E9 203 Compressed Sensing & Sparse SP.
26 Apr. 2021

Last time:
- Mutual coherence & spark
- Uniqueness via coherence & ℓ_1-coherence
- Analysis of OMP via coherence & ℓ_1-coherence

Today:
- Analysis of BP via coherence & ℓ_1-coherence
- Analysis of thresholding algos.

Analysis of Basis Pursuit

Any condition guaranteeing success of all v : $\mathrm{supp}(v) \subset S$ using OMP with $|S|$ iterations also guarantees success of BP for all vecs supported on S.

This is ∵

$$\boxed{\max_{j \in S} |(A^H r)_j| > \max_{\ell \in \bar{S}} |(A^H r)_\ell| \quad \forall\, 0 \neq r \in \{Az : \mathrm{supp}(z) \subseteq S\}}$$

implies the NSP.

Given $v \in \mathcal{N}(A) \setminus \{0\}$, we have

$$A_S v_S = -A_{\bar{S}} v_{\bar{S}}$$

$$\Rightarrow \|v_S\|_1 = \|A_S^\dagger A_S v_S\|_1 = \|A_S^\dagger A_{\bar{S}} v_{\bar{S}}\|_1$$

$$\leq \underbrace{\|A_S^\dagger A_{\bar{S}}\|_{1\to 1}}_{<1 \text{ [Exercise]}} \|v_{\bar{S}}\|_1 < \|v_{\bar{S}}\|_1.$$

$$\left(\|B\|_{1\to 1} = \max_{z \neq 0} \frac{\|Bz\|_1}{\|z\|_1} \right)$$

Thus, we get the following result:

Thm. 5.15 $A \in \mathbb{C}^{m \times N}$, ℓ_2 normalized cols.
If $\mu_1(s) + \mu_1(s-1) < 1$, then every s-sparse $x \in \mathbb{C}^N$ is exactly recovered from $y = Ax$ via BP.

Proof: [Thm. 4.5]: nec. & suff. t.p.t. A satisfies $NSP(s)$, i.e.,

$$\|v_S\|_1 < \|v_{\bar{S}}\|_1 \quad \forall\, 0 \neq v \in \mathcal{N}(A) \text{ and any } S \subset [N],\ |S| = s.$$

Let $a_1, \ldots, a_N$ denote the cols of A.

$v \in \mathcal{N}(A) \Rightarrow \sum_{j=1}^{N} v_j a_j = 0.$

Taking the inner product with a_i, $i \in S$,

$$v_i = v_i \langle a_i, a_i \rangle = -\sum_{\substack{j=1 \\ j \neq i}}^{N} v_j \langle a_j, a_i \rangle$$

$$= -\sum_{\ell \in \bar{S}} v_\ell \langle a_\ell, a_i \rangle - \sum_{\substack{j \in S \\ j \neq i}} v_j \langle a_j, a_i \rangle.$$

$$|v_i| \leq \sum_{\ell \in \bar{S}} |v_\ell| |\langle a_\ell, a_i \rangle| + \sum_{\substack{j \in S \\ j \neq i}} |v_j| |\langle a_j, a_i \rangle|$$

Sum over all $i \in S$, interchange summations:

$$\|v_S\|_1 = \sum_{i \in S} |v_i| \leq \sum_{\ell \in \bar{S}} |v_\ell| \sum_{i \in S} |\langle a_\ell, a_i \rangle| + \sum_{j \in S} |v_j| \sum_{\substack{i \in S \\ i \neq j}} |\langle a_j, a_i \rangle|$$

$$\leq \sum_{\ell \in \bar{S}} |v_\ell| \mu_1(s) + \sum_{j \in S} |v_j| \mu_1(s-1)$$

$$= \mu_1(s) \|v_{\bar{S}}\|_1 + \mu_1(s-1) \|v_S\|_1$$

$$\Rightarrow (1 - \mu_1(s-1)) \|v_S\|_1 \leq \mu_1(s) \|v_{\bar{S}}\|_1$$

Thus, if $1 - \mu_1(s-1) > \mu_1(s)$,
$\|v_S\|_1 < \|v_{\bar{S}}\|_1$ is satisfied. ☑

We have seen that

$$\mu_1(s) \geq s \sqrt{\frac{N-m}{m(N-1)}}, \quad s \leq \sqrt{N-1}$$

So if $\mu_1(s) + \mu_1(s-1) < 1$ with $\boxed{m \leq \frac{(2s-1)^2}{2}}$
and $s < \sqrt{N-1}$, and if $N \geq 2m$, we must have

$$1 > \mu_1(s) + \mu_1(s-1) \geq (2s-1) \sqrt{\frac{N-m}{m(N-1)}}$$

$$\geq \sqrt{\frac{2(N-m)}{N-1}} \geq \sqrt{\frac{N}{N-1}}, \text{ a contradiction.}$$

