E9 203 Compressed Sensing & Sparse SP

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Last time:

- Coherence: $A \in \mathbb{C}^{m \times N}$, ℓ_2 normalized cols

$$\mu = \mu(A) = \max_{1 \le i \ne j \le N} |\langle a_i, a_j \rangle|$$

- ℓ_1-coherence: $s \in [N-1]$,

$$\mu_1(s) = \max_{i \in [N]} \max \left\{ \sum_{j \in S} |\langle a_i, a_j \rangle|, S \subset [N], |S| = s, i \notin S \right\}$$

$$= \max_{\substack{S \subset [N] \\ |S| = s}} \max_{j \notin S} \sum_{i \in S} |\langle a_i, a_j \rangle|.$$

- Thm. 5.3. $\forall$ s-sparse $x \in \mathbb{C}^N$,

$$(1-\mu_1(s-1))\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\mu_1(s-1))\|x\|_2^2$$

If $\mu_1(s-1) < 1$, $A_S^H A_S$ is invertible for each $S \subset [N], |S| \le s$.

- Cor. 5.4. If $\mu_1(s) + \mu_1(s-1) < 1$, for each set $S \subset [N]$, $|S| \le 2s$, $A_S^H A_S$ is invertible & A_S is injective.
- Thm. 5.7. $\boxed{\mu \geq \sqrt{\frac{(N-m)}{m(N-1)}}}$.
- Also, $\mu_1(s) \geq s \sqrt{\frac{N-m}{m(N-1)}}$ for $s \le \sqrt{N-1}$.
- Spark $(A) \triangleq$ Smallest k s.t. A has a set of k LD cols

(Rank(A) = Largest k s.t. some set of k cols of A is LI).

$$\text{Spark}(A) = \min_{x \ne 0} \|x\|_0 \quad \text{s.t. } Ax = 0.$$

(If all cols of A are LI, spark $(A) \triangleq \infty$.)

- Today:
  - Properties of spark
  - Coherence-based guarantees for sparse recovery.

$A \in \mathbb{C}^{m \times N}$, $N \geq m$. Properties of Spark:

(a) Spark$(A) = m+1 \Rightarrow$ rank$(A) = m$

(b) Spark$(A) = 1$ iff A has an all zero col.

(c) spark $(A) \le$ rank$(A)+1 \le m+1$

(d) If $Ax = b$ has a soln. $x^\#$ s.t. $\|x^\#\|_0 < \frac{\text{spark}(A)}{2}$, then $x^\#$ is the sparsest possible soln.

(e) Spark $(A) \geq 1 + \frac{1}{\mu(A)}$, $\mu(A) = \max_{i \ne j} |\langle a_i, a_j \rangle|$

(f) Related: Kruskal rank (K-rank) of a matrix

K-rank(A) = max. k s.t. any k cols. of A are LI
= Spark$(A) - 1$.

Lemma: $A \in \mathbb{C}^{m \times N}$, unit ℓ_2 norm cols.

$$\text{spark}(A) \geq 1 + \frac{1}{\mu(A)}.$$

Proof: Let $G = A^H A$, and recall

$$G_{ii} = 1, \quad i \in [N]$$
$$|G_{ij}| \le \mu(A), \quad i,j \in [N], i \ne j$$

Take an arbitrary set S of s cols of A.

$$G_S \triangleq A_S^H A_S.$$

Clearly, $\sum_{j \ne i} |G_{S_{ij}}| \le \mu(A)(s-1)$

If $\mu(A)(s-1) < |G_{S_{ii}}| = 1$, then G_S is strictly diagonally dominant, and by Gersgorin disc thm., G_S is p.d. $\Rightarrow$ the corresp. s cols of A are LI.

Thus, if $s < 1 + \frac{1}{\mu(A)}$, then any s cols of A are LI. $\Rightarrow$ min. # LD cols of A is $\geq 1 + \frac{1}{\mu(A)}$.

$$\Rightarrow \text{Spark}(A) \geq 1 + \frac{1}{\mu(A)}. \quad \square$$

Thm. (Uniqueness via mutual coherence):

If $Ax = y$ has a soln. x satisfying $\|x\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right)$, then this soln. is necessarily the sparsest soln.

