $\mu_\Phi = O_\epsilon((t\log t)^{-\frac{1}{4}})$, $\overline{\mu}_\Phi^2 = O_\epsilon(\frac{1}{t})$ and $\|\Phi\|_2^2 = O(\frac{M}{t})$ suffice for Φ to satisfy $(t,\delta,\epsilon)-$StRIP with $t < \frac{1}{\epsilon}$. Now, using the properties (p_1'), (p_2') and (p_3'), we see that $\Phi_{m\times M}$ satisfies $(t,\delta,\epsilon)-$StRIP with $t < \frac{1}{\epsilon}$ for $m = O_\epsilon(t)$.

3) Proof of Theorem 3: We know that, when a unit norm frame $\Psi_{m\times M}$ satisfies the conditions in Theorem 15, it has the the $(t,\alpha,\epsilon)-$SInCoP with $\alpha = \beta/\log(2M/\epsilon)$. Using properties (p_1'), (p_2') and (p_3'), and taking $m = O(t(\log(\frac{M}{\epsilon}))^{3/2})$ and $\alpha = O((\log(\frac{2M}{\epsilon}))^{-1})$, it is easy to check that $\Phi_{m\times M}$ satisfies the above relations and hence, satisfies the $(t,\alpha,\epsilon)-$SInCoP.

4) Proof of Theorem 5: Before going into the proof, note that a matrix is a UNTF if and only if it satisfies the following three properties (see Section II):

1) Its rows are orthogonal;
2) All row norms are the same;
3) All columns have unit norm.

We show that $(\Psi \circledast T)$ possesses the above three properties, and therefore it is a UNTF. Note that,

1) the rows of $(\Psi \circledast T)$ are orthogonal because T is a UNTF
2) the norm of each row of $(\Psi \circledast T)$ is the same, as Ψ has same number of ones in each row and T is a tight frame
3) each column of $(\Psi \circledast T)$ has unit norm as each column of T has unit norm.

This proves that $(\Psi \circledast T)$ is a UNTF.

5) Proof of Theorem 6: From Theorem 5, we know that $(\Psi \circledast T)$ is a UNTF of size $m \times MK$. It remains to show that coherence of $(\Psi \circledast T)$ is at most $\frac{r\alpha^2}{k}$.

Each column of Ψ produces a submatrix of size $m \times K$ of $(\Psi \circledast T)$. Hence, we have such M blocks of submatrices in $(\Psi \circledast T)$. In any given block of $(\Psi \circledast T)$, the maximum modulus of inner products between any two distinct columns is μ_T. On the other hand, the maximum modulus of inner products between any two distinct columns coming from two different blocks is at most $\frac{r\alpha^2}{k}$, as Ψ has constant column weight k and the overlap between any two columns of Ψ is at most r. Therefore, the coherence of $(\Psi \circledast T)$ is at most $\max\{\frac{r\alpha^2}{k}, \mu_T\}$. In particular, if T is an orthonormal basis, $\mu_T = 0$, and the coherence of $(\Psi \circledast T)$ is at most $\frac{r\alpha^2}{k}$.

6) Proof of Theorem 7: The binary biadjacency matrix $\Phi_{m\times M}$ associated with an $(m, M, d_r, d_l, r)-$ biregular graph has the following properties:

1) As the degree of right vertices is d_r, the number of ones in each column is d_r.
2) As the degree of left vertices is d_l, the number of ones in each row is d_l.
3) As any two right vertices are connected to at most r common left vertices, the overlap between any two columns of Φ is at most r.

Therefore, Φ satisfies the conditions $(p1)$, $(p2)$ and $(p3)$ with $k = d_r$. Consequently, by Theorem 5, $(\Phi \circledast T)_{m\times MK}$ becomes a UNTF, and by Theorem 6, the coherence of $(\Phi \circledast T)$ is at most $\max\{\frac{r\alpha^2}{d_r}, \mu_T\}$.

