Group Discussion
Alternating Projections

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Organization

- Alternating convex projections
- Non-convex projections
- Alternating non-convex projections

Notations and Basic results

- $\mathbb{E}$: Euclidean space , $\mathbb{B}$: unit ball and $\mathbb{S}$: unit sphere.
- A sequence (x_k) in $\mathbb{E}$ converges linearly with rate $\kappa < 1$ to x if there is some constant α such that
  $\|x_k - x\| \leq \alpha\kappa^k \quad \forall \quad k \geq 0.$
- "$R-$linear convergence" : the infimum of all possible constants κ, is the "rate of $R-$linear convergence".
- Let $M, N \subset \mathbb{E}$. The angle between M and N as the angle between 0 and $\frac{\pi}{2}$ whose cosine is
  $c(M, N) := \max\{\langle x, y\rangle : x \in \mathbb{S} \cap M \cap (M \cap N)^{\perp}, y \in \mathbb{S} \cap N \cap (M \cap N)^{\perp}\}$
- The quantity $c(M, N)$ is well-defined unless one subspace is a subspace of the other, in which case we set $c(M, N) = 0$.

Projection, Distance and Convexity

- For closed $M \in \mathbb{E}$, the distance of x from M
  $d_M(x) = \min\{\|x - y\| : y \in M\}$
  and the projection of x onto M
  $P_M(x) = argmin\{\|x - y\| : y \in M\}$
- If M is convex, $P_M(x)$ is singleton. Otherwise, it is not for some x for sure!
- For any point $x \in M$, vectors in the cone $N_M^p(x) = \{\lambda u : \lambda \in \mathbb{R}_+; x \in P_M(x + u)\}$ are called proximal normals to M at x.
- Limits of proximal normals to M at points $x_n \in M$ approaching x are called limiting normals, and comprise the limiting normal cone $N_M(x)$.

Alternating projections on subspaces

- For affine subspaces M and N, $(P_M P_N)^n(x) \to P_{M\cap N}(x)$
- Convergence is linear at rate $(\cos\theta)^2$,
  $\|(P_M P_N)^n(x) - P_{M\cap N}(x)\| \leq (\cos\theta)^{2n-1}\|x\|$,
  where θ is the angle between M and N.
- Alternating projections naturally extends to closed convex sets M and N. $(P_M P_N)^n(x) \to P_{M\cap N}(x)$
  Convergence is linear providing $M \cap int(N) \neq \emptyset$.
- To find a point $x \in M \cap N$, with M and N closed convex sets on $\mathbb{E}$, alternating convex projections is a basic algorithm.
- Applications: statistics, finance, engineering sciences, image processing ...

Example in Finance

For symmetric matrix C, computing the nearest correlation matrix: computing the projection of C onto the intersection of S_n^+, the semi-definite positive matrices, and the matrices with ones on the diagonal.

Used as calibration for evaluating extreme risks (Stress testing)

How to compute the nearest correlation matrix ? : alternating projection.

Nonconvex heuristic

Alternating convex projections is a good method and Alternating nonconvex projections is also a popular heuristic !
Examples:

- Optics : phase retrieval of images
  Simple version : given $a_j \in \mathbb{C}^k$, find $x \in \mathbb{C}^k$, so that
  $|\langle a_j, x\rangle| = b_j \quad j = 1, \cdots, m$
  with alternative projections onto
  $M = \{(x, z) \in \mathbb{C}^k \times \mathbb{C}^m : Ax = z\}$
  $N = \{(x, z) : |z_j| = b_j, \quad j = 1, \cdots, m\}$.
- Control : low-order control design
  affine M is $n \times n$ symmetric matrices.
  N is positive semidefinite matrices of rank r.

Easy non-convex projections

For closed non-convex $M \in \mathbb{R}^n$, the projection $P_M(x)$ is somewhere nonsingleton. But projection may still be easy. Examples:

- Single quadratic constraint
  $M = \{x \in \mathbb{R}^n : x^T Ax + b^T x = c\}$
  Projection is analogous to trust-region sub problems, solvable with a special Newton method.
- Rank constraint:
  $M = \{X \in \mathbb{R}^{n \times m} : \text{rank}(X) = r\}$
  To project, find a singular value decomposition $X = UDV$ and zero all but the first r largest singular values in D.

