E9 203 Compressed Sensing and Sparse Signal Processing

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Recap:

Defn. The Gelfand m-width of a subset K of a normed space $\mathcal{X}$ is

$$\boxed{d^m(K,\mathcal{X}) = \inf_{A\in\mathbb{R}^{m\times N}} \sup_{x\in K\cap\mathcal{N}(A)} \|x\|}$$

Defn. The compressive m-width of a subset K of a normed space $\mathcal{X}$ is

$$\boxed{E^m(K,\mathcal{X}) = \inf\left\{\sup_{x\in K}\|x-\Delta(Ax)\|,\ \begin{matrix}A:\mathcal{X}\to\mathbb{R}^m \text{ linear,}\\ \Delta:\mathbb{R}^m\to\mathcal{X}\end{matrix}\right\}.}$$

Δ : reconstruction map, arbitrary

A : nonadaptive

Thm. 10.5 For $1<p\le 2$ and $m<N$, $\exists$ consts. $c_1, c_2>0$ dep. only on p s.t.

$$\boxed{c_1\min\left\{1,\frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}} \le d^m(B_1^N,\ell_p^N) \le c_2\min\left\{1,\frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-1/p}}$$

$\Rightarrow$ Matching upper and lower bounds.

Cor. 10.6 For $1<p\le 2$ and $m<N$, the compressive width satisfies, for some consts. $c_1', c_2'>0$,

$$c_1'\min\left\{1,\frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}} \le E^m(\underbrace{B_1^N}_{K},\underbrace{\ell_p^N}_{\mathcal{X}}) \le c_2'\min\left\{1,\frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}}.$$

(Saw the proof in the prev. class.)

Prop. 10.7 Let $1<p\le 2$. Suppose $\exists$ $A\in\mathbb{R}^{m\times N}$ and $\Delta:\mathbb{R}^m\to\mathbb{R}^N$ s.t. $\forall\, x\in\mathbb{R}^N$,

$$\|x-\Delta(Ax)\|_p \le \frac{C}{s^{1-\frac{1}{p}}}\sigma_s(x)_1 .$$

Then, for some consts. $c_1, c_2>0$ dep. only on C,

$$m \ge c_1 s\ln\left(\frac{eN}{s}\right),\ \text{provided } s>c_2 .$$

(Saw the proof in the prev. class.)

Cor. 10.8 If the $2s^{th}$ RIC of $A\in\mathbb{R}^{m\times N}$ satisfies $\delta_{2s}<0.6246$ (say), then necessarily

$$m \ge C s\ln\left(\frac{eN}{s}\right)$$

for some const. $C>0$ dep. only on δ_{2s}.

What remains is to prove Thm. 10.5:

For $1<p\le 2$ and $m<N$, $\exists$ consts. $c_1, c_2>0$ dep. only on p s.t.

$$c_1\min\left\{1,\frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}} \le d^m(B_1^N,\ell_p^N) \le c_2\min\left\{1,\frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}}.$$

Upper bound:

Simply use Thm. 6.12: If the $2s^{th}$ RIC of A satisfies $\delta_{2s}<0.6246$, the soln. $x^\#$ of

$$\min_{z\in\mathbb{C}^N}\|z\|_1 \quad \text{s.t.}\quad y=Az$$

satisfies $\|x-x^\#\|_p < \frac{C}{s^{1-\frac{1}{p}}}\sigma_s(x)_1$.

This $\Rightarrow$

$$\left[E^m(B_1^N,\ell_p^N) \le \frac{C}{s^{1-\frac{1}{p}}}\sup_{x\in B_1^N}\sigma_s(x)_1\right]$$

$$\le \frac{C}{s^{1-1/p}}$$

The last ineq. follows from Thm. 9.2 ($A\in\mathbb{R}^{m\times N}$ satisfies RIC $<\delta$ if it has iid subgaussian entries and $m\ge 2C\delta^{-2}s\ln\left(\frac{eN}{s}\right)$)

$m\ge 2C\delta^{-2}s\ln\left(\frac{eN}{s}\right)$ is equivalent to (see Lemma C.6) s is of the order $\frac{m}{\ln\left(\frac{eN}{m}\right)}$.

Then, using the proof of Cor. 10.6,

$$d^m(B_1^N,\ell_p^N) \le E^m(B_1^N,\ell_p^N) \le c'\left\{\frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}}$$

Also, using $\|x\|_p\le\|x\|_1$, $x\in\mathbb{R}^N$ in the defn. of the Gelfand m-width,

$$d^m(B_1^N,\ell_p^N) \le d^0(B_1^N,\ell_1^N) = 1 .$$

Thus, we have,

$$d^m(B_1^N,\ell_p^N) \le c_2\min\left\{1,\frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}}.$$

See text for a detailed, alternative, self-contained proof.

