E9 203 Compressed Sensing & Sparse Signal Processing

05 Mar. 2021

Last time:

1. For any $q > p > 0$ and any $x \in \mathbb{C}^N$, the inequality

$$\sigma_s(x)_q \leq \frac{c_{p,q}}{s^{\frac{1}{p}-\frac{1}{q}}} \|x\|_p$$

holds, with $c_{p,q} \triangleq \left[\left(\frac{p}{q}\right)^{\frac{p}{q}} \left(1 - \frac{p}{q}\right)^{\left(1-\frac{p}{q}\right)} \right]^{\frac{1}{p}} \leq 1.$

2. Minimal # meas.

$\boxed{y = Ax}$, $y \in \mathbb{C}^m$, $A \in \mathbb{C}^{m \times N}$, $x \in \mathbb{C}^N$ and s-sparse, $m < N$.

a. Recover all s-sparse vecs.

b. Given x, $\exists A$ s.t. x is the unique soln. to (P_0)

(P_0) : $\min\limits_{z \in \mathbb{C}^N} \|z\|_0$ s.t. $Az = y$ (where $y = Ax$).

3. Property ⊛: Any $z \in \mathcal{N}(A)$ has at least $2s+1$ nonzero entries.

When ⊛ holds, if $Ax = b$ has a soln. with at most s nonzero entries, it is necessarily the unique sparsest soln.

- Every s-sparse vec. x_0 is uniquely recoverable by solving (P_0).
- The mapping $S \triangleq \{x \in \mathbb{C}^N, \|x\|_0 \leq s\} \longrightarrow Y \triangleq \{y \in \mathbb{C}^m; \ y = Ax \text{ for some } x \in S\}$ is one-to-one but not onto (injective but not surjective).
- Cannot infer whether ⊛ is satisfied using simple rank properties.

Ex. $\underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\underline{b}}$

Any 2 cols are LI $\Rightarrow$ any $x' \in \mathcal{N}(A)$ must have 3 nonzero entries. $\left\lfloor \frac{3}{2} \right\rfloor = 1 \Rightarrow A$ can uniquely recover 1-sparse vecs.

Suppose $\underline{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ unique sparsest soln.

Suppose $\underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow$ No 1-sparse soln.

$\Rightarrow$ Not all $\underline{b} \in \mathbb{R}^m$ will have a corresp. 1-sparse x s.t. $Ax = \underline{b}$.

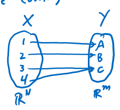
- Different 1-sparse vecs map to different $\underline{b}$s
- There are $\underline{b}$ s.t. $\underline{b} \neq Ax$ for any 1-sparse x.

Set theory:

$x \longrightarrow Ax$ $\quad x \in \mathbb{R}^N$ to $Ax \in \mathbb{R}^m$. $\quad m < N$.

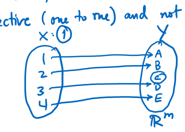
suppose A has rank m.

The mapping is not one-to-one (non-injective) and surjective (onto):

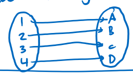


The mapping from $x \in \{\tilde{x} \in \mathbb{R}^N, \|\tilde{x}\|_0 \leq s\}$ to y in $\mathbb{R}^m$ via $y = Ax$.

[If ⊛ is satisfied (every $z \in \mathcal{N}(A)$ has at least $2s+1$ nonzeros)] is injective (one to one) and not surjective (onto)



The mapping from $\{x \in \mathbb{R}^N, x$ is a rank-m subspace of $\mathbb{R}^N\}$ (and rank$(A) = m$) to Ax is injective and surjective (1-1 and onto)



Recovery of all s-sparse vecs.

$A \in \mathbb{C}^{m \times N}$, $S \subset [N]$

$A_S \triangleq$ Column submatrix of A (cols. indexed by S) $m \times |S|$.

$x \in \mathbb{C}^N$

x_S can mean
- $\in \mathbb{C}^{|S|}$ shortened vec.
- $\in \mathbb{C}^N$, $(x_S)_\ell = x_\ell$ for $\ell \in S$ and 0 otherwise.

Thm. Given $A \in \mathbb{C}^{m \times N}$, the foll. are equivalent:

(a) Every s-sparse $x \in \mathbb{C}^N$ is the unique s-sparse soln. of $Az = Ax$.

(b) The null space $\mathcal{N}(A)$ does not contain any $2s$-sparse vecs. except $\{\underline{0}\}$

$$\mathcal{N}(A) \cap \{z \in \mathbb{C}^N : \|z\|_0 \leq 2s\} = \{\underline{0}\}.$$

(c) For every $S \subset [N]$, $|S| \leq 2s$,

A_S is injective as a map from $\mathbb{C}^{|S|} \to \mathbb{C}^m$

(d) Every set of $2s$ cols of A is LI.

Thus, if it is possible to reconstruct every s-sparse vec $x \in \mathbb{C}^N$ from $y = Ax \in \mathbb{C}^m$, then (a) holds $\Leftrightarrow$ (d) holds. This $\Rightarrow$ rank$(A) \geq 2s$

rank$(A) \leq m \Rightarrow m \geq 2s$.

Thus $m \geq 2s$ is necessary. But is it sufficient?

Thm. [2.14]: For any $N \geq 2s$, $\exists A \in \mathbb{C}^{m \times N}$ with $m = 2s$

s.t. every s-sparse $x \in \mathbb{C}^N$ can be recovered from $y = Ax$ by solving (P_0).

Proof: Fix $t_N > t_{N-1} > \cdots > t_2 > t_1 > 0$ and consider

$$A = \begin{bmatrix} 1 & 1 & & 1 \\ t_1 & t_2 & \cdots & t_N \\ \vdots & \vdots & & \vdots \\ t_1^{2s-1} & t_2^{2s-1} & \cdots & t_N^{2s-1} \end{bmatrix} \in \mathbb{C}^{m \times N}$$

Let $S = \{ j_1 < \cdots < j_{2s} \}$ be an index set, $|S| = 2s$.

Then $A_S \in \mathbb{C}^{2s \times 2s}$ is $(\text{Vandermonde matrix})^T$:

$$\det A_S = \det \begin{bmatrix} 1 & \cdots & 1 \\ t_{j_1} & & t_{j_{2s}} \\ \vdots & & \vdots \\ t_{j_1}^{2s-1} & & t_{j_{2s}}^{2s-1} \end{bmatrix} = \prod_{\substack{k < l \\ k, l = 1, 2, \dots, 2s}} (t_{j_l} - t_{j_k}) > 0.$$

A_S is nonsingular $\forall S \Rightarrow A_S$ is injective, i.e., (c) of the prev. thm. holds.

$\Rightarrow$ Every s-sparse x is the unique s-sparse soln. to $Az = Ax$, so it can be uniquely recovered by solving (P_0). $\boxtimes$

$$(P_0): \min \|x\|_0 \quad \text{s.t.} \quad y = \underset{m \times N}{A} x, \quad m < N$$

For a genie that can solve (P_0) to succeed for all possible s-sparse vecs in $\mathbb{C}^N$, $m \geq 2s$ is necessary and sufficient.