General Classes of Performance Lower Bounds for Parameter Estimation—Part II: Bayesian Bounds

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***Abstract*—In this paper, a new class of Bayesian lower bounds is proposed. Derivation of the proposed class is performed via projection of each entry of the vector-function to be estimated on a Hilbert subspace of $\mathcal{L}_2$. This Hilbert subspace contains linear transformations of elements in the domain of an integral transform, applied on functions used for computation of bounds in the Weiss–Weinstein class. The integral transform generalizes the traditional derivative and sampling operators, used for computation of existing performance lower bounds, such as the Bayesian Cramér–Rao, Bayesian Bhattacharyya, and Weiss–Weinstein bounds. It is shown that some well-known Bayesian lower bounds can be derived from the proposed class by specific choice of the integral transform kernel. A new lower bound is derived from the proposed class using the Fourier transform kernel. The proposed bound is compared with other existing bounds in terms of signal-to-noise ratio (SNR) threshold region prediction in the problem of frequency estimation. The bound is shown to be computationally manageable and provides better prediction of the SNR threshold region, exhibited by the maximum *a posteriori* probability (MAP) and minimum-mean-square-error (MMSE) estimators.**

***Index Terms*—Bayesian bounds, maximum *a posteriori* probability (MAP) estimator, mean-square-error bounds, minimum-mean-square-error (MMSE) estimator, parameter estimation, performance bounds, threshold signal-to-noise ratio (SNR), Weiss–Weinstein class.**

I. Introduction

LOWER BOUNDS on the mean square error (MSE) of estimators enable performance prediction and constitute a benchmark for performance evaluation, in the MSE sense. There are three main categories of lower bounds on the MSE of estimators: 1) non-Bayesian bounds for cases where the model parameters are deterministic; 2) Bayesian bounds for cases where the parameters are random; and 3) hybrid bounds for cases where the observation model contains deterministic and random parameters.

In Part I [1], a new class of non-Bayesian bounds was derived for the case of unbiased estimators by projecting each entry of the vector of estimation error on a Hilbert subspace of $\mathcal{L}_2$. This Hilbert subspace contains linear transformations of elements in the domain of an integral transform of the likelihood-ratio (LR) function. In this part, a new class of Bayesian bounds is derived by projecting each entry of the vector-function to be estimated on a Hilbert subspace of $\mathcal{L}_2$, which contains linear transformations of elements in the domain of an integral transform, applied on functions used for computation of bounds in the Weiss–Weinstein class.

Bayesian lower bounds can be partitioned into two categories: 1) the Ziv–Zakai class [2], derived from a binary hypothesis testing problem, and 2) the Weiss–Weinstein class [3], derived from the covariance inequality [4]. The Ziv–Zakai class contains the Ziv–Zakai [2], Bellini–Tartara [5], Chazan–Zakai–Ziv [6], Weinstein [7], extended Ziv–Zakai [8], and Bell [9] bounds. The Weiss–Weinstein class contains the Bayesian Cramér–Rao (BCR) [10], Bayesian Bhattacharyya (BBH) [3], Bobrovsky–Zakai (BZ) [11], Reuven–Messer (RM) [12], Weiss–Weinstein (WW) [13], Bayesian Abel (BA) [14], and the combined Cramér–Rao/Weiss–Weinstein (CRWW) [15] bounds. Since the RM bound is originally hybrid, we note that it is contained in the Weiss–Weinstein class only under the assumption that all the parameters to be estimated are random.

Applications of Bayesian bounds to several estimation problems can be found in [16]–[18], where the Ziv–Zakai and Weiss–Weinstein bounds were utilized for analyzing estimation performance in an underwater acoustic scenario. The Ziv–Zakai class has been applied in other applications, such as time-delay estimation [19], direction-of-arrival (DOA) estimation [20], [21] and digital communication [22]. In [23] the Weiss–Weinstein bound was applied to data aided carrier estimation. In [15] the CRWW was applied for target bearing tracking.

In this paper, we are concerned with the Weiss–Weinstein class. Bounds in the Weiss–Weinstein class are derived using particular functions, defined on the observation and parameter spaces, which are orthogonal to any function of the observations. Finding such particular functions, for which tight and computationally manageable bounds are obtained, is not an easy task. Therefore, to this day, only a limited variety of bounds in the Weiss–Weinstein class have been introduced. Moreover, in computation of bounds, such as the RM, WW, BA, and the CRWW bounds, it is usually required to evaluate these particular functions at multiple test-points for obtaining tight bounds. This fact consequences inversion of large matrices, and in addition, numerical search methods should be utilized for optimal selection of these test-points.

In this paper, we propose to overcome these disadvantages by the establishment of a new class of Bayesian lower bounds. We begin by showing that Bayesian lower bounds can be derived via projections of each entry of the vector-function to be estimated on some Hilbert spaces. Let $\mathcal{L}_2$ denote the Hilbert space of functions, defined on the observation and parameter spaces, with finite second-order statistical moments, and let $\mathcal{H}_\phi \subset \mathcal{L}_2$ denote the space of functions, which are orthogonal to any function of

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the observations. We show that a class of Bayesian bounds can be derived via projections of each entry of the vector-function to be estimated on closed subspaces of $\mathcal{H}_\phi$, where projection on $\mathcal{H}_\phi$ itself yields the tightest Bayesian lower bound, which is given by the MSE of the minimum mean square error (MMSE) estimator. Modification of these subspaces yields a variety of lower bounds.

Hence, let $\boldsymbol{\psi}$ denote a vector-function, comprised of functions in $\mathcal{H}_\phi$. Using the framework described above, it is shown that the bounds in the Weiss–Weinstein class are obtained by projection of each entry of the vector-function to be estimated on a closed subspace of $\mathcal{H}_\phi$, spanned by the entries of $\boldsymbol{\psi}$. By choosing different $\boldsymbol{\psi}$'s, different subspaces are obtained and hence a variety of bounds can be derived. However, finding functions in $\mathcal{H}_\phi$ from which $\boldsymbol{\psi}$ is comprised, such that tight and computationally manageable bounds are obtained, is not an easy task. Existing bounds are based on $\boldsymbol{\psi}$, which is comprised of *derivatives* and *samples* of functions in $\mathcal{H}_\phi$, drawn from a small set, denoted by $\mathcal{S}$. This fact consequences a limited variety of bounds. Moreover, in many cases, the functions in $\mathcal{S}$ are evaluated at many test points in order to obtain tight bounds at the expense of high computational complexity. We note that this unification approach of bounds in the Weiss–Weinstein class is related to the approach described in [24], according to which lower bounds in the Weiss–Weinstein class can be obtained via constrained minimization over $\mathcal{L}_2$. The relation between these approaches is discussed in this paper.

