E9 203 Compressed Sensing & Sparse SP

09 Apr. 2021.

Last time:

- Basis pursuit:

$$\boxed{(P_1): \min_{z \in \mathbb{C}^N} \|z\|_1 \text{ s.t. } Az = y.}$$

- NSP (s): $\|v_S\|_1 < \|v_{\overline{S}}\|_1 \quad \forall\, v \in \mathcal{N}(A)\setminus\{\underline{0}\}$
- Result: $x \in \mathbb{C}^N$ supported on S is the unique soln. of (P_1) with $y = Ax$ iff A satisfies NSP(S).
- Stability: $\mathrm{SNSP}_\rho(S)$: $\|v_S\|_1 \le \rho \|v_{\overline{S}}\|_1 \quad \forall\, v \in \mathcal{N}(A)$ $(\rho \in (0,1))$.

  $\mathrm{SNSP}_\rho(s)$: $\mathrm{SNSP}_\rho(S)\ \forall\, S \subset [N]$ with $|S| \le s$.

Thm. $A \in \mathbb{C}^{m \times N}$ satisfies the stable NSP with const. $0 < \rho < 1$ relative to S iff

$$\boxed{(*) \quad \|z - x\|_1 \le \frac{(1+\rho)}{(1-\rho)} \left(\|z\|_1 - \|x\|_1 + 2\|x_{\overline{S}}\|_1 \right) \quad \forall\, x, z \in \mathbb{C}^N \text{ with } Az = Ax.}$$

Let S = idx. set corresp. s largest mag. coeffs in x

$\Rightarrow \|x_{\overline{S}}\|_1 = \sigma_s(x)_1$.

Let $x^\#$ be the minimizer of (P_1) w/ constraint $Az = Ax$

Then, $\|x^\#\|_1 \le \|x\|_1$, $Ax^\# = Ax$.

$\Rightarrow$ If A sat. stable NSP of const $\rho \in (0,1)$ rel. to S, and if the above thm holds, $\underline{x}^\#$ is a "candidate" z

$$\Rightarrow \|x^\# - x\|_1 \le \left(\frac{1+\rho}{1-\rho}\right) \Big(\underbrace{\|x^\#\|_1 - \|x\|_1}_{-\mathrm{ve}} + 2 \underbrace{\|x_{\overline{S}}\|_1}_{\sigma_s(x)_1} \Big)$$

$$\Rightarrow \boxed{\|x^\# - x\|_1 \le 2\left(\frac{1+\rho}{1-\rho}\right)\sigma_s(x)_1}$$

Thus, if A sat. SNSP, recovery is "stable", i.e., its error dep. on $\sigma_s(x)_1$ & ρ.

Today: - We prove the above thm.
- Robust NSP

In order to prove the thm., need the foll. Lemma:

Lemma. Given $S \subset [N]$, $x, z \in \mathbb{C}^N$

$$\|(x-z)_{\overline{S}}\|_1 \le \|z\|_1 - \|x\|_1 + \|(x-z)_S\|_1 + 2\|x_{\overline{S}}\|_1$$

Proof: $\|x\|_1 = \|x_{\overline{S}}\|_1 + \|x_S\|_1$

$\le \|x_{\overline{S}}\|_1 + \|(x-z)_S\|_1 + \|z_S\|_1$

$\|(x-z)_{\overline{S}}\|_1 \le \|x_{\overline{S}}\|_1 + \|z_{\overline{S}}\|_1$

Adding,

$$\|(x-z)_{\overline{S}}\|_1 \le \|z\|_1 - \|x\|_1 + \|(x-z)_S\|_1 + 2\|x_{\overline{S}}\|_1.$$

∎

Proof of the theorem:

$$\|v_S\|_1 \le \rho \|v_{\overline{S}}\|_1 \quad \forall\, v \in \mathcal{N}(A)$$

$\Leftrightarrow$

$$\left[\begin{array}{l} \|z - x\|_1 \le \dfrac{(1+\rho)}{(1-\rho)} \left(\|z\|_1 - \|x\|_1 + 2\|x_{\overline{S}}\|_1 \right) \quad \text{—}(*) \\ \forall\, x, z \in \mathbb{C}^N \text{ with } Az = Ax. \end{array}\right.$$

$\boxed{\Leftarrow}$ Suppose A satisfies $(*)$ $\forall\, x, z \in \mathbb{C}^N$ with $Az = Ax$

Given $v \in \mathcal{N}(A)$, $A(v_S + v_{\overline{S}}) = 0$

$$A\underbrace{v_S}_{x} = -Av_{\overline{S}} \qquad z = -v_{\overline{S}}$$

Using $(*)$

$$\|v\|_1 \le \frac{(1+\rho)}{(1-\rho)} \left(\|v_{\overline{S}}\|_1 - \|v_S\|_1 \right)$$

$$(1-\rho)\left(\|v_S\|_1 + \|v_{\overline{S}}\|_1\right) \le (1+\rho)\left(\|v_{\overline{S}}\|_1 - \|v_S\|_1\right)$$

$$[(1-\rho) + (1+\rho)]\|v_S\|_1 \le [(1+\rho) - (1-\rho)]\|v_{\overline{S}}\|_1$$

$$\|v_S\|_1 \le \rho \|v_{\overline{S}}\|_1$$

i.e., the stable NSP relative to set S is satisfied.

