E9 203 Compressed Sensing & Sparse Signal Processing

12 Apr. 2021

Last time: $\boxed{y = Ax + e, \quad \|e\| \le \eta}$

Robust recovery (noisy measurements)

$$(P_{1,\eta}): \min_{z \in \mathbb{C}^N} \|z\|_1 \quad \text{s.t.} \quad \|Az - y\| \le \eta.$$

Some norm; $\eta > 0$.

Defn. (Robust NSP):

$A \in \mathbb{C}^{m\times N}$ satisfies the robust NSP w.r.t. $\|\cdot\|$ with consts $0<\rho<1$ and $\tau>0$ relative to set S if

$$\|v_S\|_1 \le \rho \|v_{\bar{S}}\|_1 + \tau \|Av\| \quad \forall v \in \mathbb{C}^N.$$

- RNSP of order s if … relative to any $S \subset [N]$, $|S| \le s$.

Thm. 4.19

Suppose A sat. RNSP(s) of consts $0<\rho<1$ & $\tau>0$. Then, for any $x \in \mathbb{C}^N$, a soln. $x^\#$ of $(P_{1,\eta})$ with $y = Ax+e$ and $\|e\| \le \eta$ approximates x with ℓ_1 error:

$$\boxed{\|x - x^\#\|_1 \le 2\left(\frac{1+\rho}{1-\rho}\right)\sigma_s(x)_1 + \frac{4\tau}{1-\rho}\eta.}$$

The proof of the thm. follows from the foll. result:

Thm. 4.20

$A \in \mathbb{C}^{m\times N}$ sat. RNSP rel. to set S w/ consts. $0<\rho<1$ and $\tau>0$ iff

$$\|z-x\|_1 \le \frac{1+\rho}{1-\rho}\left(\|z_S\|_1 - \|x\|_1 + 2\|x_{\bar{S}}\|_1\right) + \frac{2\tau}{1-\rho}\|A(z-x)\|$$

$\forall x, z \in \mathbb{C}^N$. $\boxed{\text{Proof: Exercise}}$

Today: A second main result on robust recovery

Defn. Given $q \ge 1$, $A \in \mathbb{C}^{m\times N}$ is said to satisfy the ℓ_q-robust NSP of order s w.r.t. $\|\cdot\|$ with consts. $0<\rho<1$ and $\tau>0$ if $\forall$ set $S \subset [N]$ with $|S| \le s$,

$$\|v_S\|_q \le \frac{\rho}{s^{(1-1/q)}}\|v_{\bar{S}}\|_1 + \tau\|Av\| \quad \forall v \in \mathbb{C}^N.$$

Previous:

$$\|v_S\|_1 \le \rho\|v_{\bar{S}}\|_1 + \tau\|Av\| \quad \forall v \in \mathbb{C}^N.$$

Can s.t. $1 \le p \le q$,

$$\boxed{\|v_S\|_p \le \left(s^{\frac{1}{p}-\frac{1}{q}}\right)\|v_S\|_q} \qquad \text{[HW]}$$

As a consequence,

$$\|v_S\|_p \le \frac{\rho}{s^{1-1/p}}\|v_{\bar{S}}\|_1 + \tau s^{\left(\frac{1}{p}-\frac{1}{q}\right)}\|Av\| \quad \forall v \in \mathbb{C}^N.$$

Thus, ℓ_q-RNSP implies the previous RNSP, modulo a change of $\|\cdot\| \leftarrow s^{\frac{1}{p}-\frac{1}{q}}\|\cdot\|$. Further, for $1 \le p \le q$, ℓ_q-RNSP $\Rightarrow$ ℓ_p-RNSP and the ℓ_q-RNSP is a stronger requirement than the prev. RNSP.

Thm. 4.22

Suppose $A \in \mathbb{C}^{m\times N}$ satisfies the ℓ_2 RNSP of order s, with consts. $0<\rho<1$ and $\tau>0$. Then, for any $x \in \mathbb{C}^N$ a soln $x^\#$ of $(P_{1,\eta})$ with $\|\cdot\| = \|\cdot\|_2$, $y = Ax+e$, $\|e\|_2 \le \eta$ approximates the vec. x with ℓ_p error

$$\boxed{\|x - x^\#\|_p \le \frac{C}{s^{1-\frac{1}{p}}}\sigma_s(x)_1 + D\left(s^{\frac{1}{p}-\frac{1}{2}}\right)\eta},$$

$1 \le p \le 2$, for some consts. $C, D > 0$ depending only on ρ and τ.

