E9 203 Compressed Sensing & Sparse SP

Last time:

$[N] = \{1, 2, \dots, N\}$

$|S| = \text{Card}\, S = \#$ elems. in set S

$\bar{S} = S^c = [N] \setminus S$

$x \in \mathbb{C}^N$, $\text{supp}(x) = \{ j \in [N],\ x_j \neq 0 \}$

$x \in \mathbb{C}^N$ is *s-sparse* if $\|x\|_0 = |\text{supp}(x)| \leq s$.

For $p > 0$, $\boxed{\sigma_s(x)_p \triangleq \inf \{ \|x - z\|_p,\ z \in \mathbb{C}^N,\ z \text{ is } s\text{-sparse} \}}$

- l_p error in the best s-sparse approx of x
- inf achieved by an s-sparse $z \in \mathbb{C}^N$ whose nonzero entries equal the s-largest absolute entries of x
- z need not be unique, but the optimum z is indep. of p.

Prop. For any $q > p > 0$ and any $x \in \mathbb{C}^N$

$$\sigma_s(x)_q \leq \frac{1}{s^{\frac{1}{p} - \frac{1}{q}}} \|x\|_p .$$

E.g., when $p = 1$ and $q = 2$,

$$\underbrace{\sigma_s(x)_2}_{l_2 \text{ error in the best } s\text{-term approx.}} \leq \frac{1}{\sqrt{s}} \underbrace{\|x\|_1}_{l_1\text{-norm of } x \text{ itself.}}$$

Can tighten the above bound:

Thm. [2.5] For any $q > p > 0$ and any $x \in \mathbb{C}^N$, the inequality

$$\sigma_s(x)_q \leq \frac{c_{p,q}}{s^{\frac{1}{p} - \frac{1}{q}}} \|x\|_p \qquad \left[\text{Earlier: } \sigma_s(x)_q \leq \frac{1}{s^{\frac{1}{p} - \frac{1}{q}}} \|x\|_p \right]$$

holds with

$$c_{p,q} \triangleq \left[\left(\frac{p}{q}\right)^{p/q} \left(1 - \frac{p}{q}\right)^{\left(1 - \frac{p}{q}\right)} \right]^{\frac{1}{p}} \leq 1 .$$

$p/q < 1$

Remark: when $p = 1$, $q = 2$, we get

$$\sigma_s(x)_2 \leq \frac{1}{2\sqrt{s}} \|x\|_1 .$$

(Equality when $x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{2s \times 1}$: $\sigma_s(x)_2 = \sqrt{s}$, $\|x\|_1 = 2s$.)

Proof: Let x^* be a noninc. rearrangement of $x \in \mathbb{C}^N$.

Let $\alpha_j \triangleq (x_j^*)^p$, $r \triangleq q/p > 1$. $\qquad (x_j^*)^q = (\underbrace{x_j^{*p}}_{\alpha_j})^{q/p}$

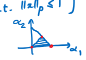
Then, $(\sigma_s(x)_q)^q = \sum_{j=s+1}^{N} (x_j^*)^q = \sum_{j=s+1}^{N} \alpha_j^r$

Define $f(\alpha_1, \dots, \alpha_N) = \sum_{j=s+1}^{N} \alpha_j^r$ $\qquad$ Convex fn. of $\alpha_1 \cdots \alpha_N$ because $r > 1$. [Check!]

Thus, $(\sigma_s(x)_q)^q \leq \max\limits_{\alpha_1 \cdots \alpha_N} f(\alpha_1, \dots, \alpha_N)$

$$\boxed{\begin{matrix} \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N \geq 0 \\ \alpha_1 + \alpha_2 + \cdots + \alpha_N \leq 1 \end{matrix}}$$

[Sufficient to show the ineq. for all x s.t. $\|x\|_p \leq 1$]



The constraint set is a convex set.
- In fact, it is a polygon

The max. of a convex fn. over a convex set occurs at an extreme point. $\Rightarrow$ Vertex of the polygon.

Vertices are obtained: set N out of the $N+1$ ineq. to $=$.

① $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$, $f(\alpha_1 \cdots \alpha_N) = 0$.

