E9 203 Compressed Sensing & Sparse SP

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Last time:

- Analysis of BP via RIC

  $\delta_{2s} < \frac{1}{3} \Rightarrow$ BP successful.

- Thm 6.12 $A \in \mathbb{C}^{m \times N}$. If $\delta_{2s} < \frac{4}{\sqrt{41}} \approx 0.6246$, then for any $x \in \mathbb{C}^N$ and $y \in \mathbb{C}^m$ with $\|Ax - y\|_2 \le \eta$, a soln $x^\#$ of

$$\min_{z \in \mathbb{C}^N} \|z\|_1 \quad \text{s.t.} \quad \|Az - y\|_2 \le \eta$$

approximates x with errors

$$\|x - x^\#\|_1 \le C\, \sigma_s(x)_1 + D\sqrt{s}\, \eta$$

$$\|x - x^\#\|_2 \le \frac{C}{\sqrt{s}}\, \sigma_s(x)_1 + D\eta$$

where consts. $C, D > 0$ depend only on δ_{2s}.

Proof: Exercise. It is an easy consequence of:

Thm. 6.13 If the $2s^{th}$ RIC of $A \in \mathbb{C}^{m \times N}$ satisfies

$$\delta_{2s} < \frac{4}{\sqrt{41}} \approx 0.6246,$$

then A satisfies the ℓ_2 RNSP of order s with constants $0 < \rho < 1$ and $\tau > 0$ depending only on δ_{2s}.

Exercise: Read the proof of Thm. 6.13 in the text.

Today: - Analysis of thresholding based algos

- (Time permitting) Analysis of OMP.

Remark: Recall our prev. result on the success of BP (P_1):

(Thm. 4.5): Given $A \in \mathbb{C}^{m \times N}$, every s-sparse $x \in \mathbb{C}^N$ is the unique sol. of (P_1) $\left[\min_z \|z\|_1 \text{ s.t. } Az = y\right]$ with $y = Ax$ iff A satisfies NSP(s).

$\Rightarrow$ Sparse recovery is preserved if the measurements are reshuffled, rescaled or added.

However: these operations may deteriorate the RIC!

Reshuffling preserves RIC: $\delta_s(UA) = \delta_s(A)$ for any unitary $U \in \mathbb{C}^{m \times m}$.

But adding a measurement or rescaling may $\uparrow$ RIC!

This is one limitation of RIC based analysis.

$\rightarrow$ See discussion in the text

Analysis of Thresholding based Algos

Recall IHT: Start with $x^0 \in \mathbb{C}^N$ (typically $= 0$).

Generate the sequence x^n:

$$x^{n+1} = H_s\left(x^n + A^H(y - Ax^n)\right)$$

H_s: retains the s largest abs. entries.

Intuitively, δ_s is small, $\|Ax\|_2^2 \approx \|x\|_2^2$ for s-sparse x $\Rightarrow$ A is "like" an orthonormal matrix.

$$\Rightarrow A^H(y - Ax^n) = A^H A(x - x^n) \approx x - x^n.$$

$$\Rightarrow H_s(x^n + A^H(y - Ax^n)) \approx H_s(x^n + x - x^n) = H_s(x) = x.$$

Thm 6.15 Suppose the $(3s)^{th}$ RIC of $A \in \mathbb{C}^{m \times N}$ satisfies $\boxed{\delta_{3s} < \frac{1}{2}}$. Then, for any s-sparse $x \in \mathbb{C}^N$, the seq. x^n defined by the IHT algo with $y = Ax$ converges to x.

To prove the thm., need:

Lemma 6.16 Given $u, v \in \mathbb{C}^N$ and $S \subset [N]$

$$\boxed{|\langle u, (I - A^H A)v\rangle| \le \delta_t \|u\|_2 \|v\|_2 \quad \text{if } |\mathrm{supp}(u) \cup \mathrm{supp}(v)| \le t}$$

$$\boxed{\|((I - A^H A)v)_S\|_2 \le \delta_t \|v\|_2 \quad \text{if } |S \cup \mathrm{supp}(v)| \le t.}$$

Proof: Let $T \triangleq \mathrm{supp}(u) \cup \mathrm{supp}(v)$. Then,

$$\begin{aligned}
|\langle u, (I - A^H A)v\rangle| &= |\langle u, v\rangle - \langle Au, Av\rangle| \\
&= |\langle u_T, v_T\rangle - \langle A_T u_T, A_T v_T\rangle| \\
&= |\langle u_T, (I - A_T^H A_T) v_T\rangle| \\
&\le \|u_T\|_2 \|(I - A_T^H A_T) v_T\|_2 \\
&\le \|u_T\|_2 \underbrace{\|I - A_T^H A_T\|_{2\to 2}} \|v_T\|_2 \\
&\le \delta_t \|u_T\|_2 \|v_T\|_2 = \delta_t \|u\|_2 \|v\|_2.
\end{aligned}$$

For the second inequality,

$$\|((I - A^H A)v)_S\|_2^2 = \langle \underbrace{((I - A^H A)v)_S}, (I - A^H A)v\rangle$$

Take as u in the first ineq.:

$$\Big(|\langle u, (I - A^H A)v\rangle| \le \delta_t \|u\|_2 \|v\|_2 \quad \text{if } |\underbrace{\mathrm{supp}(u)}_{=S} \cup \mathrm{supp}(v)| \le t\Big)$$

$$\le \delta_t \|((I - A^H A)v)_S\|_2 \|v\|_2.$$

Dividing by $\|((I - A^H A)v)_S\|_2$ completes the proof $\square$.