$\boxed{(2s-1) \geq \sqrt{2m}}$

$\boxed{m \geq c s^2 \text{ is a fundamental bottleneck with coherence based bounds.}}$

Analysis of thresholding based algorithms

Basic thresholding:

$$S^\# = L_s(A^H y)$$

$$x^\# = \arg\min_{z \in \mathbb{C}^N} \{ \|y - Az\|_2,\ \mathrm{supp}(z) \subseteq S^\# \}$$

Prop. 3.7: $x \in \mathbb{C}^N$ is recoverable via BT iff

$$\min_{j \in S} |(A^H y)_j| > \max_{\ell \in \bar{S}} |(A^H y)_\ell|.$$

Thm. 5.16 $A \in \mathbb{C}^{m \times N}$, ℓ_2 normalized cols.
$x \in \mathbb{C}^N$, $\mathrm{supp}(x) = S$, $|S| = s$.

If $\boxed{\mu_1(s) + \mu_1(s-1) < \frac{\min_{i \in S} |x_i|}{\max_{i \in S} |x_i|}}$,

then x is exactly recovered via BT.

Proof: $a_1, \ldots, a_N$: cols of A, $\|a_i\|_2 = 1$.
Using Prop. 3.7, we need t.s.t. for any $j \in S$, $\ell \in \bar{S}$,

$$|\langle Ax, a_j \rangle| > |\langle Ax, a_\ell \rangle|$$

$$|\langle Ax, a_\ell \rangle| = \left| \sum_{i \in S} x_i \langle a_i, a_\ell \rangle \right|$$

$$\leq \sum_{i \in S} |x_i| |\langle a_i, a_\ell \rangle| \leq \left(\max_{i \in S} |x_i| \right) \cdot \mu_1(s)$$

$$|\langle Ax, a_j \rangle| = \left| \sum_{i \in S} x_i \langle a_i, a_j \rangle \right|$$

$$\geq |x_j| - \sum_{\substack{i \in S \\ i \neq j}} |x_i| |\langle a_i, a_j \rangle|$$

$$\geq \min_{i \in S} |x_i| - \left(\max_{i \in S} |x_i| \right) \mu_1(s-1)$$

Thus, when $\mu_1(s) + \mu_1(s-1) < \frac{\min_{i \in S} |x_i|}{\max_{i \in S} |x_i|}$,

$$|\langle Ax, a_j \rangle| - |\langle Ax, a_\ell \rangle|$$

$$\geq \min_{i \in S} |x_i| - (\mu_1(s) + \mu_1(s-1)) \max_{i \in S} |x_i| > 0.$$ ☑

Analysis of iterative hard thresholding: See Ex. 5.10 [HW]

Analysis of hard thresholding pursuit:

HTP: Init: $x^0 = 0$, $S^0 = \phi$
Iterate:
[HTP_1]: $S^{n+1} = L_s(x^n + A^H(y - Ax^n))$
[HTP_2]: $x^{n+1} = \arg\min_{z \in \mathbb{C}^N} \{ \|y - Az\|_2,\ \mathrm{supp}(z) \subseteq S^{n+1} \}$
Until STOP.

Thm. 5.17 $A \in \mathbb{C}^{m \times N}$, ℓ_2 normalized cols.
If $2\mu_1(s) + \mu_1(s-1) < 1$, every s-sparse $x \in \mathbb{C}^N$ is exactly recovered from $y = Ax$ after at most s iterations of hard thresholding pursuit.

Proof: Consider indices $j_1, j_2, \ldots, j_N$ s.t.
$|x_{j_1}| \geq |x_{j_2}| \geq \cdots \geq |x_{j_s}| > |x_{j_{s+1}}| = \cdots = |x_{j_N}| = 0.$
Will s.t. for $0 \leq n \leq s-1$, the set $\{j_1, \ldots, j_{n+1}\}$ is included in S^{n+1} defined by [HTP_1] with $y = Ax$ as the largest abs. entries of

[HTP_1]: $S^{n+1} = L_s(x^n + A^H(y - Ax^n))$

$$z^{n+1} \triangleq x^n + A^H(y - Ax^n) = x^n + A^H A(x - x^n) \quad \text{—①}$$

(This will ⇒ $S^s = S = \mathrm{supp}(x)$ and hence $x^s = x$ by [HTP_2].)