Lower bound: Want to show

Thm. 10.10 There is a const. $c>0$ s.t., for $1<p\le\infty$ and $m<N$,

$$d^m(B_1^N,\ell_p^N) \ge c\min\left\{1,\frac{\ln\left(\frac{eN}{m}\right)}{m}\right\}^{1-\frac{1}{p}}.$$

The proof relies on two results.

Lemma 10.12 Given integers $s<N$, $\exists$

$$\boxed{n \ge \left(\frac{N}{4s}\right)^{s/2}} \quad \text{— (*1)}$$

subsets $S_1, S_2, \dots, S_n$ of $[N]$ s.t. each S_j has cardinality s and

$$|S_i\cap S_j| < \frac{s}{2}\quad \forall\, i\ne j \quad \text{— (*2)}$$

Proof: WLOG assume $s\le\frac{N}{4}$, else $n=1$ is sufficient.

Let $\mathcal{B} \triangleq$ Family of subsets of $[N]$ with cardinality s.

Draw $S_1\in\mathcal{B}$.

Collect $\mathcal{A}_1 = \{S\in\mathcal{B} : |S_1\cap S|\ge\frac{s}{2}\}$

We have

$$|\mathcal{A}_1| = \sum_{k=\lceil\frac{s}{2}\rceil}^{s}\binom{s}{k}\binom{N-s}{s-k}$$

$$\le 2^s\max_{\lceil\frac{s}{2}\rceil\le k\le s}\binom{N-s}{s-k}$$

$$= 2^s\binom{N-s}{\lfloor s/2\rfloor}$$

Any $S\in\mathcal{B}\setminus\mathcal{A}_1$ satisfies $|S_1\cap S|<\frac{s}{2}$

Draw $S_2\in\mathcal{B}\setminus\mathcal{A}_1$ provided $|\mathcal{B}\setminus\mathcal{A}_1|\ne 0$

Collect $\mathcal{A}_2 = \{S\in\mathcal{B}\setminus\mathcal{A}_1 : |S_2\cap S|\ge\frac{s}{2}\}$

By the same argument as before,

$$|\mathcal{A}_2| \le 2^s\binom{N-s}{\lfloor s/2\rfloor}$$

Observe that any $S\in\mathcal{B}\setminus(\mathcal{A}_1\cup\mathcal{A}_2)$ satisfies

$$|S_1\cap S|<\frac{s}{2},\quad |S_2\cap S|<\frac{s}{2}.$$

Repeat the procedure to select $S_1, S_2, \dots, S_n$ until $\mathcal{B}\setminus(\mathcal{A}_1\cup\mathcal{A}_2\cup\dots\cup\mathcal{A}_n)$ is empty.

Thus, $|S_i\cap S_j|<\frac{s}{2}\ \forall\, i\ne j$ is satisfied by construction.

Moreover,

$$n \ge \frac{|\mathcal{B}|}{\max_{1\le i\le n}|\mathcal{A}_i|} \ge \frac{\binom{N}{s}}{2^s\binom{N-s}{\lfloor s/2\rfloor}}$$

$$= \frac{1}{2^s}\frac{N(N-1)\cdots(N-s+1)}{(N-s)(N-s-1)\cdots(N-s-\lfloor s/2\rfloor+1)}\cdot\frac{1}{s(s-1)\cdots(\lfloor\frac{s}{2}\rfloor+1)}$$

$$\ge \frac{1}{2^s}\frac{N(N-1)\cdots(N-\lceil\frac{s}{2}\rceil+1)}{s(s-1)\cdots(s-\lceil\frac{s}{2}\rceil+1)}$$

(Replace $(N-\lceil\frac{s}{2}\rceil)\cdots(N-s+1)$ with smaller numbers $(N-s)\cdots(N-s-\lfloor\frac{s}{2}\rfloor+1)$)

$$\ge \frac{1}{2^s}\left(\frac{N}{s}\right)^{\lceil\frac{s}{2}\rceil} \ge \left(\frac{N}{4s}\right)^{s/2},$$

which shows that (*1) is fulfilled, completing the proof. ☐

The second result we need to show:

Thm. 10.11 Given $A\in\mathbb{R}^{m\times N}$, if every $2s$-sparse $x\in\mathbb{R}^N$ is a minimizer of $\|z\|_1$ s.t. $Ax=Az$, then

$$m \ge c_1 s\ln\left(\frac{eN}{s}\right)$$

where $c_1=\frac{1}{\ln 9}$ and $c_2=4$.