In order to overcome the disadvantages in existing bounds in the Weiss–Weinstein class, a new class of Bayesian lower bounds is proposed. Following the basic idea, which was first presented in the conference paper [25], the bounds in this class are obtained via projection of each entry of the vector-function to be estimated on a closed subspace of $\mathcal{H}_\phi$. This subspace contains linear transformations of elements in the domain of an integral transform, which is applied on the functions in the set $\mathcal{S}$. By modifying the kernel of the integral transform, different subspaces of $\mathcal{H}_\phi$ are derived, and as a consequence, a variety of bounds is obtained.

The integral transform generalizes the *derivative* and *sampling* operators used for computation of some well-known bounds in the Weiss–Weinstein class. Hence, it is shown that by specific choice of kernels, some well-known bounds in the Weiss–Weinstein class can be derived from the proposed class. In the paper we show that the proposed class is a subclass of the Weiss–Weinstein class. In comparison to the Weiss–Weinstein class, instead of modifying $\boldsymbol{\psi}$, we use a fixed set of functions in $\mathcal{H}_\phi$ (the set $\mathcal{S}$) and modify only the kernel of the integral transform.

In order to obtain tight and computationally manageable bounds, we look for kernels, such that the significant information in the functions in $\mathcal{S}$ is "compressed" into few elements in the domain of the integral transform. In searching for this kind of "compressing" integral transform, we note that in cases where the spectra of the functions in $\mathcal{S}$ are concentrated in a small subset of the frequency domain, the significant information in these functions can be "compressed" into a few frequency components via the Fourier transform. Motivated by this fact, a new lower bound is derived from the proposed class using the kernel of the Fourier transform. It is shown that the proposed bound is computationally manageable and provides better prediction of SNR threshold region exhibited by the maximum *a posteriori* probability (MAP) and MMSE estimators, in the problem of frequency estimation.

The paper is organized as follows: In Section II, it is shown that Bayesian lower bounds in the Weiss–Weinstein class can be obtained via projections of each entry of the vector-function to be estimated, on a closed subspace of $\mathcal{H}_\phi$. In Section III, we show that the bounds in the Weiss–Weinstein class are obtained by projection of each entry of the vector-function to be estimated on a specific closed subspace of $\mathcal{H}_\phi$. In Section IV, a new class of Bayesian lower bounds is derived by applying an integral transform on the functions in $\mathcal{S}$. The relations of the proposed class to the Weiss–Weinstein class is discussed as well. In Section V, it is shown that some well-known Bayesian MSE bounds can be derived from the proposed class by modifying the kernel of the integral transform. In Section VI, a new bound is derived from the proposed class using the kernel of the Fourier transform. In Section VII, the proposed bound is compared with some other known bounds in terms of threshold SNR prediction in the problem of frequency estimation. Section VIII, summarizes the main points of this contribution.

II. Bayesian Lower Bounds Based on Projections in Some Hilbert Subspaces of $\mathcal{L}_2$

Let $\mathcal{L}_2$ denote the Hilbert space of functions defined on the observation and parameter spaces with finite second-order statistical moments, and let $\mathcal{H}_\phi \subset \mathcal{L}_2$ be the space of functions, which are orthogonal to any function of the observations. In this section, it is shown that Bayesian bounds can be derived via projections of each entry of the vector-function to be estimated on closed subspaces of $\mathcal{H}_\phi$. We begin by stating some definitions and assumptions, which will be used in this paper. Afterwards, the Hilbert subspace, $\mathcal{H}_\phi$, is constructed. It is then shown that projection of each entry of the vector-function to be estimated on any closed subspace of $\mathcal{H}_\phi$, denoted by $\mathcal{H}_\varphi$, yields an estimator-independent lower bound on the MSE of any estimator. As a special case, if $\mathcal{H}_\varphi = \mathcal{H}_\phi$ it is shown that the tightest Bayesian lower bound, given by MSE of the MMSE estimator, is obtained.

A. Definitions and Assumptions

1) **Parameter space**:
We assume that the parameter space, $\boldsymbol{\Theta}$, is a subset of $\mathbb{R}^M$ with finite Lebesgue-measure, λ.
2) **Function to be estimated**:
The estimation of $\mathbf{g}(\boldsymbol{\theta})$, where $\mathbf{g}: \boldsymbol{\Theta} \rightarrow \mathbb{R}^L$ is deterministic known and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ is random unknown, is considered. We note that all the functions used in this paper are assumed to be measurable [26].
3) **Observation space**:
An observation space of points, $\mathbf{x}$, is denoted by $\mathcal{X}$.
4) **Probability measure and probability density function**:
Let $\mathcal{P}$ denote a probability measure on $\mathcal{X} \times \boldsymbol{\Theta}$, and let μ denote a σ-finite positive measure on $\mathcal{X}$. It is assumed

that $\mathcal{P}$ is absolutely continuous w.r.t. the product measure $\mu \times \lambda$, such that the Radon–Nikodym derivative [26]

$$f(\mathbf{x}, \boldsymbol{\theta}) = \frac{d\mathcal{P}(\mathbf{x}, \boldsymbol{\theta})}{d\mu(\mathbf{x}) d\boldsymbol{\theta}} \tag{1}$$

exists. The function $f(\mathbf{x}, \boldsymbol{\theta})$ is the joint probability density function (PDF) of $\mathbf{x}$ and $\boldsymbol{\theta}$.

