E9 203 Compressed Sensing & Sparse SP

Last time: Underdetermined linear systems

$$\boxed{y = Ax}, \quad x \in \mathbb{R}^N,\ A \in \mathbb{R}^{m \times N},\ y \in \mathbb{R}^m$$

(i) $m << N$, (ii) A has full (row) rank.

$J(x)$: Cost : $\mathbb{R}^N \to \mathbb{R}$

$$\min_{x \in \mathbb{R}^N} J(x) \quad \text{s.t. } y = Ax$$

$J(x) = \|x\|_2^2$, unique globally optimal closed form soln.

$J(x) = \|x\|_p^p$, $p > 1$, " " " soln. but not closed form for $p \neq 2$.

$J(x) = \|x\|_1$, convex but not strictly convex

- Set of solns. form a bounded convex set
- $\exists$ at least 1 soln. which has at most m nonzeros.

$J(x) = \|x\|_p^p$, $p < 1$: Nonconvex cost.

ℓ_1 min. $\equiv$ linear program.

[Chapter 2]:

Notation: $[N] = \{1, 2, \dots, N\}$

$|S|$ or $\text{card}(S) \triangleq$ Cardinality of set S

$\bar{S}$ or $S^c = [N] \setminus S$.

For $x \in \mathbb{C}^N$,

$$\text{supp}(x) = \{ j \in [N] : x_j \neq 0 \}.$$

$x \in \mathbb{C}^N$ is s-sparse if

$$\|x\|_0 = |\text{supp}(x)| \leq s.$$

$$\|x\|_p^p = \sum_{j=1}^N |x_j|^p \xrightarrow{p \to 0} \sum_{j=1}^N \mathbb{1}_{\{x_j \neq 0\}} = |\{ j \in [N] : x_j \neq 0\}|.$$

<u>Defn.</u> For $p > 0$, the ℓ_p error in the best s-term approx to $x \in \mathbb{C}^N$ is

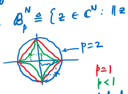
$$\sigma_s(x)_p \triangleq \inf \{ \|x - z\|_p,\ z \in \mathbb{C}^N,\ z \text{ } s\text{-sparse} \}.$$

The infemum is achieved by an s-sparse vec. $z \in \mathbb{C}^N$ whose nonzero entries equal the s-largest absolute entries of x.

- z need not be unique
- The optimum z is independent of p.

Informally, $x \in \mathbb{C}^N$ is "compressible" if $\sigma_s(x)_p$ decays very quickly with s. Turns out: this happens if

$$x \in B_p^N \triangleq \{ z \in \mathbb{C}^N : \|z\|_p \leq 1 \} \text{ for small } p > 0.$$



B_p^N, $p < 1$ is a good model for compressible vecs.

<u>Defn.</u> A nonincreasing rearrangement of $x \in \mathbb{C}^N$ is a vec. $x^* \in \mathbb{R}_+^N$ for which

$$x_1^* \geq x_2^* \geq \dots \geq x_N^* \geq 0$$

and there is a permutation $\pi : [N] \to [N]$ with $x_j^* = |x_{\pi(j)}|$, $j = 1, 2, \dots, N$.

For $0 < p < q$,

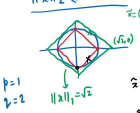
$$\|x\|_p \geq \|x\|_q \qquad [\, \|x\|_1 \geq \|x\|_2 \,]$$

Opposite ineq:

$$\|x\|_1 \leq \sqrt{N} \|x\|_2 \qquad [\, N \text{ is the dim. of } x \,]$$

$$\|x\|_2 \leq \sqrt{N} \|x\|_\infty$$

$\tilde{x} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$: $\|\tilde{x}\|_2 = 1$, $\|\tilde{x}\|_1 = \sqrt{2}$



$\|x\|_2 = 1$

$\|x\|_1 = 1$

$\hat{x} = (\frac{1}{2}, -\frac{1}{2})$. $\|\hat{x}\|_2 = \frac{1}{\sqrt{2}}$, $\|\hat{x}\|_1 = 1$

$\|\hat{x}\|_1 \leq \sqrt{2} \|\hat{x}\|_2$

$\hat{\hat{x}} = (0, -1)$. $\|\hat{\hat{x}}\|_2 = 1$, $\|\hat{\hat{x}}\|_1 = 1$

$\|\hat{\hat{x}}\|_1 \leq \sqrt{2} \|\hat{\hat{x}}\|_2$

For $0 < p < q$, the sphere with radius $\sqrt{N}$ and $\|\cdot\|_p$ inscribes the sphere of radius 1 with $\|\cdot\|_q$.