$\boxed{\Rightarrow}$ Suppose A sat. SNSP(S) w/ const $\rho \in (0,1)$

Consider $x, z \in \mathbb{C}^N$ with $Az = Ax$.

$v = z - x \in \mathcal{N}(A)$.

$\mathrm{SNSP}_\rho(S) \Rightarrow \|v_S\|_1 \le \rho \|v_{\overline{S}}\|_1$

From the Lemma,

$$\|v_{\overline{S}}\|_1 \le \|z\|_1 - \|x\|_1 + \|v_S\|_1 + 2\|x_{\overline{S}}\|_1$$

$$\|v_{\overline{S}}\|_1 \le \|z\|_1 - \|x\|_1 + \rho\|v_{\overline{S}}\|_1 + 2\|x_{\overline{S}}\|_1$$

Since $\rho < 1$,

$$\|v_{\overline{S}}\|_1 \le \frac{1}{1-\rho}\left(\|z\|_1 - \|x\|_1 + 2\|x_{\overline{S}}\|_1 \right)$$

$$\|v\|_1 = \|v_S\|_1 + \|v_{\overline{S}}\|_1 \le (1+\rho)\|v_{\overline{S}}\|_1$$

$$\Rightarrow \|v\|_1 \le \left(\frac{1+\rho}{1-\rho}\right)\left(\|z\|_1 - \|x\|_1 + 2\|x_{\overline{S}}\|_1 \right)$$

as desired. ∎

Remark: $A \in \mathbb{C}^{m \times N}$, consider $S \subset [N]$, $|S| = s$.

Define an operator R_S on $\underline{v \in \mathcal{N}(A)}$ by

$$R_S(v) = v_S.$$

NSP $\Rightarrow$ $\|v_S\|_1 < \|v_{\overline{S}}\|_1 \quad \forall\, v \in \mathcal{N}(A)\setminus\{\underline{0}\}$.

$\mu \triangleq \max\{\|R_S\|_{1\to 1} : S \subset [N], |S| = s\} < \frac{1}{2}$.

$$\|R_S\|_{1\to1} = \max_{v \in \mathcal{N}(A)\setminus\{0\}} \frac{\|R_S(v)\|_1}{\|v\|_1}$$

$$\frac{\|v_S\|}{\|v_S\|_1 + \|v_{\overline{S}}\|_1} < \frac{1}{2}$$

$\Rightarrow$ A satisfies the SNSP with const. $\rho \triangleq \frac{\mu}{1-\mu} < 1$.

Thus, stability of BP comes for free if A satisfies NSP(s). However, if μ is close to $\frac{1}{2}$, ρ is close to 1, and hence, the const. $\frac{2(1+\rho)}{(1-\rho)}$ can be large. [Rem. 4.16].

Noisy measurements:

$$y \approx Ax, \quad \|Ax - y\| \le \eta$$

Some norm, e.g. ℓ_2 norm. some $\eta \ge 0$.

We will consider the foll. variant of (P_1):

$$(P_{1,\eta}): \min_{z \in \mathbb{C}^N} \|z\|_1 \text{ s.t. } \|Az - y\| \le \eta.$$

Defn: $A \in \mathbb{C}^{m \times N}$ satisfies the robust NSP w.r.t. $\|\cdot\|$ with constants $0 < \rho < 1$ and $\tau > 0$ relative to set S if

$$\|v_S\|_1 \leq \rho \|v_{\bar{S}}\|_1 + \tau \|Av\| \quad \forall v \in \mathbb{C}^N.$$

It is said to satisfy RNSP of order s (RNSP(s)) with consts. $0 < \rho < 1$ and $\tau > 0$ if it satisfies RNSP (S) with consts. ρ, τ relative to any set $S \subset [N]$, $|S| \leq s$.

Remark: Do not require $v \in \mathcal{N}(A)$.

Thm. [4.19]: Suppose A satisfies RNSP(s) with consts $0 < \rho < 1$ and $\tau > 0$. Then, for any $x \in \mathbb{C}^N$, a soln $x^\#$ of $(P_{1,\eta})$ with $y = Ax + e$ and $\|e\| \leq \eta$ approximates x with ℓ_1 error

$$\boxed{\|x - x^\#\|_1 \leq 2\left(\frac{1+\rho}{1-\rho}\right)\sigma_s(x)_1 + \frac{4\tau}{1-\rho}\eta}.$$

Thm. [4.20]: $A \in \mathbb{C}^{m \times N}$ satisfies RNSP (S) with consts. $0 < \rho < 1$ and $\tau > 0$ iff

$$\|z - x\|_1 \leq \left(\frac{1+\rho}{1-\rho}\right)\left(\|z_S\|_1 - \|x\|_1 + 2\|x_{\bar{S}}\|_1\right) + \frac{2\tau}{1-\rho}\|A(z-x)\|$$

$\forall\, x, z \in \mathbb{C}^N$.

[Don't need $Az = Ax$].

See text for the proof of Thm. 4.19, and show that thm. 4.20 follows from Thm. 4.19.