7) Proof of Theorem 8: We consider the finite field $\mathbb{F}_p = \{f_1, f_2, \ldots, f_p\}$ where p is a prime or a prime power. Let S_r^p be the collection of polynomials of degree at most r, where $r \leq p-1$. It is easy to check that the cardinality of S_r^p is $|S^p| = p^{r+1}$. We fix any ordered $k-$tuple $z \in \mathbb{F}_p^k$ with $r < k \leq p$. For simplicity, we consider $z = (f_1, \ldots, f_k)$. For $P \in S_r^p$, we form an ordered $k-$tuple after evaluating P at each of the points of z, i.e., $d^P := \big(P(f_1), \cdots, P(f_k)\big)$. From the $k-$tuple d^P we form a binary vector v^P of length pk using

$$v^P(p(i-1)+n) = \begin{cases} 1, & \text{if } P(f_i) = f_j \\ 0, & \text{otherwise} \end{cases}$$

where $1 \leq i \leq k, 1 \leq j \leq p$. We construct a binary matrix V^p of size $pk \times p^{r+1}$ by taking v^P as its columns for all $P \in S^p$.

It can be verified that the matrix V^p satisfies the following properties.

1) V^p has k row-blocks of size p each. Each column v^P of V^p has exactly k ones.
2) Let $V^p(q,:)$ be the qth row of V^p. Then, there exist unique nonnegative integers i and j such that $q = ip + j$ and $j \leq p-1$. According to the construction, the number of ones in $V^p(q,:)$ is the same as the cardinality of the set $S(q) = \{P \in S_r^p : P(f_i) = f_j\}$. Now, $P \in S(q)$ implies that P can be written as $P = (x - f_i)Q + f_j$, for some $Q \in S_{r-1}^p$. Therefore, we get $|S(q)| = |S_{r-1}^p| = p^r$. Hence, $V^p(q,:)$ contains p^r ones. Notice that the number of ones in qth row is independent of q. As a result, every row of V^p contains p^r ones.
3) The density of V^p is $\frac{1}{p}$.
4) For $P_i \neq P_j$, there are at most r common points between any two distinct $k-$tuples d^{P_i} and d^{P_j}. This is true because P_i and P_j have at most r common roots. Consequently, there is at most r overlap between any two distinct columns of V^p.

Therefore, V^p is a $(k, p^r, r)-$binary matrix, and the coherence of V^p is at most $\frac{r}{k}$.

8) Proof of Theorem 9: Let us first take $m = pq$, where p and q are two distinct primes or prime powers. With $r < k \leq \min\{p, q\}$, we apply the composition rule on the matrices $(V_r^p)_{pk\times p^{r+1}}$ and $(V_r^q)_{qk\times q^{r+1}}$ as in Section V-A to obtain a new binary matrix $V_r^{p,q} = V_r^p * V_r^q$ of size $pqk \times (pq)^{r+1}$. From the properties of V_r^p and V_r^q in Section V-A, it is easy to see that $V_r^{p,q}$ satisfies the following properties:

1) $V_r^{p,q}$ has k row-blocks of size pq each. Each row block contains a single one. Hence, there are exactly k ones in each column.
2) From the composition rule given in [33], it can be derived that $V_r^{p,q}$ has $(pq)^r$ ones in each row because every row of V_r^p and V_r^q contain p^r and q^r ones, respectively.
3) The overlap between any two distinct columns of $V_r^{p,q}$ is at most r.
4) The density of $V_r^{p,q}$ is $\frac{1}{pq}$.
5) Therefore, $V_r^{p,q}$ is a $(k, (pq)^r, r)-$binary matrix of size $pqk \times (pq)^{r+1}$ and consequently, the coherence $\mu_{V_r^{p,q}}$ of $V_r^{p,q}$ is at most $\frac{r}{k}$.

Let $m = p_1 \ldots p_t$, where $p_1, \ldots, p_t$ are distinct primes or prime powers and $r < k \leq \min\{p_1, \ldots, p_t\}$. Now, using the composition rule for the product of two primes recursively, we can obtain V_r^m as a $(k, m^r, r)-$binary matrix of size $mk \times m^{r+1}$ and hence, the coherence of V_r^m is at most $\frac{r}{k}$.

9) Proof of Theorem 12: Suppose an orthogonal basis $T_{k\times k}$ contains an all ones column (e.g., T is a DFT matrix or Hadamard matrix). Then, the columns of Φ are contained in $(\Phi \circledast T)$. Without loss of generality, assume that the first M columns of $(\Phi \circledast T)$ are Φ itself. Suppose the nonzero entries of the k-sparse vector x are contained within its first M indices. Then, the linear systems $y = (\Phi \circledast T)x$ and $y = \Phi\tilde{x}$ are equivalent, where $\tilde{x}$ is a truncated version of x containing its first M entries. Therefore, according to Theorems 4 and 11, we can infer sparse recovery guarantees of Φ from $(\Phi \circledast T)$.

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