5) **The Hilbert space of $\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})$**:
The Hilbert space of absolutely square integrable functions, $\zeta : \mathcal{X} \times \boldsymbol{\Theta} \to \mathbb{C}$, w.r.t. $\mathcal{P}$ is denoted by $\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})$. The inner-product of $\zeta(\mathbf{x}, \boldsymbol{\theta})$ and $\zeta'(\mathbf{x}, \boldsymbol{\theta})$ in $\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})$ is defined by

$$\begin{aligned} &\langle \zeta(\mathbf{x}, \boldsymbol{\theta}), \zeta'(\mathbf{x}, \boldsymbol{\theta}) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \\ &\quad \triangleq \int_{\mathcal{X} \times \boldsymbol{\Theta}} \zeta(\mathbf{x}, \boldsymbol{\theta}) \zeta'^*(\mathbf{x}, \boldsymbol{\theta}) d\mathcal{P}(\mathbf{x}, \boldsymbol{\theta}) \\ &\quad \triangleq \mathrm{E}_{\mathbf{x}, \boldsymbol{\theta}} [\zeta(\mathbf{x}, \boldsymbol{\theta}) \zeta'^*(\mathbf{x}, \boldsymbol{\theta})] \end{aligned} \tag{2}$$

where $(\cdot)^*$ and $\mathrm{E}_{\mathbf{x}, \boldsymbol{\theta}}[\cdot]$ denote the complex-conjugate and the expectation w.r.t. $f(\mathbf{x}, \boldsymbol{\theta})$, respectively. Hence, $\zeta(\mathbf{x}, \boldsymbol{\theta}) \in \mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})$ if and only if the squared norm

$$\|\zeta(\mathbf{x}, \boldsymbol{\theta})\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \triangleq \langle \zeta(\mathbf{x}, \boldsymbol{\theta}), \zeta(\mathbf{x}, \boldsymbol{\theta}) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \tag{3}$$

is finite.

6) **Estimation error and MSE**:
Let $\hat{\mathbf{g}} : \mathcal{X} \to \mathbb{R}^L$ denote an estimator of $\mathbf{g}(\boldsymbol{\theta})$. The vector of estimation error and the MSE matrix are given by

$$\mathbf{e}_{\hat{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\theta}) \triangleq \mathbf{g}(\boldsymbol{\theta}) - \hat{\mathbf{g}}(\mathbf{x}), \tag{4}$$

and

$$\mathbf{MSE}(\hat{\mathbf{g}}(\mathbf{x})) \triangleq \mathrm{E}_{\mathbf{x}, \boldsymbol{\theta}} \left[\mathbf{e}_{\hat{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\theta}) \mathbf{e}^H_{\hat{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\theta}) \right] \tag{5}$$

respectively, where it is assumed that $\hat{\mathbf{g}}(\mathbf{x}), \mathbf{g}(\boldsymbol{\theta}) \in \mathcal{L}_2^L(\mathcal{X} \times \boldsymbol{\Theta})$ and

$$\mathcal{L}_2^L(\mathcal{X} \times \boldsymbol{\Theta}) \triangleq \underbrace{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta}) \times \cdots \times \mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})}_{L}.$$

B. Construction of the Hilbert Subspace, $\mathcal{H}_\phi \subset \mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})$

In this subsection, the following Hilbert subspace of $\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})$ is constructed.

$$\mathcal{H}_\phi \triangleq \left\{ \phi(\mathbf{x}, \boldsymbol{\theta}) \in \mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta}) \right. \left. : \int_{\boldsymbol{\Theta}} \phi(\mathbf{x}, \boldsymbol{\theta}) f(\mathbf{x}, \boldsymbol{\theta}) d\boldsymbol{\theta} = 0 \text{ for a.e. } \mathbf{x} \in \mathcal{X} \right\} \tag{6}$$

where it is assumed that $\mathcal{H}_\phi$ is complete, i.e., any Cauchy sequence in $\mathcal{H}_\phi$ converges to a limit in $\mathcal{H}_\phi$. Observing (6), one can notice that

$$\zeta(\mathbf{x}) \perp \mathcal{H}_\phi, \forall \zeta(\mathbf{x}) \in \mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta}). \tag{7}$$

C. Bayesian Lower Bounds Based on Projections on Closed Subspaces of $\mathcal{H}_\phi$

Let $\hat{\mathbf{g}} : \mathcal{X} \to \mathbb{R}^L$ denote an estimator of $\mathbf{g}(\boldsymbol{\theta})$. In this subsection, it is shown that projection of each entry of $\mathbf{g}(\boldsymbol{\theta})$ on any closed subspace of $\mathcal{H}_\phi$ yields a lower bound on $\mathbf{MSE}(\hat{\mathbf{g}}(\mathbf{x}))$.

Theorem 1: Let $\hat{\mathbf{g}}(\mathbf{x}) \in \mathcal{L}_2^L(\mathcal{X} \times \boldsymbol{\Theta})$ denote an estimator of $\mathbf{g}(\boldsymbol{\theta})$, and let

$$\begin{aligned} \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \triangleq [&p_{\mathcal{J}}([\mathbf{g}(\boldsymbol{\theta})]_0 \,|\, \mathcal{H}_\varphi), \ldots, \\ &p_{\mathcal{J}}([\mathbf{g}(\boldsymbol{\theta})]_{L-1} \,|\, \mathcal{H}_\varphi)]^T \end{aligned} \tag{8}$$

where $p_{\mathcal{J}}([\mathbf{g}(\boldsymbol{\theta})]_l \,|\, \mathcal{H}_\varphi)$ is the projection of $[\mathbf{g}(\boldsymbol{\theta})]_l$ on $\mathcal{H}_\varphi \subseteq \mathcal{H}_\phi$, and $\mathcal{H}_\varphi$ is closed. Then

$$\begin{aligned} \mathbf{MSE}(\hat{\mathbf{g}}(\mathbf{x})) &\succeq \mathbf{C}_{\mathcal{H}_\varphi} \\ &\triangleq \mathrm{E}_{\mathbf{x}, \boldsymbol{\theta}} \left[\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \mathbf{p}^H_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \right]. \end{aligned} \tag{9}$$