<u>Hölder's inequality</u>

$$\sum_{i=1}^N |a_i| |b_i| \leq \left(\sum_{i=1}^N |a_i|^t \right)^{\frac{1}{t}} \left(\sum_{i=1}^N |b_i|^{\frac{t}{t-1}} \right)^{(1 - \frac{1}{t})}.$$

Set $|a_i| = |x_i|^p$, $|b_i| = 1$, $r = \frac{q}{p} > 1$, $t = r$

$$\sum_{i=1}^N |x_i|^p \leq \left(\sum_{i=1}^N (|x_i|^p)^{\frac{q}{p}} \right)^{\frac{p}{q}} \left(\sum_{i=1}^N 1^{\frac{q}{q-p}} \right)^{1 - p/q}$$

$\frac{t}{t-1} = \frac{q/p}{\frac{q}{p} - 1}$, $1 - \frac{1}{t} = 1 - \frac{p}{q}$

$$\text{RHS} = \left(\sum_{i=1}^N |x_i|^q \right)^{p/q} N^{1 - p/q}$$

$$\text{LHS} = \|x\|_p^p$$

$$\Rightarrow \|x\|_p = \left(\sum_{i=1}^N |x_i|^p \right)^{\frac{1}{p}} \leq \underbrace{\left(\sum_{i=1}^N |x_i|^q \right)^{\frac{1}{q}}}_{\|x\|_q} \cdot N^{\frac{1}{p} - \frac{1}{q}}$$

$$\boxed{\|x\|_p \leq N^{\left(\frac{1}{p} - \frac{1}{q}\right)} \cdot \|x\|_q} \quad 0 < p < q$$

$$\|x\|_q \leq \|x\|_p \leq N^{\left(\frac{1}{p} - \frac{1}{q}\right)} \cdot \|x\|_q, \quad 0 < p < q.$$

In general, $\exists$ c, C s.t.

$$c \|x\|_p \leq \|x\|_q \leq C \|x\|_p$$

<u>Prop.</u> For any $q > p > 0$ and any $x \in \mathbb{C}^N$,

$$\sigma_s(x)_q \leq \frac{1}{s^{\left(\frac{1}{p} - \frac{1}{q}\right)}} \cdot \|x\|_p.$$

(Will use this t.s.t. $m = 2s$ is nec. for recovering all s-sparse vecs. from $y = Ax$)

when $p = 1$, $q = 2$

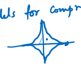
$$\sigma_s(x)_2 \leq \frac{1}{\sqrt{s}} \|x\|_1$$

when $q = 2$, $p < 2$,

$1 \quad \|x\|_p$

$$\sigma_s(x)_2 \leq \frac{\quad}{s^{\frac{1}{p}-\frac{1}{2}}} \quad {}_p$$

$\Rightarrow$ Unit balls in the ℓ_p quasi-norm for $p \leq 1$ are good models for compressible vectors.



Proof: Let $x^* \in \mathbb{R}_+^N$ be a noninc. rearrangement of $x \in \mathbb{C}^N$. Then,

$$\sigma_s(x)_q^q = \sum_{j=s+1}^{N} (x_j^*)^q \leq \underbrace{\underbrace{(x_s^*)^{q-p}} \sum_{j=s+1}^{N} (x_j^*)^p}_{RHS}$$

$$(x_j^*)^q = (x_j^*)^{q-p} (x_j^*)^p$$

$$\leq (x_s^*)^{q-p} (x_j^*)^p \qquad j \geq s$$

$$RHS \leq \Big(\underbrace{\frac{1}{s} \sum_{j=1}^{s} (x_j^*)^p} \Big)^{\frac{q-p}{p}} \sum_{j=s+1}^{N} (x_j^*)^p .$$

$\underbrace{(x_j^*)^p, \ j = 1, 2, \ldots s}$ are all $\geq (x_s^*)^p$

avg. $\geq (x_s^*)^p$

$(\text{avg.})^{\frac{q-p}{p}} \geq (x_s^*)^{q-p}$

$$\leq \left(\frac{1}{s} \|x\|_p^p \right)^{\frac{q-p}{p}} \cdot \|x\|_p^p \qquad \text{(Just adds more nneg terms)}$$

$$= \left(\frac{1}{s}\right)^{\frac{q}{p}-1} \cdot \|x\|_p^{q-p} \cdot \|x\|_p^p$$

Thus,

$$\sigma_s(x)_q^q \leq \frac{1}{s^{\frac{q}{p}-1}} \|x\|_p^q$$

$$\sigma_s(x)_q \leq \frac{1}{s^{\frac{1}{p}-\frac{1}{q}}} \cdot \|x\|_p \qquad \square$$