E9 203 Compressed Sensing & Sparse SP

12 May 2021

Last time:

- Analysis of greedy algos via RIC.

Today:

- To show that Gaussian matrices satisfy RIP.

<u>Recap</u>: Two Lemmas

<u>Lemma 1</u>. $A \in \mathbb{R}^{m \times N}$, $A_{ij} \overset{iid}{\sim} \mathcal{N}(0, \frac{1}{m})$.

Then, for any <u>fixed</u> $x \in \mathbb{R}^N$,

$$\Pr\left\{ \left| \|Ax\|_2^2 - \|x\|_2^2 \right| \geq \varepsilon \|x\|_2^2 \right\} \leq 2 e^{-m c_0(\varepsilon)},$$

$\forall\ 0 < \varepsilon < 1$, where $c_0(\varepsilon) = \frac{\varepsilon^2}{4} - \frac{\varepsilon^3}{6}$.

[$c_0(\varepsilon) = \frac{\varepsilon^2}{12}$ also works].

<u>Remark</u>: $\mathbb{E}\{\|Ax\|_2^2\} = \|x\|_2^2$: $\mathrm{var}(A_{ij}) = \frac{1}{m}$ preserves the length of x in the <u>expected sense</u>.

<u>Lemma 2</u>. Let $\varepsilon \in (0,1)$ be given. Then, $\exists$ a set of points ("net of points") $Q \subset \mathbb{R}^k$ s.t. $\|q\|_2 = 1\ \forall\ q \in Q$, and $|Q| \leq \left(\frac{3}{\varepsilon}\right)^k$, and for any $x \in \mathbb{R}^k$, $\|x\|_2 = 1$, $\min\limits_{q \in Q} \|x - q\|_2 \leq \varepsilon$.

<u>Remark</u>: Upper bd. on the min. # pts we need to take on the unit l_2 ball s.t. an arbitrary pt on the unit l_2 ball is within ε of at least one of the pts.

<u>Thm</u>. Let $0 < \delta < 1$, $A \in \mathbb{R}^{m \times N}$, $A_{ij} \overset{iid}{\sim} \mathcal{N}(0, \frac{1}{m})$.

If $m \geq \kappa_1 k \log\left(\frac{N}{k}\right)$, Then A satisfies RIP of order k with the prescribed δ with prob. exceeding

$$\boxed{1 - \left[2 e^{-\kappa_2 m} \left(\frac{42 e}{\delta}\right)^k\right]}$$

where κ_1 is arbitrary and $\kappa_2 = \frac{\delta^2}{24} - \frac{1}{\kappa_1}$.

<u>Remark</u>: If $m = \kappa_1 k \log\left(\frac{N}{k}\right)$, the prob. becomes

$$1 - 2 \underbrace{e^{-\left(\frac{\delta^2}{24} - \frac{1}{\kappa_1}\right)\kappa_1 k \log\left(\frac{N}{k}\right)} \left(\frac{42e}{\delta}\right)^k}_{\left(\frac{N}{k}\right)^{-\left(\frac{\delta^2 \kappa_1}{24} - 1\right)k} \left(\frac{42e}{\delta}\right)^k}$$

$$\left(\frac{42e}{\delta} \cdot \frac{1}{(N/k)^{\left(\frac{\delta^2 \kappa_1}{24} - 1\right)}}\right)^k$$

<u>Remark</u>: By choosing κ_1 large enough, can ensure $\kappa_2 > 0$, and further, for large enough m, $e^{-\kappa_2 m} \left(\frac{42e}{\delta}\right)^k \ll 1$ holds.

<u>Proof</u>: Define $\Sigma_k = \{x \in \mathbb{R}^N, \|x\|_0 \leq k\}$.

Need t.s.t.

$$(1-\delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|_2^2 \quad \forall x \in \Sigma_k \quad \text{w.h.p}$$

(1) Suffices to consider $\|x\|_2 = 1$.

(2) Fix $T \subset [N]$, $|T| = k$, and let

$$\mathcal{X}_T \triangleq \{x \in \mathbb{R}^N, \mathrm{supp}(x) = T\} \subset \Sigma_k.$$

Choose a finite set of pts $Q_T \subset \mathcal{X}_T$ s.t. $\|q\|_2 = 1$ $\forall\ q \in Q_T$ and $\forall\ x \in \mathcal{X}_T$ with $\|x\|_2 = 1$,

$$\min_{q \in Q_T} \|x - q\|_2 \leq \frac{\delta}{14}.$$

Lemma 2 $\Rightarrow$ Can choose Q_T s.t. $|Q_T| \leq \left(\frac{42}{\delta}\right)^k$.