Proof: The Cauchy–Schwartz inequality [26] implies that $\forall \mathbf{r} \in \mathbb{C}^L$

$$\begin{aligned} &\left| \langle \mathbf{r}^H \mathbf{e}_{\hat{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\theta}), \mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \right|^2 \\ &\quad \le \|\mathbf{r}^H \mathbf{e}_{\hat{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\theta})\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \\ &\quad\quad \times \|\mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \end{aligned} \tag{10}$$

where by the definition of $\mathbf{e}_{\hat{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\theta})$ in (4)

$$\begin{aligned} &\langle \mathbf{r}^H \mathbf{e}_{\hat{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\theta}), \mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \\ &\quad = \langle \mathbf{r}^H \mathbf{g}(\boldsymbol{\theta}), \mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \\ &\quad\quad - \langle \mathbf{r}^H \hat{\mathbf{g}}(\mathbf{x}), \mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})}. \end{aligned} \tag{11}$$

According to the Hilbert projection theorem, stated in Appendix A

$$\begin{aligned} &\langle \mathbf{r}^H \mathbf{g}(\boldsymbol{\theta}), \mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \\ &\quad = \langle \mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi), \mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \\ &\quad = \|\mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})}, \forall \mathbf{r} \in \mathbb{C}^L. \end{aligned} \tag{12}$$

Moreover, since according to (12) $\mathbf{r}^H \hat{\mathbf{g}}(\mathbf{x}) \perp \mathcal{H}_\phi$ and $\mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \in \mathcal{H}_\phi$, then

$$\langle \mathbf{r}^H \hat{\mathbf{g}}(\mathbf{x}), \mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} = 0. \tag{13}$$

Hence, substitution of (12) and (13) into (11) yields

$$\begin{aligned} &\langle \mathbf{r}^H \mathbf{e}_{\hat{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\theta}), \mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \\ &\quad = \|\mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})}. \end{aligned} \tag{14}$$

Therefore, according to (10) and (14), $\forall \mathbf{r} \in \mathbb{C}^L$

$$\begin{aligned} &\|\mathbf{r}^H \mathbf{e}_{\hat{\mathbf{g}}}(\mathbf{x}, \boldsymbol{\theta})\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \\ &\quad \ge \|\mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})}. \end{aligned} \tag{15}$$

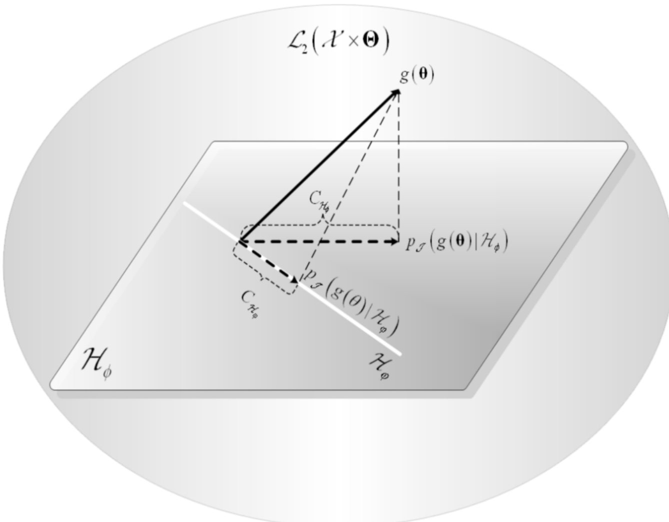

Fig. 1. Geometric interpretation of Bayesian bounds for the one-dimensional case i.e., $L = 1$. The spaces $\mathcal{L}_2(\mathcal{X} \times \mathbf{\Theta})$, $\mathcal{H}_\phi$ and $\mathcal{H}_\varphi$ are illustrated by the spheroid, plane and the white axis, respectively. The terms $p_{\mathcal{J}}(g(\boldsymbol{\theta})\,|\,\mathcal{H}_\phi)$ and $p_{\mathcal{J}}(g(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi)$ (marked by thick dashed arrows) denote the projections of $g(\boldsymbol{\theta})$ (marked by a thick solid arrow) on $\mathcal{H}_\phi$ and $\mathcal{H}_\varphi \subseteq \mathcal{H}_\phi$, respectively. The terms $C_{\mathcal{H}_\phi}$ and $C_{\mathcal{H}_\varphi}$ are the squared norms of $p_{\mathcal{J}}(g(\boldsymbol{\theta})\,|\,\mathcal{H}_\phi)$ and $p_{\mathcal{J}}(g(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi)$ in $\mathcal{L}_2(\mathcal{X} \times \mathbf{\Theta})$, respectively.

Thus, by (2) and (3) it is implied that $\forall \mathbf{r} \in \mathbb{C}^L$

$$\mathbf{r}^H \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{e}_{\hat{\mathbf{g}}}(\mathbf{x},\boldsymbol{\theta})\mathbf{e}_{\hat{\mathbf{g}}}^H(\mathbf{x},\boldsymbol{\theta})\right]\mathbf{r} \geq \mathbf{r}^H \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi)\mathbf{p}_{\mathcal{J}}^H(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi)\right]\mathbf{r}. \quad (16)$$

Finally, since the matrices $\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\mathbf{e}_{\hat{\mathbf{g}}}(\mathbf{x},\boldsymbol{\theta})\mathbf{e}_{\hat{\mathbf{g}}}^H(\mathbf{x},\boldsymbol{\theta})]$ and $\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi)\mathbf{p}_{\mathcal{J}}^H(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi)]$ are Hermitian, it is implied by (5) and (16) that the semi-inequality in (9) holds. ■

Geometric interpretation of $\mathbf{C}_{\mathcal{H}_\varphi}$ is depicted in Fig. 1 for the one-dimensional case, i.e., $L = 1$.

D. The Tightest Bayesian Lower Bound

In this subsection, it is shown that projection of each entry of $\mathbf{g}(\boldsymbol{\theta})$ on $\mathcal{H}_\varphi = \mathcal{H}_\phi$ yields the tightest Bayesian lower bound, which is given by the MSE matrix of the MMSE estimator.