Proof: Consider the quotient space

$$\mathcal{X} = \{[x]\triangleq x+\mathcal{N}(A),\ x\in\mathbb{R}^N\}$$

which is normed with

$$\|[x]\| \triangleq \inf_{v\in\mathcal{N}(A)}\|x-v\|_1,\quad x\in\mathbb{R}^N.$$

ℓ_1^N a VS, $\mathcal{N}(A)$ subspace

Equivalence relationship

$x\sim y$ if $x-y\in\mathcal{N}(A)$.

Any $x\in\mathcal{N}(A)\sim\underline{0}$.

Equivalence class denoted by

$[x] = \underline{x}+\mathcal{N}(A)$

$= \{x+n,\ n\in\mathcal{N}(A)\}$

... $2s$-sparse $x\in\mathbb{R}^N$, every $z\in x-v$ with

Given a $2s$-sparse ... $v \in \mathcal{N}(A)$ satisfies $Az = Ax$. Hence, by the assump. of the thm.,

$$\underline{\underline{\| [x] \|_1 = \|x\|_1}} .$$

Let $S_1, \dots, S_n$ be the sets introduced in Lemma 10.12, and define s-sparse vecs. $x^1 \dots x^n \in \mathbb{R}^N$ with unit l_1 norms by

$$x^i_k = \begin{cases} 1/s & \text{if } k \in S_i \\ 0 & \text{if } k \notin S_i \end{cases}$$

For $1 \le i \neq j \le n$

$$\| [x^i] - [x^j] \| = \| [x^i - x^j] \| = \|x^i - x^j\|_1$$

since $x^i - x^j$ is $2s$-sparse.

Also, $\|x^i - x^j\|_1 > 1$

$$\because \ | x^i_k - x^j_k | = \begin{cases} 1/s & \text{if } k \in S_i \Delta S_j \\ 0 & \text{else} \end{cases}$$

where $S_i \Delta S_j \triangleq \underbrace{(S_i \cup S_j)}_{\ge \frac{3s}{2}} \setminus \underbrace{(S_i \cap S_j)}_{< \frac{s}{2}}$

$$| S_i \Delta S_j | > s .$$

Hence, $\| [x^i] - [x^j] \| > 1 \quad \forall \ 1 \le i \neq j \le n$

Thus, $\{ [x^1], \dots, [x^n] \}$ is a 1-separating subset of the unit sphere $\mathcal{X}$.

From $\boxed{\text{Prop. C3}}$ in the appendix, this $\Rightarrow$ $n \le 3^m$.

Lemma 10.12

$(*1): n \ge \left(\frac{N}{4s} \right)^{s/2}$

Thus, from $(*1)$, we get

$$\left(\frac{N}{4s} \right)^{s/2} \le 3^m$$

$$\Rightarrow \quad \frac{s}{2} \ln \left(\frac{N}{4s} \right) \le m \ln 3$$

$$\Rightarrow \quad m \ge \frac{s}{\ln 9} \ln \left(\frac{N}{4s} \right),$$ which is the desired inequality. ☒

$\alpha [x] = [\alpha x]$

$[x] + [y] = [x + y]$.

Set of all affine subsets of l_1^N that are parallel to $\mathcal{N}(A)$.

$\mathcal{X} = \{ [x] \triangleq x + \mathcal{N}(A), x \in \mathbb{R}^N \}$

$\mathcal{X}$ has dimension $r = \text{rank}(A) \le m$.

Prop. C3 $\Rightarrow$ $n \le 3^r \le 3^m$.

Let $\|\cdot\|$ be some norm on $\mathbb{R}^N$ and $\mathcal{U}$ be a subset of the unit ball $B = \{ x \in \mathbb{R}^N, \|x\| \le 1 \}$

Then, the packing and covering # satisfy, for $t > 0$,

$$\mathcal{N}(\mathcal{U}, \|\cdot\|, t) \le \mathcal{P}(\mathcal{U}, \|\cdot\|, t) \le \left(1 + \frac{2}{t} \right)^N .$$

Let T be a subset of a metric space $(\mathcal{X}, d)$

For $t > 0$, the covering # $\mathcal{N}(T, d, t)$ is the smallest integer $\mathcal{N}$ s.t. T can be covered with balls $B(x_\ell, t) \triangleq \{ x \in \mathcal{X}, d(x, x_\ell) \le t \}$, $x_\ell \in T$, $\ell \in [\mathcal{N}]$, i.e.,

$$T \subset \bigcup_{\ell=1}^{\mathcal{N}} B(x_\ell, t)$$

The set of pts. $\{ x_1, \dots, x_{\mathcal{N}} \}$ is called a t-covering.

The packing # $\mathcal{P}(T, d, t)$ is defined, for $t > 0$, as the maximal integer P s.t. there are pts $x_\ell \in T$, $\ell \in [P]$ which are t-separated, i.e., $d(x_\ell, x_k) > t$ for all $l, k \in [P]$, $k \neq \ell$.