Repeat $\forall \binom{N}{k}$ choices for T, and collect Q_T together.

$$Q = \bigcup_{T: |T| = k} Q_T$$

$$\binom{N}{k} = \frac{N(N-1)\cdots(N-k+1)}{k!} \leq \frac{N^k}{k!} \leq \left(\frac{eN}{k}\right)^k.$$

$$\Rightarrow |Q| \leq \left(\frac{eN}{k}\right)^k \left(\frac{42}{\delta}\right)^k = \left(\frac{42eN}{k\delta}\right)^k.$$

<u>Lemma 1</u> $\Rightarrow$ For each $q \in Q$, $\forall\ 0 < \varepsilon < 1$,

$$\left.\begin{aligned} &\Pr\{(1-\varepsilon)\|q\|_2^2 \leq \|Aq\|_2^2 \leq (1+\varepsilon)\|q\|_2^2\} \\ &\quad \geq 1 - 2e^{-m c_0(\varepsilon)}. \end{aligned}\right\}$$

$\Rightarrow$ Taking the union bound,

$$\left.\begin{aligned} &(1-\varepsilon)\|q\|_2^2 \leq \|Aq\|_2^2 \leq (1+\varepsilon)\|q\|_2^2 \ \boxed{\forall}\ q \in Q \\ &\text{w.p.} \geq 1 - 2e^{-m c_0(\varepsilon)} \cdot \left(\frac{42eN}{k\delta}\right)^k. \end{aligned}\right\}$$

Let $\varepsilon = \delta/\sqrt{2}$.

Also note $\log\left[\left(\frac{42eN}{k\delta}\right)^k\right] \leq k\left[\log\left(\frac{N}{k}\right) + \log\frac{42e}{\delta}\right]$

$(\because m \geq \kappa_1 k \log \frac{N}{k})$ $\leq \frac{m}{\kappa_1} + k \log \frac{42e}{\delta}$.

$$1 - 2e^{-m c_0(\varepsilon)} \left(\frac{42eN}{k\delta}\right)^k \geq 1 - 2e^{-\frac{m\delta^2}{24}} \cdot e^{\frac{m}{\kappa_1} + k\log\left(\frac{42e}{\delta}\right)}$$

$(\because c_0(\varepsilon) = \varepsilon^2/12)$

$$\geq 1 - 2\left(\frac{42e}{\delta}\right)^k e^{-m\underbrace{\left(\frac{\delta^2}{24} - \frac{1}{\kappa_1}\right)}_{\kappa_2}}.$$

So we have the reqd. failure prob. But we need t.s.t. A satisfies RIP with the prescribed δ.

Let δ_k be the smallest # s.t.

$$\|Ax\|_2 \leq \sqrt{1+\delta_k}\, \|x\|_2 \quad \forall x \in \Sigma_k, \|x\|_2 \leq 1.$$

Need t.s.t. $\delta_k \leq \delta$.

Lemma 2 $\Rightarrow$ $\forall x \in \Sigma_k$, $\|x\|_2 = 1$, can pick $q \in Q$ s.t. $\|x - q\|_2 \leq \frac{\delta}{14}$.

In fact, $q \in \Sigma_k$ as well ($\because$ $\mathrm{supp}(x) = T$, then $x = x_T \Rightarrow$ can pick $q \in Q_T$ s.t. $\|x - q\|_2 \leq \frac{\delta}{14}$, $\mathrm{supp}(q) = T$ as well.)

$$
\begin{aligned}
\|Ax\|_2 &\le \|Aq\|_2 + \|A(x-q)\|_2 \qquad (\Delta\text{le}) \\
&\le \sqrt{1+\tfrac{\delta}{\sqrt{2}}}\,\|q\|_2 + \sqrt{1+\delta_k}\,\|x-q\|_2 \\
&\le \left(\sqrt{1+\tfrac{\delta}{\sqrt{2}}} + \sqrt{1+\delta_k}\,\tfrac{\delta}{14}\right)
\end{aligned}
$$

Since, by defn. δ_k is the smallest # s.t.

$$\|Ax\|_2 \le \sqrt{1+\delta_k}\,\|x\|_2 \quad \forall x \in \Sigma_k,$$

$$\sqrt{1+\delta_k} \le \sqrt{1+\tfrac{\delta}{\sqrt{2}}} + \sqrt{1+\delta_k}\,\tfrac{\delta}{14}.$$

$$\Rightarrow \sqrt{1+\delta_k} \le \frac{\sqrt{1+\frac{\delta}{\sqrt{2}}}}{1-\frac{\delta}{14}} \underset{\uparrow \text{HW}}{\le} \sqrt{1+\delta} \Rightarrow \delta_k \le \delta.$$

as desired.

(Why we chose $\frac{\delta}{14}$)

For the lower bound,

$$
\begin{aligned}
\|Ax\|_2 &\ge \|Aq\|_2 - \|A(q-x)\|_2 \\
&\ge \sqrt{1-\tfrac{\delta}{\sqrt{2}}} - \sqrt{1+\delta}\,\tfrac{\delta}{14}. \\
&\underset{\uparrow \text{HW}}{\ge} \sqrt{1-\delta}
\end{aligned}
$$

Thus, given $0<\delta<1$, $A \in \mathbb{R}^{m\times N}$, $A_{ij} \overset{iid}{\sim} \mathcal{N}(0,\frac{1}{m})$,

$\forall x \in \Sigma_k$, $(1-\delta)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\delta)\|x\|_2^2$

holds w.p. exceeding

$$1 - 2\left(\frac{42e}{\delta}\right)^k e^{-k_2 m}$$

where $k_2 = \frac{\delta^2}{24} - \frac{1}{k_1}$, provided $m \ge k_1 k \log\left(\frac{N}{k}\right)$.

$\boxtimes$