Theorem 2: Let

$$\hat{\mathbf{g}}_{\mathrm{MMSE}}(\mathbf{x}) \triangleq \mathrm{E}_{\boldsymbol{\theta}\,|\,\mathbf{x}}[\mathbf{g}(\boldsymbol{\theta})] \quad (17)$$

denote the MMSE estimator of $\mathbf{g}(\boldsymbol{\theta})$, where $\mathrm{E}_{\boldsymbol{\theta}\,|\,\mathbf{x}}[\cdot]$ is the conditional expectation given $\mathbf{x}$. Then

$$\mathbf{MSE}(\hat{\mathbf{g}}_{\mathrm{MMSE}}(\mathbf{x})) = \mathbf{C}_{\mathcal{H}_\phi} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\phi)\mathbf{p}_{\mathcal{J}}^H(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\phi)\right]. \quad (18)$$

Proof: Since $\hat{\mathbf{g}}_{\mathrm{MMSE}}(\mathbf{x})$, $\mathbf{g}(\boldsymbol{\theta}) \in \mathcal{L}_2^L(\mathcal{X} \times \mathbf{\Theta})$, then by using (6) and (17), one can verify that

$$\mathbf{e}_{\hat{\mathbf{g}}_{\mathrm{MMSE}}}(\mathbf{x},\boldsymbol{\theta}) \triangleq \mathbf{g}(\boldsymbol{\theta}) - \hat{\mathbf{g}}_{\mathrm{MMSE}}(\mathbf{x}) \in \mathcal{H}_\phi^L \quad (19)$$

where $\mathcal{H}_\phi^L \triangleq \underbrace{\mathcal{H}_\phi \times \cdots \times \mathcal{H}_\phi}_{L}$. According to the Hilbert projection theorem, stated in Appendix A, there exists a unique $\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\phi) \in \mathcal{H}_\phi^L$, such that

$$\mathbf{r}^H(\mathbf{g}(\boldsymbol{\theta}) - \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\phi)) \perp \mathcal{H}_\phi, \quad \forall \mathbf{r} \in \mathbb{C}^L. \quad (20)$$

Hence, according to (7) and (19), one can notice that (20) holds for

$$\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\phi) = \mathbf{e}_{\hat{\mathbf{g}}_{\mathrm{MMSE}}}(\mathbf{x},\boldsymbol{\theta}). \quad (21)$$

Therefore, according to (5), it is concluded that the equality in (18) holds. ■

Geometric interpretation of $\mathbf{C}_{\mathcal{H}_\phi}$ is depicted in Fig. 1 for the one-dimensional case, i.e., $L = 1$.

In conclusion, by Theorems 2 and 3, it is implied that $\forall \hat{\mathbf{g}}(\mathbf{x}) \in \mathcal{L}_2^L(\mathcal{X} \times \mathbf{\Theta})$

$$\mathbf{MSE}(\hat{\mathbf{g}}(\mathbf{x})) \succeq \mathbf{MSE}(\hat{\mathbf{g}}_{\mathrm{MMSE}}(\mathbf{x})) = \mathbf{C}_{\mathcal{H}_\phi} \succeq \mathbf{C}_{\mathcal{H}_\varphi}. \quad (22)$$

Hence, by modifying $\mathcal{H}_\varphi$, a variety of bounds can be obtained, where $\mathbf{C}_{\mathcal{H}_\phi}$ is the tightest lower bound on the MSE of any estimator in $\mathcal{L}_2^L(\mathcal{X}\times\mathbf{\Theta})$. However, according to (17), calculation of $\mathbf{C}_{\mathcal{H}_\phi}$ requires the derivation of $\hat{\mathbf{g}}_{\mathrm{MMSE}}(\mathbf{x})$. In many cases this task is analytically impossible and consequently $\mathbf{C}_{\mathcal{H}_\phi}$ is practically incomputable. Therefore, it is preferable to use $\mathbf{C}_{\mathcal{H}_\varphi}$ instead of $\mathbf{C}_{\mathcal{H}_\phi}$, though it is less tight than $\mathbf{C}_{\mathcal{H}_\phi}$. In the following section, we use the framework presented above, for derivation of the Weiss–Weinstein class of bounds.

III. The Weiss–Weinstein Class

In this section, it is shown that the Weiss–Weinstein class of bounds is obtained via projection of each entry of $\mathbf{g}(\boldsymbol{\theta})$ on a closed subspace of $\mathcal{H}_\phi$, which contains linear combinations of elements in $\mathcal{H}_\phi$. Derivation of some existing bounds in the Weiss–Weinstein class is described, and their disadvantages are discussed as well. Finally, the relation between this approach for unification of bounds in the Weiss–Weinstein class and the approach described in [24] is discussed.

A. Construction of a Closed Subspace of $\mathcal{H}_\phi$

In this subsection, a closed subspace of $\mathcal{H}_\phi$ is constructed in the following manner. Let $\boldsymbol{\psi} : \mathcal{X} \times \mathbf{\Theta} \to \mathbb{C}^N$, where $\boldsymbol{\psi} \in \mathcal{H}_\phi^N$ and $\mathcal{H}_\phi^N \triangleq \underbrace{\mathcal{H}_\phi \times \cdots \times \mathcal{H}_\phi}_{N}$. The following space is constructed:

$$\mathcal{H}_\varphi^{(\boldsymbol{\psi})} \triangleq \{\varphi_{\mathbf{a},\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta}) \triangleq \mathbf{a}^H\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta}) : \mathbf{a} \in \mathbb{C}^N\}. \quad (23)$$

Since any Cauchy sequence in $\mathcal{H}_\varphi^{(\boldsymbol{\psi})}$ converges to a limit in $\mathcal{H}_\varphi^{(\boldsymbol{\psi})}$, it is concluded that $\mathcal{H}_\varphi^{(\boldsymbol{\psi})}$ is complete, and hence, it is a closed subspace of $\mathcal{H}_\phi$.

