Optimal Stopping Theory and Opportunistic Transmission Scheduling

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Examples[Ferguson, 2006]

- **Optimal Stopping Theory (OST):** It is concerned with the *problem of choosing a time* to take a given action based on sequentially observed random variables in order to *maximize (minimize)* an expected payoff (cost).
- Examples
  1. Maximizing the average in coin tossing problem.
  2. House selling Problem.
  3. Classical secretary problem.

(1). Ch.1. http://www.math.ucla.edu/ tom/Stopping/Contents.html

Introduction

Stopping rule problems are defined by two objects

1. A sequence of random variables, $X_1, X_2, \ldots$, whose joint distribution is assumed to be known.
2. A sequence of real-valued reward functions (may be -ve or even -∞),

$$y_0, y_1(x_1), y_2(x_1, x_2), \ldots, y_\infty(x_1, x_2, \ldots)$$

where,

$y_0 :=$ reward received if you choose not to take any observation.

$y_1(x_1) :=$ reward for stopping at 1^{st}-stage after observing x_1.

3. **Goal:** To choose a stopping time to maximize the *expected* reward.

Stopping Rule

- A (randomized) stopping rule is a sequence of probabilities of stopping and is represented as,

$$\Phi = (\phi_0, \phi_1(x_1), \phi_2(x_1, x_2), \ldots).$$

- Probability of stopping at stage n, given that you have observed $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, is given by,

$$0 \leq \phi_n(x_1, \ldots, x_n) \leq 1 \quad \forall \quad n.$$

- For non-randomized stopping rules,

$$\phi_n(x_1, \ldots, x_n) = 0 \text{ or } 1 \quad \forall \quad n.$$

Probability Mass Function (pmf) of Stopping Time N

The pmf of N given $X = x = (x_1, x_2, \dots)$ is denoted by,

$$\Psi = (\psi_0, \psi_1, \psi_2, \dots, \psi_\infty).$$

Where

$$\psi_n(x_1, \dots, x_n) = P(N = n | X = x) \quad \text{for } n = 0, 1, 2, \dots$$

This may be related to stopping rule as follows,

$$\psi_0 = \phi_0$$

$$\psi_1(x_1) = (1 - \phi_0)\phi_1(x_1)$$

$$\vdots$$

$$\psi_n(x_1, \dots, x_n) = \left[\prod_{j=1}^{n-1} (1 - \phi_j(x_1, \dots, x_j))\right] \phi_n(x_1 \dots, x_n)$$

Problem

Problem, then, is to choose a stopping rule Φ to maximize the expected return, $V(\Phi)$, given as,

$$V(\Phi) = E\left[y_N(x_1, \ldots, x_N)\right]$$

$$V(\Phi) = E\left[\sum_{j=0}^{=\infty} \psi_j(x_1, \ldots, x_j) y_j(x_1, \ldots, x_j)\right].$$

" $= \infty$" corresponds to the case when stopping never occurs.

Finite Horizon Problems (FHP)[Ferguson, 2006]

- If it is compulsory to stop after observing $x_1, \ldots, x_T$, we say the problem has horizon T
- FHP may be obtained as a special case of the general problem by setting,

$$y_{T+1} = \ldots = y_\infty = -\infty$$

- Such problems can be solved by method of *Backward Induction*

(2). Ch.2. http://www.math.ucla.edu/ tom/Stopping/Contents.html

Backward Induction

Define

$$V_T^{(T)}(x_1, \ldots x_T) = \max\{y_j(x_1, \ldots, x_j), A\}$$

Where,

$$A = E\left(V_{j+1}^{(T)}(x_1, \ldots x_j, X_{j+1}) | X_1 = x_1, \ldots, X_j = x_j\right),$$

is the expected return obtained by continuing and using the optimal rule for stages $j+1$ through T, given that we have observed $X_1 = x_1, \ldots, X_j = x_j$, and

$$V_j^{(T)}(x_1, \ldots x_j),$$

represents the maximum return one can obtain starting from stage J and having observed $X_1 = x_1, \ldots, X_j = x_j$.