Compare with uniqueness via spark:

[If $Ax = y$ has a soln. x satisfying $\|x\|_0 < \frac{\text{spark}(A)}{2}$, then it is necessarily the sparsest soln.

$\mu \gtrsim \frac{1}{\sqrt{m}} \Rightarrow$ the bound in the thm. can never be larger than $\frac{\sqrt{m}}{2}$. Spark, on the other hand, can $= m+1$.

So, uniqueness via mutual coh. is much weaker than uniqueness via spark.

Similarly, uniqueness via ℓ_1-coherence:

If $\mu_1(s) < 1$, then all $(s+1)$ subsets of cols. of A are LI. Hence,

$$\text{spark}(A) \geq \min_{1 \le s \le m} \{ s \mid \mu_1(s-1) \geq 1 \}$$

Thm. (Uniqueness via ℓ_1-coherence):

If $Ax = y$ has a soln. x s.t.

$$\|x\|_0 < \frac{1}{2}\left[\min_{1 \le s \le m} \{ s \mid \mu_1(s-1) \geq 1 \} \right]$$

then x is necessarily the sparsest soln.

Now, since $\mu \geq \sqrt{\frac{N-m}{m(N-1)}}$

$$\mu_1(s) \geq s\sqrt{\frac{N-m}{m(N-1)}}, \quad s \le \sqrt{N-1},$$

$\Rightarrow$ For large N, the lower bound on $\mu \sim \frac{1}{\sqrt{m}}$, $\mu_1(s) \sim \frac{s}{\sqrt{m}}$

$$\mu_1(s) < 1 \Rightarrow \boxed{s < o(\sqrt{m})}$$

" Quadratic bottleneck".

Analysis of OMP via ℓ_1-coherence:

Thm. $A \in \mathbb{C}^{m \times N}$, ℓ_2-normalized cols.

If $\mu_1(s) + \mu_1(s-1) < 1$, then every s-sparse $x \in \mathbb{C}^N$ is exactly recovered from $y = Ax$ after at most s iterations of OMP.

Proof: Let $a_1, a_2, \ldots, a_N$ denote the cols of A, ℓ_2 normalized ($\|a_i\|_2 = 1$, $i \in [N]$.)

[Recall Prop. 3.5 :

Given $A \in \mathbb{C}^{m \times N}$, every $0 \neq x \in \mathbb{C}^N$, $\mathrm{supp}(x) = S$, $|S| \le s$, is successfully recovered from $y = Ax$ after at most s iterations of OMP iff

A_S is injective and

$$\max_{j \in S} |(A^H r)_j| > \max_{\ell \in \bar{S}} |(A^H r)_\ell|$$

$\forall\ 0 \neq r \in \{ Az,\ \mathrm{supp}(z) \subset S \}$.]

Let $r \triangleq \sum_{i \in S} r_i a_i$, let $k \in S$ s.t.

$$|r_k| = \max_{i \in S} |r_i| > 0 \quad \because r \neq 0.$$

For $\ell \in \bar{S}$

$$\begin{aligned} |\langle r, a_\ell \rangle| &= \Big| \sum_{i \in S} r_i \langle a_i, a_\ell \rangle \Big| \\ &\le \sum_{i \in S} |r_i| \, |\langle a_i, a_\ell \rangle| \\ &\le |r_k| \, \mu_1(s) \end{aligned}$$

On the other hand,

$$\begin{aligned} |\langle r, a_k \rangle| &= \Big| \sum_{i \in S} r_i \langle a_i, a_k \rangle \Big| \\ &\ge |r_k| \, |\langle a_k, a_k \rangle| - \sum_{\substack{i \in S \\ i \neq k}} |r_i| \, |\langle a_i, a_k \rangle| \\ &\ge |r_k| - |r_k| \mu_1(s-1). \end{aligned}$$

Thus, if $\mu_1(s) + \mu_1(s-1) < 1$, then

$$\max_{j \in S} |\langle r, a_j \rangle| > \max_{\ell \in \bar{S}} |\langle r, a_\ell \rangle|$$

is satisfied. The injectivity of A_S follows from Cor. 5.4. $\square$