Proof of Thm. 6.15

Suffices to find a const. $0 \le \rho < 1$ s.t.

$$\|x^{n+1} - x\|_2 \le \rho \|x^n - x\|_2, \quad n \ge 0$$

since, by induction, this $\Rightarrow$

$$\|x^n - x\|_2 \le \rho^n \|x^0 - x\|_2 \longrightarrow 0 \text{ as } n \to \infty.$$

$$\Big(x^{n+1} = H_s\big(\underbrace{x^n + A^H(y - Ax^n)}_{\triangleq u^n}\big)\Big)$$

By defn., the s-sparse vec. x^{n+1} is at least as good an approx. to

$$u^n = x^n + A^H A(x - x^n)$$

than the s-sparse vec x.

$$\|u^n - x^{n+1}\|_2^2 \le \|u^n - x\|_2^2$$

$$\begin{aligned}
\|u^n - x^{n+1}\|_2^2 &= \|(u^n - x) + (x - x^{n+1})\|_2^2 \\
&= \|(u^n - x) - (x^{n+1} - x)\|_2^2 \\
&= \|u^n - x\|_2^2 + \|x^{n+1} - x\|_2^2 - 2\,\mathrm{Re}\langle u^n - x, x^{n+1} - x\rangle
\end{aligned}$$

$$\Rightarrow \|x^{n+1} - x\|_2^2 \le 2\,\mathrm{Re}\langle u^n - x, x^{n+1} - x\rangle$$

Using Lemma 6.16, $u^n - x = x^n - x + A^H A(x - x^n) = (I - A^H A)(x^n - x)$

$$\mathrm{Re}\langle u^n - x, x^{n+1} - x\rangle = \mathrm{Re}\left(\langle (I - A^H A)(x^n - x), x^{n+1} - x\rangle\right)$$

(x^n, x, x^{n+1} are all s-sparse, so the union of their support is of size at most $3s$)

$$\le \delta_{3s} \|x^n - x\|_2 \|x^{n+1} - x\|_2$$

If $\|x^{n+1} - x\|_2 = 0$, nothing to prove. If $\|x^{n+1} - x\|_2 > 0$, divide by it to get

$$\|x^{n+1} - x\|_2 \le 2\delta_{3s} \|x^n - x\|_2$$

$\underbrace{}_{<1} \Rightarrow$ the result follows. $\blacksquare$

<u>Remark:</u> $\delta_{3s} \leq 2\delta_{2s} + \delta_s \leq 3\delta_{2s}$.

If $\delta_{2s} < 1/6$, it is guaranteed that $\delta_{3s} < \frac{1}{2}$, so IHT will succeed.

Related: See Ex. 6.19, 6.20, 6.21 in the text.

Next time: Analysis of HTP via RIC.

<u>Recall HTP:</u>

Start with $x^0 \in \mathbb{C}^N$

$$S^{n+1} = L_s\left(x^n + A^H(y - Ax^n)\right)$$

$\leftarrow$ Idx set of s largest abs. entries

$$x^{n+1} = \underset{z \in \mathbb{C}^N}{\arg\min} \left\{ \|y - Az\|_2 ,\ \operatorname{supp}(z) \subseteq S^{n+1} \right\}.$$

<u>Thm. 6.18</u> Suppose δ_{3s} of $A \in \mathbb{C}^{m \times N}$ satisfies

$$\delta_{3s} < \frac{1}{\sqrt{3}} \approx 0.5773.$$

Then, for $\underline{x} \in \mathbb{C}^N$, $\underline{e} \in \mathbb{C}^m$, $S \subset [N]$ with $|S| = s$, the seq. x^n defined by IHT or HTP with $y = Ax + e$ satisfies, for any $n \geq 0$,

$$\|x^n - x_S\|_2 \leq \rho^n \|x^0 - x_S\|_2 + \tau \|Ax_{\bar{S}} + e\|_2$$

where

$$\rho = \sqrt{3}\,\delta_{3s} < 1, \quad \tau \leq \frac{2.18}{1-\rho} \quad \text{for IHT}$$

$$\rho = \sqrt{\frac{2\delta_{3s}^2}{1-\delta_{3s}^2}} < 1, \quad \tau \leq \frac{5.15}{1-\rho} \quad \text{for HTP}.$$

$$\sqrt{\frac{2\delta_{3s}^2}{1-\delta_{3s}^2}} < \underline{\underline{\sqrt{3}\,\delta_{3s}}}$$

Proof: See text.