The above thm. is a conseq. of:

Thm. 4.25

Given $1 \le p \le q$, suppose $A \in \mathbb{C}^{m\times N}$ satisfies ℓ_q-RNSP of order s with consts. $0<\rho<1$ & $\tau>0$. Then, for any $x, z \in \mathbb{C}^N$

$$\|z-x\|_p \le \frac{C}{s^{1-\frac{1}{p}}}\left(\|z\|_1 - \|x\|_1 + 2\sigma_s(x)_1\right) + D s^{\frac{1}{p}-\frac{1}{q}}\|A(z-x)\|$$

where $C = \frac{(1+\rho)^2}{1-\rho}$ and $D = \left(\frac{3+\rho}{1-\rho}\right)\tau$.

Remark 1: For $p=1$ & $p=2$, the error bound is

$$\left(\|x - x^\#\|_p \le \frac{C}{s^{1-1/p}}\sigma_s(x)_1 + D\left(s^{\frac{1}{p}-\frac{1}{2}}\right)\eta\right)$$

$$\|x - x^\#\|_1 \le C\sigma_s(x)_1 + D\sqrt{s}\,\eta$$

$$\|x - x^\#\|_2 \le \frac{C}{\sqrt{s}}\sigma_s(x)_1 + D\eta.$$

Remark 2: Note that, regardless of the ℓ_p-space in which the error is estimated (evaluated), $\sigma_s(x)_1$ appears on the RHS. Why does it not involve the ℓ_p, $p>1$ error in approximating x by an s-sparse vec? Turns out (see Chap. 11) that such a bound is impossible in the (m, N) of interest in CS.

Also, recall Thm. 2.5: For any $q > p > 0$ and any $x \in \mathbb{C}^N$, $\sigma_s(x)_q \le \frac{c_{pq}}{s^{\frac{1}{p}-\frac{1}{q}}}\|x\|_p \leftarrow$

with $c_{pq} \triangleq \left[\left(\frac{p}{q}\right)^{\frac{p}{q}}\left(1-\frac{p}{q}\right)^{\left(1-\frac{p}{q}\right)}\right]^{\frac{1}{p}} \le 1.$

The unit ball B_q^N in ℓ_q norm, $q<1$ is a good model for compressible vecs.

$\Rightarrow$ if $\|x\|_q \le 1$ for $q<1$, and if $p \ge 1$,

Thm. 2.5 reduces to

$$(*) \quad \boxed{\sigma_s(x)_p \le s^{\frac{1}{p}-\frac{1}{q}}} \qquad (-\text{ve power of } s)$$

When $\eta = 0$, the bound in Thm. 4.22 becomes

$$\|x - x^\#\|_p \le \boxed{\frac{C}{s^{1-\frac{1}{p}}}\boxed{\sigma_s(x)_1}}$$

$$\leq \frac{C}{s^{1-1/p}}\, s^{1-\frac{1}{q}} \quad \text{[Using (*) w/ } p=1)$$

$$= \underline{C\, s^{\frac{1}{p}-\frac{1}{q}}}, \quad 1 \leq p \leq 2.$$

Thus, the decay rate of $\|x - x^{\#}\|_p$ is the same as the decay rate of the best s-term approx of x in the ℓ_p-norm. Hence, $\frac{\sigma_s(x)_1}{s^{1-\frac{1}{p}}}$ is not significantly worse than $\sigma_s(x)_p$.

Remark 3: ℓ_q RNSP:

$$\left(\|v_S\|_q \leq \frac{\rho}{s^{1-1/q}} \|v_{\bar{S}}\|_1 + \tau \|Av\| \quad \forall v \in \mathbb{C}^N \right)$$

Bounds of the above form are necessary to obtain bounds of the form

$$\|x - x^{\#}\|_q \leq \frac{C}{s^{1-\frac{1}{q}}} \sigma_s(x)_1 + D\eta \quad \text{—(**)}$$

where $x^{\#}$ is the minimizer of $(P_{1,\eta})$ with $y = Ax + e$, and $\|e\| \leq \eta$. In fact, given $v \in \mathbb{C}^N$, $S \subset [N]$, $|S| \leq s$, using (**) with $x = v$, $e = -Av$ & $\eta = \|Av\|$, so that $x^{\#} = 0$ (solves $(P_{1,\eta})$)

$$\|v\|_q \leq \frac{C}{s^{1-\frac{1}{q}}} \|v_{\bar{S}}\|_1 + D\|Av\|$$

$$\|v_S\|_q \leq \frac{C}{s^{1-1/q}} \|v_{\bar{S}}\|_1 + D\|Av\|$$

So the ℓ_q-RNSP requirement is directly related to the type of bounds we have derived.