B. Derivation of the Weiss–Weinstein Class

In this subsection, the Weiss–Weinstein class of bounds is derived in the following manner. First, the vector of projections

$$\mathbf{p}_{\mathcal{J}}\left(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}\right) \triangleq \left[p_{\mathcal{J}}\left([\mathbf{g}(\boldsymbol{\theta})]_0\,|\,\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}\right),\ldots,\; p_{\mathcal{J}}\left([\mathbf{g}(\boldsymbol{\theta})]_{L-1}\,|\,\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}\right)\right]^T \quad (24)$$

is obtained. According to the Hilbert projection theorem stated in Appendix A, $p_{\mathcal{J}}([\mathbf{g}(\boldsymbol{\theta})]_l\,|\,\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})})$ is the unique solution of the following system of equations:

$$\left\langle p_{\mathcal{J}}\left([\mathbf{g}(\boldsymbol{\theta})]_l\,|\,\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}\right), \varphi_{\mathbf{a},\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta})\right\rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})} = \left\langle [\mathbf{g}(\boldsymbol{\theta})]_l, \varphi_{\mathbf{a},\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta})\right\rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})}, \quad \forall \varphi_{\mathbf{a},\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}. \quad (25)$$

Hence, since $p_{\mathcal{J}}([\mathbf{g}(\boldsymbol{\theta})]_l\,|\,\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}) \in \mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}$, let

$$\varphi_{\tilde{\mathbf{a}}_l,\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta}) \triangleq p_{\mathcal{J}}\left([\mathbf{g}(\boldsymbol{\theta})]_l\,|\,\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}\right) = \tilde{\mathbf{a}}_l^H \boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta}) \quad (26)$$

where the second equality in (26) stems from (23). According to (2), (25) and (26)

$$\tilde{\mathbf{a}}_l^H \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\psi}^H(\mathbf{x},\boldsymbol{\theta})]\mathbf{a} = \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[[\mathbf{g}(\boldsymbol{\theta})]_l \boldsymbol{\psi}^H(\mathbf{x},\boldsymbol{\theta})]\mathbf{a}, \quad \forall \mathbf{a} \in \mathbb{C}^N. \quad (27)$$

Therefore, it is concluded from (27) that

$$\tilde{\mathbf{a}}_l = \mathrm{E}^{-1}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\psi}^H(\mathbf{x},\boldsymbol{\theta})]\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta})[\mathbf{g}(\boldsymbol{\theta})]_l] \quad (28)$$

where it is assumed that $\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\psi}^H(\mathbf{x},\boldsymbol{\theta})]$ is nonsingular. Hence, substitution of (28) into (26) and using the definition of $\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})})$ in (24) yields

$$\mathbf{p}_{\mathcal{J}}\left(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}\right) = \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\mathbf{g}(\boldsymbol{\theta})\boldsymbol{\psi}^H(\mathbf{x},\boldsymbol{\theta})]\mathrm{E}^{-1}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\psi}^H(\mathbf{x},\boldsymbol{\theta})]\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta}). \quad (29)$$

Second, according to (9) and (29), the Weiss–Weinstein class [3] is given by

$$\begin{aligned}\mathbf{C}_{\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}} &\triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{p}_{\mathcal{J}}\left(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}\right)\mathbf{p}_{\mathcal{J}}^H\left(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}\right)\right] \\ &= \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\mathbf{g}(\boldsymbol{\theta})\boldsymbol{\psi}^H(\mathbf{x},\boldsymbol{\theta})]\mathrm{E}^{-1}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\psi}^H(\mathbf{x},\boldsymbol{\theta})] \\ &\quad \times \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta})\mathbf{g}^T(\boldsymbol{\theta})]. \end{aligned} \quad (30)$$

By modifying $\boldsymbol{\psi}(\cdot,\cdot)$, the subspace $\mathcal{H}_{\varphi}^{(\boldsymbol{\psi})}$ is modified and a variety of bounds can be obtained from (30).

C. Existing Bounds in the Weiss–Weinstein Class

Existing bounds in the Weiss–Weinstein class can be derived by choosing $\boldsymbol{\psi}(\cdot,\cdot)$, which is composed of *derivatives* and *samples* of functions drawn from the set

$$\mathcal{S} \triangleq \{\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}), \nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})\} \quad (31)$$

where $\boldsymbol{\tau} \in \boldsymbol{\Lambda}$, $\boldsymbol{\Lambda}$ is a subset of $\mathbb{R}^M$ with finite Lebesgue-measure

$$\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \triangleq \rho(\mathbf{x};\boldsymbol{\theta}+\boldsymbol{\tau},\boldsymbol{\theta}) - 1, \quad (32)$$

$$\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \triangleq \rho^{\beta(\boldsymbol{\tau})}(\mathbf{x};\boldsymbol{\theta}+\boldsymbol{\tau},\boldsymbol{\theta}) - \rho^{(1-\beta(\boldsymbol{\tau}))}(\mathbf{x};\boldsymbol{\theta}-\boldsymbol{\tau},\boldsymbol{\theta}), 0 < \beta(\boldsymbol{\tau}) < 1, \quad (33)$$

$$\rho(\mathbf{x};\boldsymbol{\xi},\boldsymbol{\theta}) \triangleq \begin{cases} \frac{f(\mathbf{x},\boldsymbol{\xi})}{f(\mathbf{x},\boldsymbol{\theta})}, & f(\mathbf{x},\boldsymbol{\theta}) > 0 \\ 0, & \text{otherwise} \end{cases}. \quad (34)$$

$\boldsymbol{\xi} \in \boldsymbol{\Theta}$, and it is assumed that $\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})$, $\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \in \mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})\forall\boldsymbol{\tau}\in\boldsymbol{\Lambda}$. Similar to the proof of Proposition 1 in Part I [1], it can be shown using Holder's inequality [26], and the Tonelli and Fubini theorems [26], that $\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})$, $\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \in \mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})\forall\boldsymbol{\tau}\in\boldsymbol{\Lambda}$ implies that $\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})$, $\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \in \mathcal{L}_1(\boldsymbol{\Lambda})$, for a.e. $\mathbf{x}\in\mathcal{X}$ and $\boldsymbol{\theta}\in\boldsymbol{\Theta}$, where $\mathcal{L}_1(\boldsymbol{\Lambda})$ denotes the space of absolutely integrable functions on $\boldsymbol{\Lambda}$. We note that given $\boldsymbol{\tau}\in\boldsymbol{\Lambda}$, $\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \in \mathcal{H}_{\phi}$ s.t. the condition $\int_{\boldsymbol{\Theta}} f(\mathbf{x},\boldsymbol{\theta})d\boldsymbol{\theta} = \int_{\boldsymbol{\Theta}} f(\mathbf{x},\boldsymbol{\theta}+\boldsymbol{\tau})d\boldsymbol{\theta}$. One can also notice that $\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \in \mathcal{H}_{\phi}\forall\boldsymbol{\tau}\in\boldsymbol{\Lambda}$.