Spectral sets and Projection

For permutation-invariant $K \subset \mathbb{R}^n$, the spectral set of symmetric matrices
$\lambda^{-1}(K) = \{X \in S_n : (\lambda_1(X), \lambda_2(X), \cdots, \lambda_n(X)) \in K\}$.
Examples:

- $K = R^n_+$ gives the positive semi-definite cone S^+_n.
- $K = \{x : \|x\|_\infty = r\}$ gives $\{X : \lambda_{\max}(X) = r\}$

Theorem

If $y \in P_K(x)$ and U orthogonal, then
$U^T Diag(y) U \in P_{\lambda^{-1}(K)}(U^T Diag(x) U)$

Prox-regular spectral sets

Transfer of structure: if K is invariant by permutation of entries

- K convex $\Rightarrow \lambda^{-1}(K)$ convex.
- K prox-regular $\Rightarrow \lambda^{-1}(K)$ prox-regular.
- General notion of prox-regularity : P_M is locally unique.
- prox-regular spectral sets have locally all the good properties. (Ex: manifolds ...)

Prox-regular spectral sets in practice

Many spectral sets in alternative nonconvex projections

- Numerical algebra: nonnegative inverse eigenvalue problem
  For $\overline{\lambda}$ given, find $X \in M \cap N$,
  $M = \{X \in \mathbb{R}^{n\times n} : \lambda(X) = \overline{\lambda}\}$

$N = \{X \in \mathbb{R}^{n\times n} : X_{ij} \geq 0\}$.

- Image processing: design of tight frames
  Find the associated Gram matrix $X \in M \cap N$
  $M = \{X \in \mathbb{C}^{n\times n} : \lambda(X) = (\frac{n}{d}, \cdots, \frac{n}{d}, 0, \cdots, 0)\}$
  $N = \{X \in \mathbb{C}^{n\times n} : X_{ii} = 1, \|X\|_\infty \leq \mu\}$.

Alternating non-convex projections

Theorem

(local linear convergence) For closed sets $M, N \subset \mathbb{R}^n$. Assume

- *strong regularity holds at $\overline{x} \in M \cap N$*
- *M is super-regular at $\overline{x}$*
- *initial x_0 near $\overline{x}$*

Then alternating projection method converges $R-$linearly to $M \cap N$.

Comments:

- Super-regular sets: convex sets, smooth manifolds
- The convergence rate is $\cos\theta$, where θ is the minimal angle between $N_M(\overline{x})$ and $-N_N(\overline{x})$
- Rate is $(\cos\theta)^2$ if both M and N are super-regular.

Strong regularity

Definition

Strong regularity: $N_M(\bar{x}) \cap -N_N(\bar{x}) = \{0\}$, in other words, the minimal angle between $N_M(\bar{x})$ and $-N_N(\bar{x})$ is $\theta > 0$.

Examples

- The intersection of two smooth manifolds is strongly regular $\Leftrightarrow$ the manifolds are transverse
- The intersection of two convex sets is strongly regular $\Leftrightarrow$ no separating hyperplane

Definition

(transversality). Suppose M and N are two C^k-manifolds around a point $x \in M \cap N$. We say that M and N are transverse at x if $T_M(x) + T_N(x) = E$, where $T_M(x)$ is the tangent space to M at $x \in M$.

Super-regularity

Definition

(Super-regularity) A closed set $X \subset \mathbb{E}$ is super-regular at a point $z \in X$ when, for all $\delta > 0$, if distinct points $w, x \in X$ are sufficiently near z, then their difference $w - x$ makes an angle of at least $\frac{\pi}{2} - \delta$ with any nonzero normal $v \in N_X(x)$.

Examples of super-regular sets:

- convex sets
- smooth manifolds
- prox-regular sets
- constraint sets with Mangasarian-Fromovitz
- nearly convex sets
- subsmooth hypomonotone

prox-regular $\subset$ super-regular $\subset$ regular

References

- A. Lewis and J. Malick, "Alternating projections on manifolds," Mathematics of Operations Research, 2007.
- A. Lewis, R. Luke, and J. Malick, "Local convergence of nonconvex averaged and alternating projections," Foundations of Computational Mathematics, 2008.

Thank You