Opportunistic Scheduling [Poulakis, 2013]

- Considering basic channel capacity equation

$$R = W \log_2 \left(1 + \frac{g.P_{T_x}}{N_o.W} \right) \quad (1)$$

$$\Rightarrow P_{T_x} \propto \frac{1}{g}$$

- Good channel conditions are explored to get better utilization of energy.
- OST is used to find the optimal time instants, to transmit with minimum energy, depending on channel conditions.

(3). Marios I. Poulakis etal. , "Channel-Aware Opportunistic Transmission Scheduling for Energy-Efficient Wireless links" *IEEE Trans. of Vehicular Technology*, vol.62, pp.192-204, January 2013.

Problem Setup

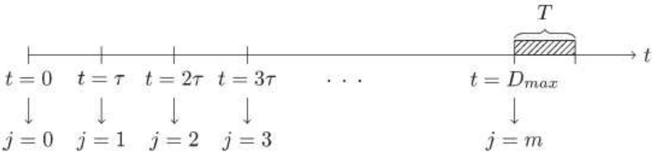

Figure: Problem setup for single-hop point-to-point wireless link

Assumptions:

- Pdf of channel under consideration is known.
- Transmitter is aware of instantaneous CSI at the receiver.
- $\tau >$ channel coherence time,
  and $T <$ channel coherence time.

Energy-Efficient Opportunistic Transmission Scheduler (E^2OTS)

- First, we consider that the OTS problem is executed for one time $(E^2OTS) - I$.
- The problem is to choose a stopping rule, $1 \leq N \leq m$, to minimize the expected energy consumption, $E[E_N]$, of the device. Where,

$$E_N = P_N.T + N.E_c = \left(\frac{2^{\frac{R}{W}} - 1}{g_N} \right) .N_oWT + N.E_c \qquad (2)$$

  where, $E_c =$ energy required for channel measurement
- Finite horizon problem with horizon D_{max}.

Multithreshold Policy for $E^2OTS - I$

Using the backward induction to find the optimal stopping rule, we write

$$V_j^{(m)} = \min\{P_j T, A_{m-j}\} + E_c, \tag{3}$$

where,

$$A_{m-j} = E\left[V_{j+1}^{(m)}((g_1, \ldots, g_j, G_{j+1}) | G_1 = g_1, \ldots, G_j = g_j)\right]. \tag{4}$$

Hence, the optimal stopping rule suggests stopping and transmitting at stage j if

$$P_j T \leq A_{m-j}.$$

Multithreshold Policy for $E^2OTS - I$ (contd.)

An average cost of continuing can be considered to be associated with each stage j, given as,

$$P_{th,j} = \frac{A_{m-j}}{T} \quad \text{for} \quad j = 0, 1, \ldots, m-1, \tag{5}$$

$$P_{th,m} = P_{max} = \frac{A_0}{T}. \tag{6}$$

Using backward induction we can compute A_{m-j} for each individual stage, as following

$$\begin{aligned} A_{m-j} &= E \min\left[PT, A_{m-j-1}\right] + E_c \quad \text{for} \quad j = 0, \ldots, m-1, \\ &= \int_0^{\frac{A_{m-j-1}}{T}} pT dF_P + \int_{\frac{A_{m-j-1}}{T}}^{P_{max}} A_{m-j-1} dF_P + E_c \end{aligned} \tag{7}$$

where, $F_P(p)$ is P_{max} normalized cdf of transmission power.

Multithreshold Policy for $E^2OTS - I$ (contd.)

The optimal thresholds associated with each stage j, can be calculated as,

$$P^*_{th,j} = \int_0^{P^*_{th,j}} p dF_P + P^*_{th,j+1} - P^*_{th,j+1} F_P(P^*_{th,j+1}) + \frac{E_c}{T} \quad (8)$$

for $j = 0, \ldots, m-1$, and

$$P^*_{th,m} = \frac{A_0}{T} = P_{max}. \quad (9)$$

The policy that minimizes the energy consumption for $E^2OTS - I$ can be given as

$$\texttt{if } P_j \leq P^*_{th,j} \rightarrow \texttt{ transmit at } j$$

$$\texttt{else } \rightarrow \texttt{ postpone}$$

$E^2OTS - II$: Rate of Return

- Problem of $E^2OTS - I$ is repeated for L rounds.

$$\{E_{N_1}, \ldots, E_{N_L}\} \rightarrow \text{Cost Sequence}$$

$$\{N_1, \ldots, N_L\} \rightarrow \text{Stopping time sequence.}$$

With, $1 \leq N_\ell \leq m \quad \text{for} \quad \ell = 1, \ldots, L.$

- **Aim:** To minimize the average energy consumption per unit time, i.e. the average power consumption (rate of return).