For example, the Bayesian Cramér–Rao bound [10] is obtained by choosing

$$\boldsymbol{\psi}_{\mathrm{BCR}}(\mathbf{x},\boldsymbol{\theta}) = \left(\left.\frac{\partial \nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})}{\partial\boldsymbol{\tau}}\right|_{\boldsymbol{\tau}=\mathbf{0}}\right)^T = \left(\frac{\partial \log f(\mathbf{x},\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right)^T \quad (35)$$

s.t. some conditions on $\mathbf{g}(\boldsymbol{\theta})$ and $f(\mathbf{x},\boldsymbol{\theta})$, which are detailed in Section V, where the partial derivative w.r.t. $\boldsymbol{\theta} = [\theta_0,\ldots,\theta_{M-1}]^T$ is defined as $\frac{\partial}{\partial\boldsymbol{\theta}} \triangleq [\frac{\partial}{\partial\theta_0},\ldots,\frac{\partial}{\partial\theta_{M-1}}]$.

The Reuven–Messer bound [12] is obtained by choosing

$$\boldsymbol{\psi}_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta}) = [\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_0),\ldots,\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_{N-1})]^T \quad (36)$$

where $\boldsymbol{\tau}_n$, $n = 0,\ldots,N-1$, denote test points in $\boldsymbol{\Lambda}$.

The Weiss–Weinstein bound [13] is obtained by choosing

$$\boldsymbol{\psi}_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta}) = [\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_0),\ldots,\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_{N-1})]^T. \quad (37)$$

One can notice that due to the fact that the set $\mathcal{S}$ contains only two functions and $\boldsymbol{\psi}(\cdot,\cdot)$ is composed of only *derivatives* and *samples* of functions in $\mathcal{S}$, a small variety of $\boldsymbol{\psi}$'s can be obtained, which consequences a limited variety of bounds. Moreover, in many cases, the functions in $\mathcal{S}$ are evaluated at many test points in order to derive tight bounds at the expense of high computational complexity. In Section IV, we propose to overcome these disadvantages by the establishment of a new class of bounds. This class of bounds is based on applying an integral transform, which generalizes the *derivative* and *sampling* operators, applied on the functions in $\mathcal{S}$.

D. Relation to the Unification Approach Described in [24]

In [24], a different approach for unifying the bounds in the Weiss–Weinstein class was proposed. According to this approach, lower bounds in the Weiss–Weinstein class can be obtained via constrained minimization over $\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$. Using some equivalent sets of constraints, denoted by $\mathcal{C}_i$, $i \in 1,\ldots,4$, it was shown that the tightest Bayesian lower bound is obtained. Moreover, by specific choice of subsets of these constraints, it was proved in [24] that some well-known bounds in the Weiss–Weinstein class can be derived. In relation to the proposed unification approach described above, it can be shown by the Hilbert projection theorem, stated in Appendix A, that projection of each entry of $\mathbf{g}(\boldsymbol{\theta})$ on closed subspaces of $\mathcal{H}_\phi$ is equivalent to the constrained minimization over $\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$ using subsets of $\mathcal{C}_i$, $i\in 1,\ldots,4$. For example, it can be shown that derivation of the Bayesian Cramér–Rao bound [10] via projection on $\mathcal{H}_\varphi^{(\boldsymbol{\psi}_{\rm BCR})}\subset\mathcal{H}_\phi$, where $\boldsymbol{\psi}_{\rm BCR}(\mathbf{x},\boldsymbol{\theta})$ is defined in (35), is equivalent to the constraint minimization using $\mathcal{P}_2\subset\mathcal{C}_2$, where $\mathcal{C}_2$ and $\mathcal{P}_2$ are defined in (10) and (20) in [24], respectively.

IV. A New Class of Bayesian Lower Bounds

In this section, a new class of Bayesian lower bounds is derived via projection of each entry of $\mathbf{g}(\boldsymbol{\theta})$ on a closed subspace of $\mathcal{H}_\phi$. This subspace contains linear transformation of elements in the domain of an integral transform, which is applied on the functions in the set $\mathcal{S}$, defined in (31). Derivation of the proposed class is carried out via the following steps. First, a closed subspace of $\mathcal{H}_\phi$ is constructed. Second, the result of Theorem 2 is applied in order to derive the proposed class. The relation of the proposed class to the Weiss–Weinstein class is discussed as well.

A. Construction of a Closed Subspace of $\mathcal{H}_\phi$

In this subsection, a closed subspace of $\mathcal{H}_\phi$ is constructed via the following steps. First, let

$$\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})\triangleq[\nu_{\rm RM}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}),\nu_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})]^T \quad (38)$$

where $\nu_{\rm RM}(\cdot,\cdot,\boldsymbol{\tau})$ and $\nu_{\rm WW}(\cdot,\cdot,\boldsymbol{\tau})$ are defined in (32) and (33), respectively. An integral transform on $\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})$ is defined by

$$\begin{aligned}\boldsymbol{\eta}_{\mathbf{H}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\alpha})&\triangleq(\mathcal{T}_{\mathbf{H}}\boldsymbol{\nu})(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\alpha})\\&=\int_{\boldsymbol{\Lambda}}\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\,\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau}\end{aligned} \quad (39)$$

where $\mathbf{H}:\mathbf{V}\times\boldsymbol{\Lambda}\rightarrow\mathbb{C}^{P\times 2}$ is the kernel of $\mathcal{T}_{\mathbf{H}}$, $\mathbf{V}\subset\mathbb{R}^M$ is a measurable space with finite Lebesgue-measure, and $\boldsymbol{\alpha}\in\mathbf{V}$.