For this, it is sufficient t.p.t

$$\min_{1 \leq k \leq n+1} |z_{j_k}^{n+1}| > \max_{\ell \in \bar{S}} |z_\ell^{n+1}|. \quad \text{—②}$$

Note that, $\forall\, j \in [N]$,

$$z_j^{n+1} = x_j^n + \sum_{i=1}^{N} (x_i - x_i^n) \langle a_i, a_j \rangle$$

$$= x_j + \sum_{i \neq j} (x_i - x_i^n) \langle a_i, a_j \rangle \qquad (\because \langle a_i, a_i \rangle = 1)$$

$$\therefore\ |z_j^{n+1} - x_j| \leq \sum_{\substack{i \in S^n \\ i \neq j}} |x_i - x_i^n| \, |\langle a_i, a_j \rangle| + \sum_{\substack{i \in S \setminus S^n \\ i \neq j}} |x_i| \cdot |\langle a_i, a_j \rangle| \quad \text{—③}$$

Now, for $1 \leq k \leq n+1$ and $\ell \in \bar{S}$

$$\begin{cases} |z_{j_k}^{n+1}| \geq |x_{j_k}| - \mu_1(s) \, \|(x - x^n)_{S^n}\|_\infty - \mu_1(s) \, \|x_{S \setminus S^n}\|_\infty & \text{—④} \\ |z_\ell^{n+1}| \leq \mu_1(s) \, \|(x - x^n)_{S^n}\|_\infty + \mu_1(s) \, \|x_{S \setminus S^n}\|_\infty & \text{—⑤} \end{cases}$$

In particular, for $n = 0$, $\|(x - x^n)_{S^n}\|_\infty = 0 \Rightarrow$ ④ & ⑤ give

$$|z_{j_1}^1| \geq (1 - \mu_1(s)) \, \|x\|_\infty > \mu_1(s) \, \|x\|_\infty \geq |z_\ell^1| \quad \forall\, \ell \in \bar{S}$$

$\because\ 2\mu_1(s) < 1$. Therefore, the base case of the induction hypothesis in ② holds for $n = 0$. Assume that the hypothesis holds for $n-1$, with $n \geq 1$.

$$x^n = \underset{z \in \mathbb{C}^N}{\arg\min} \left\{ \|y - Az\|_2,\ \operatorname{supp}(z) \subset S^n \right\}$$

$\Rightarrow \{j_1, \ldots, j_n\} \subset S^n$.

Now, [HTP_2] with n replaced by $(n-1)$ gives (using Lemma 3.4)

$$\left(A^H A (x - x^n) \right)_{S^n} = 0.$$

Hence, for any $j \in S^n$, defn. ① of $z^{n+1} \Rightarrow \boxed{z_j^{n+1} = x_j^n}$,

$$\left(z^{n+1} \triangleq x^n + A^H A (x - x^n) \right)$$

then ③ $\Rightarrow$

$$\underbrace{|x_j^n - x_j|}_{z_j^{n+1}} \leq \underbrace{\mu_1(s-1)}_{j \in S^n \Rightarrow 1^{st} \text{ sum in ③ has at most } (s-1) \text{ terms}} \|(x - x^n)_{S^n}\|_\infty + \underbrace{\mu_1(s-1)}_{\substack{\text{The first index picked } \in S \\ \text{and by the induction hyp.,} \\ \text{at most } (s-1) \text{ terms in the} \\ \text{second term of ③.}}} \|x_{S \setminus S^n}\|_\infty$$

Taking max over $j \in S^n$ and rearranging $\Rightarrow$

$$\|(x - x^n)_{S^n}\|_\infty \leq \frac{\mu_1(s-1)}{(1 - \mu_1(s-1))} \|x_{S \setminus S^n}\|_\infty .$$

Substituting ④ & ⑤ $\Rightarrow$ for $1 \leq k \leq n+1$ and $\ell \in \bar{S}$

$$\left. \begin{aligned} |z_{j_k}^{n+1}| &\geq \left(1 - \frac{\mu_1(s)}{1 - \mu_1(s)} \right) |x_{j_{n+1}}| \\ |z_\ell^{n+1}| &\leq \frac{\mu_1(s)}{1 - \mu_1(s)} |x_{j_{n+1}}| . \end{aligned} \right\} \quad \text{Verify! } \|x_{S \setminus S^n}\|_\infty = |x_{j_{n+1}}|$$

Since $\boxed{\frac{\mu_1(s)}{1 - \mu_1(s)} < \frac{1}{2}}$, this $\Rightarrow$ ② holds for n too.

This completes the proof by induction. $\boxtimes$

$$\left(\min_{1 \leq k \leq n+1} |z_{j_k}^{n+1}| > \max_{\ell \in \bar{S}} |z_\ell^{n+1}| \right)$$