E9 203 Compressed Sensing & Sparse SP

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Recap:

Defn. The Gelfand m-width of a subset K of a normed space $\mathcal{X}$ (contained in $\mathbb{R}^N$) is

$$\boxed{d^m(K, \mathcal{X}) = \inf_{A \in \mathbb{R}^{m\times N}} \sup_{x \in K \cap \mathcal{N}(A)} \|x\|}$$

$\{d^m(K,\mathcal{X})\}_{m\geq 0}$ nonincreasing in m, $d^0(K,\mathcal{X}) = \sup_{x\in\mathcal{X}} \|x\|$.

Defn. The compressive m-width of a subset K of a normed space $\mathcal{X}$ is defined as

$$E^m(K,\mathcal{X}) \triangleq \inf \Big\{ \sup_{x\in K} \|x - \Delta(Ax)\|, \quad A: \mathcal{X} \to \mathbb{R}^m \text{ linear}, \ \Delta: \mathbb{R}^m \to \mathcal{X} \Big\}$$

Thm. 10.5 For $1 < p \leq 2$ and $m < N$, $\exists$ consts. $c_1, c_2 > 0$ depending only on p s.t.

$$c_1 \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}} \leq d^m(\underbrace{B_1^N}_{K}, \underbrace{\ell_p^N}_{\mathcal{X}}) \leq c_2 \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}}.$$

(Matching upper & lower bounds.)

Thm. 10.5 leads to:

Cor. 10.6 For $1 < p \leq 2$ and $m < N$, the compressive width satisfies

$$c_1' \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}} \leq E^m(B_1^N, \ell_p^N) \leq c_2' \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}}$$

for some consts. $c_1', c_2' > 0$.

From the lower estimate, can show:

Prop. 10.7 Let $1 < p \leq 2$. Suppose $\exists A \in \mathbb{R}^{m\times N}$ and $\Delta: \mathbb{R}^m \to \mathbb{R}^N$ s.t. $\forall x \in \mathbb{R}^N$,

$$\|x - \Delta(Ax)\|_p \leq \frac{C}{s^{1-1/p}} \sigma_s(x)_1.$$

Then, for some consts. $c_1, c_2 > 0$ depending only on C,

$$\boxed{m \geq c_1 s \ln\left(\frac{eN}{s}\right)}, \text{ provided } s > c_2.$$

As a conseq. of Prop. 10.7, we have the foll. result:

Cor. 10.8 If the $2s^{th}$ RIC of $A \in \mathbb{R}^{m\times N}$ satisfies $\delta_{2s} < 0.6246$ (say), then necessarily

$$m \geq c s \ln\left(\frac{eN}{s}\right)$$

for some const. $c > 0$ depending only on δ_{2s}.

Today: some proofs.

Cor. 10.6 For $1 < p \leq 2$ and $m < N$, the compressive width satisfies

$$c_1' \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}} \leq E^m(B_1^N, \ell_p^N) \leq c_2' \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}}$$

for some consts $c_1', c_2' > 0$.

Proof: The result extends the statement on Gelfand m-width of Thm. 10.5 to compressive m-widths. Hence, from Thm. 10.5, it suffices t.s.t.

$$c_1 \cdots {}^{1-1/p} \leq \boxed{d^m(B_1^N, \ell_p^N) \leq E^m(B_1^N, \ell_p^N) \leq 2 d^m(B_1^N, \ell_p^N)} \leq 2c_2 \cdots {}^{1-1/p}.$$

Now, B_1^N satisfies $-B_1^N = B_1^N$.

$d^m(K,X) = \inf_{A \in \mathbb{R}^{m\times N}} \sup_{x \in X \cap \mathcal{N}(A)} \|x\|$, $X \to B_1^N$

From the defn. of Gelfand width,

$$d^m(B_1^N, \ell_p^N) \leq \sup_{v \in B_1^N \cap \mathcal{N}(A)} \|v\| \quad \text{— (*1)}$$

where $A: \mathbb{R}^N \to \mathbb{R}^m$ is a linear map.

For any $v \in B_1^N \cap \mathcal{N}(A)$, $Av = 0 \Rightarrow$

$$\|v - \Delta(0)\| = \|v - \Delta(Av)\| \leq \sup_{x\in B_1^N} \|x - \Delta(Ax)\|$$

Since $-B_1^N = B_1^N$, $-v \in B_1^N \cap \mathcal{N}(A)$ also $\Rightarrow$

$$\|-v - \Delta(0)\| = \|-v - \Delta(-Av)\| \leq \sup_{x\in B_1^N} \|x - \Delta(Ax)\|$$

Thus, for any $v \in B_1^N \cap \mathcal{N}(A)$,

$$\|v\| = \left\| \tfrac{1}{2}(v - \Delta(0)) - \tfrac{1}{2}(-v - \Delta(0)) \right\|$$
$$\leq \tfrac{1}{2}\|v - \Delta(0)\| + \tfrac{1}{2}\|-v - \Delta(0)\|$$
$$\leq \sup_{x\in B_1^N} \|x - \Delta(Ax)\| \quad \text{— (*2)}$$

→ no longer dep. on v.

From (*1) and (*2), we get

$$d^m(B_1^N, \ell_p^N) \leq \sup_{x\in B_1^N} \|x - \Delta(Ax)\| \quad \text{— (*3)}$$

This holds for any A and $\Delta \Rightarrow$

$$d^m(B_1^N, \ell_p^N) \leq \underbrace{E^m(B_1^N, \ell_p^N)}_{\text{Inf of (*3) over } A \,\&\, \Delta}$$

which establishes the first part of the inequality.