$E^2OTS - II$: Rate of Return

- Average energy consumption per unit time can be expressed as (by law of large nos.)

$$\frac{\sum_{\ell=1}^{L} E_{N_\ell}}{\sum_{\ell=1}^{L} T_{N_\ell}} \longrightarrow \frac{E\left[E_N\right]}{E\left[T_N\right]} \tag{10}$$

Where,

$$T_N = N\tau + T. \tag{11}$$

- An optimal stopping problem of choosing a stopping rule $1 \leq N \leq m$ to minimize the ratio $\frac{E[E_N]}{E[T_N]}$.

$E^2OTS - II$: Rate of Return

Theorem 1

- If for some λ, $\inf_{N \in \mathcal{C}} E\left(E_N - \lambda T_N\right) = 0$, then $\inf_{N \in \mathcal{C}} \frac{E[E_N]}{E[T_N]} = \lambda$. Moreover, if $\inf_{N \in \mathcal{C}} E\left(E_N - \lambda T_N\right) = 0$ is attained at $N^* \in \mathcal{C}$, then N^* is optimal for minimizing $\inf_{N \in \mathcal{C}} \frac{E[E_N]}{E[T_N]}$.
- Conversely, if $\inf_{N \in \mathcal{C}} \frac{E[E_N]}{E[T_N]} = \lambda$ and if the infimum is attained at at $N^* \in \mathcal{C}$, then $\inf_{N \in \mathcal{C}} E\left(E_N - \lambda T_N\right) = 0$ and the infimum is attained at N^*.

$\mathcal{C}$ is the class of stopping rules s.t. $\mathcal{C} = \{N : N \geq 1, ET_N < \infty\}$

$E^2OTS - II$: Rate of Return

- From Theorem 1, following two minimization problems are equivalent

$$\inf_{N \in \mathcal{C}} \frac{E\left[E_N\right]}{E\left[T_N\right]} = \lambda^* \Longleftrightarrow \inf_{N \in \mathcal{C}} E\left(E_N - \lambda^* T_N\right) = 0 \tag{12}$$

- The optimal return is given by,

$$V(\lambda) = \inf_{N \in \mathcal{C}} \left[E\left[E_N\right] - \lambda E\left[T_N\right]\right] = E\left[E_{N(\lambda)}\right] - \lambda E\left[T_{N(\lambda)}\right], \tag{13}$$

where, $N(\lambda)$ is the stopping rule that achieves minimum for λ.

- Optimal rate of return, λ^*, can be found by solving $V(\lambda^*) = 0$ and hence we can find optimal stopping time $N^* = N(\lambda^*)$.

$E^2OTS - II$: Multithreshold Policy

$$\text{Let } Z_N = E_N - \lambda T_N \quad (14)$$

$$= \left(\frac{2^{\frac{R}{W}} - 1}{g_N}\right).N_oWT + N.E_c - \lambda N\tau - \lambda T \quad (15)$$

$$= \left[\left(\frac{2^{\frac{R}{W}} - 1}{g_N}\right).N_oW - \lambda\right] T + N(E_c - \lambda) \quad (16)$$

Given that we have observed $G_1 = g_1, \ldots, G_j = g_j$, the minimum rate of return at stage j

$$V_j^{(m)} = \min\{P_jT - \lambda T, A_{m-j}\} + E_c - \lambda\tau, \quad (17)$$

where,

$$A_{m-j} = E\left[V_{j+1}^{(m)}((g_1, \ldots, g_j, G_{j+1})|G_1 = g_1, \ldots, G_j = g_j)\right]. \quad (18)$$

$E^2OTS-II$: Multithreshold Policy

Hence, the optimal stopping rule suggests stopping and transmitting at stage j if

$$P_jT-\lambda T\leq A_{m-j}.$$

So the transmission power threshold is,

$$P_{th,j} = \frac{A_{m-j}}{T}+\lambda \quad \text{for} \quad j=0,1,\ldots,m-1, \qquad (19)$$

$$P_{th,m} = P_{max}=\frac{A_0}{T}+\lambda \quad \text{for} \quad j=m. \qquad (20)$$

