E9 203 Compressed Sensing & Sparse SP

10 May 2021

Last time:
- Analysis of thresholding algos via RIP

Today:
- Analysis of greedy algos via RIP

Analysis of Greedy Algos (OMP, CoSaMP).

See example in text: RIC based conditions are insufficient to establish recovery of all s-sparse vecs. in at most s iterations of OMP.

Three ways to proceed:

1. Modify OMP to go past s iterations, so that hopefully it figures out the correct support.
2. Modify OMP to reject incorrect indices (CoSaMP)
3. Make the bound on δ_s (or δ_{s+1}, etc) dependent on s.

Recall OMP:

Init: S^0 ; $x^0 = \arg\min \{ \|y - Az\|_2, \ \mathrm{supp}(z) \subset S^0 \}$

Iterate:

(Pick idx corresp. largest mag. entry)

(OMP$_1$) $\quad S^{n+1} = S^n \cup L_1 \{ A^H (y - Ax^n) \}$

(OMP$_2$) $\quad x^{n+1} = \arg\min \{ \|y - Az\|_2, \ \mathrm{supp}(z) \subset S^{n+1} \}$

For a finite # of steps.

Prop. 6.24 $A \in \mathbb{C}^{m \times N}$. $y = Ax + e$, x is s-sparse $S = \mathrm{supp}(x)$. Let (x^n) denote the seq. defined by OMP with init. S^0. Let $s^0 = |S^0|$, $s' = |S \setminus S^0|$. If $\delta_{s+s^0+12s'} < \frac{1}{6}$, then there is a const. $C > 0$ depending only on $\delta_{s+s^0+12s'}$ s.t.

$$\| y - Ax^{12s'} \|_2 \leq C \|e\|_2 .$$

Note: If $e = 0$, $S^0 = \emptyset$, the prop $\Rightarrow$ exact recovery of s-sparse vecs. in $12s$ iterations.

This is because: $\|A(x - x^{12s})\|_2 = 0$

$\Rightarrow A(x - x^{12s}) = 0$

$\Rightarrow x - x^{12s} = 0$ because $\|x - x^{12s}\|_0 \leq 13s$

and $\delta_{13s} < 1 \Rightarrow (1 - \delta_{13s}) \|x - x^{12s}\|_2^2 \leq \|A(x - x^{12s})\|_2^2 = 0$
$\leq (1 + \delta_{13s}) \|x - x^{12s}\|_2^2$

$\Rightarrow x = x^{12s}$.

Thm. 6.25 $A \in \mathbb{C}^{m \times N}$, $\delta_{13s} < \frac{1}{6}$. Then, $\exists\, C > 0$ depending only on δ_{13s} s.t. $\forall x \in \mathbb{C}^N$ & $e \in \mathbb{C}^m$, the seq. (x^n) defined by OMP with $y = Ax + e$ satisfies $\|y - Ax^{12s}\|_2 \leq C \|A x_{\bar{S}} + e\|_2$

where $S \subset [N]$, $|S| = s$.

Furthermore, if $\delta_{26s} < \frac{1}{6}$, then $\exists\, C, D > 0$ that depend only on δ_{26s} s.t. $\forall x \in \mathbb{C}^N$, $e \in \mathbb{C}^m$, the seq. (x^n) defined by OMP with $y = Ax + e$ satisfies, for any $1 \leq p \leq 2$,

$$\boxed{\| x - x^{24s} \|_p \leq \frac{C}{s^{1-\frac{1}{p}}} \sigma_s(x)_1 + D s^{\frac{1}{p} - \frac{1}{2}} \|e\|_2}.$$

Compressive Sampling Matching Pursuit (CoSaMP):

Init: $x^0 = 0$

Repeat $\quad U^{n+1} = \mathrm{supp}(x^n) \cup L_{2s}(A^H(y - Ax^n))$

$\quad u^{n+1} = \arg\min \{ \|y - Az\|_2, \ \mathrm{supp}(z) \subset U^{n+1} \}$

$\quad x^{n+1} = H_s(u^{n+1})$.