To show the second part of the inequality, observe that

$$B_1^N + B_1^N \subset 2B_1^N$$

Define the maps $A \in \mathbb{R}^{m\times N}$ and $\Delta: \mathbb{R}^m \to \ell_p^N$ such that

$$\Delta(y) \in B_1^N \cap \underbrace{A^{-1}(y)}_{\substack{\text{Inverse map of } A \in \mathbb{R}^{m\times N} \\ = \text{Set of all vecs in } \ell_p^N \text{ that map to } y}} \quad \forall y \in A(B_1^N).$$

With this choice,

$$E^m(B_1^N, \ell_p^N) \leq \sup_{x\in B_1^N} \|x - \Delta(Ax)\|$$
$$\leq \sup_{x\in B_1^N} \left[\sup_{z \in B_1^N \cap A^{-1}(Ax)} \|x - z\| \right]$$

For $x \in B_1^N$ and $z \in B_1^N \cap A^{-1}(Ax)$,

$$x - z \in B_1^N + (-B_1^N) \subset 2B_1^N$$

and $x - z \in \mathcal{N}(A)$ (Both x & z give rise to the same y $(= Ax = Az)$).

Therefore,

$$E^m(B_1^N, \ell_p^N) \leq \sup_{u \in 2B_1^N \cap \mathcal{N}(A)} \|u\|$$
$$= 2 \sup_{v \in B_1^N \cap \mathcal{N}(A)} \|v\|$$

Taking the infimum over A,

$$E^m(B_1^N, \ell_p^N) \leq 2 d^m(B_1^N, \ell_p^N),$$

which establishes the second part of the inequality, and concludes the proof. ∎

Thus, we showed that, for $1 < p \leq 2$ and $m < N$,

$$E^m(B_1^N, \ell_p^N) \asymp \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}}$$

where the notation $B \asymp A$ means $\exists\, c_1, c_2 > 0$ s.t. $c_1 A \leq B \leq c_2 A$.

The lower estimate

$$c_1 \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}} \leq E^m(B_1^N, \ell_p^N)$$

is of significance for CS. We have seen that, if $m \geq c s \ln\left(\frac{eN}{s}\right)$, there are matrices $A \in \mathbb{R}^{m\times N}$ with small RICs and reconstruction maps (e.g., BP, IHT, OMP) s.t.

$$\boxed{\|x - \Delta(Ax)\|_p \leq \frac{C}{s^{1-\frac{1}{p}}} \sigma_s(x)_1}$$

(See, e.g. Thm. 6.12). (Thus, $m \geq c s \ln\left(\frac{eN}{s}\right)$ sufficient.)

Conversely, using the above, we can now s.t. the existence of Δ and A providing such a stability result in the reconstruction error forces

$$\boxed{m \geq c' s \ln\left(\frac{eN}{s}\right)}$$

to be satisfied. This is Prop. 10.7:

Prop. 10.7 Let $1 < p \leq 2$. Suppose $\exists A \in \mathbb{R}^{m\times N}$ and $\Delta: \mathbb{R}^m \to \mathbb{R}^N$ s.t. $\forall x \in \mathbb{R}^N$

$$\|x - \Delta(Ax)\|_p \leq \frac{C}{\ldots} \sigma_s(x)_1 \quad \text{— (*4)}$$

$$\|x - \Delta(Ax)\|_p \leq \frac{C}{s^{1-1/p}} \sigma_s(x)_1 .$$

Then, for some consts. $c_1, c_2 > 0$ depending only on C,

$$\boxed{m \geq c_1 s \ln\left(\frac{eN}{s}\right)}, \quad \text{provided } s > c_2 .$$

(removed later in the text.)

Proof : (4) $\Rightarrow$

$$E^m(B_1^N, \ell_p^N) \leq \frac{C}{s^{1-1/p}} \sup_{x \in B_1^N} \sigma_s(x)_1 \leq \frac{C}{s^{1-1/p}} .$$

From Cor. 10.6, $\exists$ $c > 0$ s.t.

$$c \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-1/p} \leq E^m(B_1^N, \ell_p^N) .$$

Hence, for some const. $c' > 0$,

$$\underbrace{c' \min\left\{1, \frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}}_{*} \leq \frac{1}{s} .$$

If the min in $*$ is 1, then $s \leq \frac{1}{c'}$, and the hypothesis $s > c_2 \triangleq \frac{1}{c'}$ allows us to discard this.

Hence, the min in $*$ is $\frac{\ln\left(\frac{eN}{m}\right)}{m}$, $\Rightarrow$

$$m \geq c' s \ln\left(\frac{eN}{m}\right) .$$

(See Lemma C.6 in the text) $\Rightarrow$

$$\boxed{m \geq c_1 s \ln\left(\frac{eN}{s}\right)}, \text{ with } c_1 = \frac{c'e}{1+e} . \quad \square$$

Thus, the existence of $A \in \mathbb{R}^{m \times N}$ and $\Delta : \mathbb{R}^m \to \mathbb{R}^N$ which ensures stable recovery of all $x \in \mathbb{R}^N \Rightarrow$

$$m \geq c_1 s \ln\left(\frac{eN}{s}\right) !$$

Cor. 10.8. If the $2s^{th}$ RIC of $A \in \mathbb{R}^{m \times N}$ satisfies $\delta_{2s} < 0.6246$ (say), then necessarily

$$m \geq c s \ln\left(\frac{eN}{s}\right)$$

for some constant $c > 0$ depending only on δ_{2s}.

Proof : If $\delta_{2s} < 0.6246$ and Δ is the ℓ_1-min. (BP) reconstruction map, from Thm. 6.12,

$$\|x - \Delta(Ax)\|_2 \leq \frac{C}{\sqrt{s}} \sigma_s(x)_1$$

for some const. C dep. only on δ_{2s}. Hence, the statement of Cor. 10.8 follows from Prop. 10.7. $\square$