$E^2OTS - II$: Multithreshold Policy

Following backward induction, we can compute $A_{m-j}(\lambda)$ for each individual stage,

$$
\begin{aligned}
A_{m-j}(\lambda) &= E\min\{P_jT - \lambda T, A_{m-j-1}(\lambda)\} + E_c - \lambda\tau \quad (21)\\
&= \int_0^{\frac{A_{m-j-1}(\lambda)}{T}} (pT - \lambda T)dF_P + \int_{\frac{A_{m-j-1}(\lambda)}{T}}^{P_{max}} A_{m-j-1}(\lambda)dF_P \\
&+ E_c - \lambda\tau \quad \text{for} \quad j = 0, 1, \ldots, m-1. \quad (22)
\end{aligned}
$$

Consequently we can compute the corresponding power threshold $P_{th,j}(\lambda)$, for each stage for each λ, using (19) and (20).

$E^2OTS - II$: Optimal Threshold Policy

- Optimal policy is the collection of thresholds corresponding to optimal rate of return, λ^*, i.e,

$$P^*_{th,j} = P_{th,j}(\lambda^*) = \frac{A_{m-j}(\lambda^*)}{T} + \lambda^*$$

- The policy that minimizes the rate of return for $E^2OTS - II$ is given as,

$$\texttt{if } P_j \le P^*_{th,j} \rightarrow \texttt{ transmit at } j$$

$$\texttt{else } \rightarrow \texttt{ postpone}$$

$E^2OTS - II$: Optimal Threshold Policy

Proposition 1

Optimal power thresholds $P^*_{th,j}$ are increasing on j = 1,. . . , m i.e,

$$P^*_{th,j} \leq P^*_{th,j} \quad \text{for} \quad j = 1, \ldots, m-1.$$

proof: It is equivalent to show that $A_{i+1}(\lambda^*) \leq A_i(\lambda^*)$, for $i = 0, \ldots, m-2$. Let $A_1(\lambda^*) > A_0(\lambda^*)$. Then,

$$\begin{aligned} A_2(\lambda^*) &= E\min[PT - \lambda T, A_1(\lambda^*)] + E_c - \lambda\tau \\ &\geq E\min[PT - \lambda T, A_1(\lambda^*)] + E_c - \lambda\tau = A_1(\lambda^*) > A_0(\lambda^*) \end{aligned}$$

Therefore, inductively we have $A_m(\lambda^*) > A_0(\lambda^*) = P_{max}T - \lambda T$. As $A_m(\lambda^*) = 0 \Rightarrow \lambda^* > P_{max}$. Hence, $A_1(\lambda^*) \leq A_0(\lambda^*)$ and rest of the proof follows similarly by induction.

$E^2OTS - II$: Optimal Threshold Policy

Proposition 2

$V(\lambda^*) = 0 \Leftrightarrow A_m(\lambda^*) = 0$

Proposition 3

$A_j(\lambda)$ is continuous and monotonically decreases as λ increases from 0 to $+\infty, \quad \forall, \; 0, \ldots, m.$

For all j, $A_j(\lambda)$ goes from some positive value (for $\lambda = 0$) to $-\infty$ (for $\lambda = \infty$). Hence, $A_(\lambda) = 0$ has at least one solution.

More Reading on OTS applications

- D.Zheng, W.Ge, and J.Zhang, *"Distributed opportunistic scheduling for ad hoc networks with random access : An optimal stopping approach "*, IEEE Trans. Inf. Theory, Vol. 55, no.1, pp 205-222, Jan.2009.
- D.Zheng, M.Cao, J.Zhang, and P.R.Kumar *"Channel-aware distributed scheduling for exploiting multiuser diversity in ad hoc networks : A unified PHY/MAC approach "*, in proc. IEEE 27^{th} INFOCOM, Phoenix, AZ pp 1454-1462, Apr.2008.
- S. Chakraborty, Y.Dong, D. K. Y. Lau, and J.C.S Lui, *"On effectiveness of movement prediction to reduce energy consumption in wireless communication "*, IEEE Trans. Mobile Comput., Vol. 5, no.2, pp 157-169, Feb.2006.