② $\alpha_1 = \alpha_2 = \cdots = \alpha_k > \alpha_{k+1} = \cdots = \alpha_N = 0$, $1 \leq k \leq s$

$f(\alpha_1 \cdots \alpha_N) = 0$.

③ $\alpha_1 = \alpha_2 = \cdots = \underbrace{\alpha_k}_{s+1 \cdots k} > \alpha_{k+1} = \cdots = \alpha_N = 0$, $k > s$.

Max. when $\alpha_1 + \cdots + \alpha_N = \alpha_1 + \cdots + \alpha_k = k\alpha_1 = 1$

$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1/k \Rightarrow f(\alpha_1 \cdots \alpha_N) = (k - s) \frac{1}{k^r}$.

$$\max_{\langle \text{polygon} \rangle} f(\alpha_1 \cdots \alpha_N) = \max_{s+1 \leq k \leq N} \left(\frac{k - s}{k^r} \right).$$

Can upper bound the RHS by letting k be continuous valued $\Rightarrow k^* = \left(\frac{r}{r-1}\right) s$.

$$\max_{\langle \text{polygon} \rangle} f(\alpha_1 \cdots \alpha_N) \leq \frac{k^* - s}{(k^*)^r} = \frac{\left(\frac{r}{r-1} - 1\right) s}{\left(\frac{r}{r-1}\right)^r s^r}$$

$$\text{RHS} = \left(\frac{1}{r-1}\right) \left(1 - \frac{1}{r}\right)^r \frac{1}{s^{r-1}} = \frac{1}{r} \left(1 - \frac{1}{r}\right)^{r-1} \frac{1}{s^{r-1}} .$$

$$= c_{p,q}^q \cdot \frac{1}{s^{\left(\frac{q}{p} - 1\right)}} .$$ The result follows. ☐

Minimal # measurements

$$\underset{\mathbb{C}^m}{y} = \underset{\mathbb{C}^{m \times N}}{A}\ \underset{\mathbb{C}^N,\ s\text{-sparse}}{x} \qquad m < N$$

Q. How small can m be?

Two interpretations:

(a) All s-sparse x must be recoverable
- Fix A, draw all possible s-sparse x, and want recover all such x from $y = Ax$.

(b) Given x, A allows for its unique recovery from y.
- Fix x. Consider $\underbrace{\{ z \in \mathbb{C}^N \text{ s.t. } Az = y \}}_{\text{should have } x \text{ as its unique } s\text{-sparse element.}}$, where $y = Ax$.

A. $m \geq 2s$ for recovery of all s-sparse vecs.

$m \geq s+1$ for recovery of a given s-sparse vector.

Note that the foll. props. are equivalent:

(i) x is the unique s-sparse soln. of $Az = y$ with $y = Ax$. That is,

$\{ z \in \mathbb{C}^N : Az = Ax, \ \|z\|_0 \le s \} = \{x\}$

(ii) x is the unique soln. of

P_0 $\left[\min_{z \in \mathbb{C}^N} \|z\|_0 \ \text{ s.t. } \ Az = y. \right]$

Note that if any $z \in \mathcal{N}(A)$ has at least $2s+1$ nonzero entries, (*) then if $Ax = y$ has a soln. with at most s nonzero entries, it is necessarily the unique sparsest soln.

$\Longrightarrow$ Every s-sparse x_0 is uniquely recoverable by solving $\min_{x} \|x\|_0$ s.t. $Ax = y$. P_0.

1. $x \longrightarrow Ax$: mapping from $\mathbb{R}^N \to \mathbb{R}^m$

When (*) is satisfied, the mapping $S \to Y$

where $S = \{ x \in \mathbb{R}^N, \ \|x\|_0 \le s \}$

$Y = \{ y \in \mathbb{R}^m, \ y = Ax \text{ for some } x \in S \}$

is one-to-one but not onto.

2. Cannot infer whether (*) is satisfied using simple rank properties.

Suppose all subsets of m cols. of A are LI

$\Rightarrow$ Any $x' \in \mathcal{N}(A)$ has at least $m+1$ nonzeros.

If $y = Ax$ has a soln. with $\le \left\lfloor \frac{m+1}{2} \right\rfloor$ nonzero entries, it is the unique sparsest soln.