For a number of steps.

Thm. 6.27 Suppose A satisfies

$$\delta_{4s} < \frac{\sqrt{\sqrt{\frac{11}{3}} - 1}}{2} \approx 0.4782.$$

Then, for $x \in \mathbb{C}^N$, $e \in \mathbb{C}^m$, $S \subset [N]$, $|S| = s$, the seq. (x^n) defined by CoSaMP with $y = Ax + e$ satisfies

$$\|x^n - x_S\|_2 \leq \rho^n \|x^0 - x_S\|_2 + \tau \|A x_{\bar{S}} + e\|_2$$

where $0 < \rho < 1$ and $\tau > 0$ depend only on δ_{4s}.

Remark: If x is s-sparse and $e = 0$, $\Rightarrow x$ is exactly recovered as $n \to \infty$.

Stability and Robustness:

Thm. 6.28 Suppose A satisfies $\delta_{8s} < 0.4782$. Then, for $x \in \mathbb{C}^N$, $e \in \mathbb{C}^m$, the seq. (x^n) defined by CoSaMP with $y = Ax + e$, $x^0 = 0$, and s replaced by $2s$ satisfies, for any $n \geq 0$,

$$\|x - x^n\|_1 \leq C \sigma_s(x)_1 + D\sqrt{s} \|e\|_2 + 2\rho^n \sqrt{s} \|x\|_2$$

$$\|x - x^n\|_2 \leq \frac{C}{\sqrt{s}} \sigma_s(x)_1 + D \|e\|_2 + 2\rho^n \|x\|_2$$

where $C, D > 0$ and $0 < \rho < 1$ dep. only on δ_{8s}.

(Wang & Shim, TSP Sep. 2012):

Thm. For any s-sparse $x \in \mathbb{C}^N$, OMP recovers x from $y = Ax$ with s iterations if

$$\delta_{s+1} < \frac{1}{\sqrt{s}+1}.$$

HW: Check that the above does NOT contradict the counter-example in the text.

To show that Gaussian matrices satisfy RIP

Lemma 1: $A \in \mathbb{R}^{m \times N}$, $A_{ij} \overset{\text{iid}}{\sim} \mathcal{N}(0, \frac{1}{m})$

Then, for any fixed $x \in \mathbb{R}^N$,

$$\Pr\left\{ \left| \|Ax\|_2^2 - \|x\|_2^2 \right| \geq \varepsilon \|x\|_2^2 \right\} \leq 2 e^{-m C_0(\varepsilon)}$$

$\forall \; 0 < \varepsilon < 1$, where $C_0(\varepsilon) = \frac{\varepsilon^2}{4} - \frac{\varepsilon^3}{6}$.

[$C_0(\varepsilon) = \frac{\varepsilon^2}{12}$ also works].

Remark: $\mathbb{E}\{\|Ax\|_2^2\} = \mathbb{E}\{x^T A^T A x\} = x^T I x = \|x\|_2^2$

Hence, $\mathrm{var}\{A_{ij}\} = \frac{1}{m}$ is important, as it preserves the length of x in the expectation sense.

Lemma 2 Let $\varepsilon \in (0,1)$ be given. Then, $\exists$ a set of points, $Q \subset \mathbb{R}^k$ s.t. $\|q\|_2 = 1 \; \forall q \in Q$,

$|Q| \leq \left(\frac{3}{\varepsilon}\right)^k$, and for any $x \in \mathbb{R}^k$, $\|x\|_2 = 1$,

$$\min_{q \in Q} \|x - q\|_2 \leq \varepsilon.$$

Remark: Puts an upper bound on the min # pts we need to take on the unit ℓ_2 ball s.t. any given point on the unit ℓ_2 ball is within ε of at least one of these points. ("ε-net").