Second, given $\mathbf{H}(\cdot,\cdot)$, the following space is constructed:

$$\mathcal{H}_\varphi^{(\mathbf{H})}\triangleq\left\{\varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta})\triangleq\int_{\mathbf{V}}\mathbf{a}^H(\boldsymbol{\alpha})\boldsymbol{\eta}_{\mathbf{H}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\alpha})d\boldsymbol{\alpha}\;:\;\mathbf{a}(\boldsymbol{\alpha})\in\mathcal{L}_1^P(\mathbf{V})\right\} \quad (40)$$

where $\mathbf{a}:\boldsymbol{\Lambda}\rightarrow\mathbb{C}^P$, $\mathcal{L}_1^P(\mathbf{V})\triangleq\underbrace{\mathcal{L}_1(\mathbf{V})\times\cdots\times\mathcal{L}_1(\mathbf{V})}_{P}$, and $\mathcal{L}_1(\mathbf{V})$ denotes the space of absolutely integrable functions in $\mathbf{V}$. In Appendix B, it is shown that $\mathcal{H}_\varphi^{(\mathbf{H})}\subset\mathcal{H}_\phi$ s.t. the condition that $\forall\boldsymbol{\alpha}\in\mathbf{V}$, $\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\in\mathcal{L}_1^{P\times 2}(\boldsymbol{\Lambda})\triangleq\mathcal{L}_1^P(\boldsymbol{\Lambda})\times\mathcal{L}_1^P(\boldsymbol{\Lambda})$, i.e., $\forall\boldsymbol{\alpha}\in\mathbf{V}$, each entry of the matrix-function $\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})$ is absolutely integrable in $\boldsymbol{\Lambda}$. Hence, assuming that the above condition holds and $\mathcal{H}_\varphi^{(\mathbf{H})}$ is complete, i.e., any Cauchy sequence in $\mathcal{H}_\varphi^{(\mathbf{H})}$ converges to a limit in $\mathcal{H}_\varphi^{(\mathbf{H})}$, then $\mathcal{H}_\varphi^{(\mathbf{H})}$ is a closed subspace of $\mathcal{H}_\phi$.

B. The Proposed Class of Bounds

In this subsection, we use the result of Theorem 2 in order to derive the proposed class of bounds. Since $\mathcal{H}_\varphi^{(\mathbf{H})}$ is a closed subspace of $\mathcal{H}_\phi$, then according to (9) the proposed class is given by

$$\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H})}}\triangleq{\rm E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{p}_{\mathcal{J}}\left(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi^{(\mathbf{H})}\right)\mathbf{p}_{\mathcal{J}}^H\left(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi^{(\mathbf{H})}\right)\right]. \quad (41)$$

In Appendix C, it is shown that a closed form expression of $\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H})}}$ is given by

$$\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H})}}=\int_{\mathbf{V}}\int_{\mathbf{V}}\tilde{\mathbf{A}}^H(\boldsymbol{\alpha})\mathbf{K}_{\mathbf{H}}(\boldsymbol{\alpha},\boldsymbol{\alpha}')\,\tilde{\mathbf{A}}(\boldsymbol{\alpha}')\,d\boldsymbol{\alpha}'d\boldsymbol{\alpha} \quad (42)$$

where

$$\mathbf{K}_{\mathbf{H}}(\boldsymbol{\alpha},\boldsymbol{\alpha}')=\int_{\boldsymbol{\Lambda}}\int_{\boldsymbol{\Lambda}}\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\mathbf{K}_{\boldsymbol{\nu}}(\boldsymbol{\tau},\boldsymbol{\tau}')\mathbf{H}^H(\boldsymbol{\alpha}',\boldsymbol{\tau}')d\boldsymbol{\tau}'d\boldsymbol{\tau}, \quad (43)$$

$$\mathbf{K}_{\boldsymbol{\nu}}(\boldsymbol{\tau},\boldsymbol{\tau}')={\rm E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})\boldsymbol{\nu}^T(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}')]. \quad (44)$$

$\tilde{\mathbf{A}}(\boldsymbol{\alpha})$ is the solution of the following integral equation:

$$\int_{\mathbf{V}}\mathbf{K}_{\mathbf{H}}(\boldsymbol{\alpha},\boldsymbol{\alpha}')\tilde{\mathbf{A}}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}'=\int_{\boldsymbol{\Lambda}}\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\boldsymbol{\Gamma}(\boldsymbol{\tau})d\boldsymbol{\tau}, \quad (45)$$

and

$$\boldsymbol{\Gamma}(\boldsymbol{\tau})\triangleq{\rm E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})\mathbf{g}^T(\boldsymbol{\theta})]. \quad (46)$$

The bound in (42) constitutes a new class of lower bounds. By modifying $\mathbf{H}(\cdot,\cdot)$, the subspace $\mathcal{H}_\varphi^{(\mathbf{H})}$ is modified and a variety of bounds can be obtained from the proposed class. In order to obtain tight and computationally manageable bounds, we look for kernels, such that the significant information in $\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})$ is compressed into few elements in the domain of the integral transform. Finally, let $\mathcal{H}_\varphi$ and $\mathcal{H}'_\varphi$ denote closed subspaces of $\mathcal{H}_\phi$. According to Theorem 5 in Appendix D, if $\mathcal{H}_\varphi\supset\mathcal{H}'_\varphi$ then $\mathbf{C}_{\mathcal{H}_\varphi}\succeq\mathbf{C}_{\mathcal{H}'_\varphi}$. Therefore, it is concluded that order relation between any two bounds, $\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H}_1)}}$ and $\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H}_2)}}$ can be obtained by comparing the Hilbert subspaces $\mathcal{H}_\varphi^{(\mathbf{H}_1)}$ and $\mathcal{H}_\varphi^{(\mathbf{H}_2)}$.

C. Relation of the Proposed Class to the Weiss–Weinstein Class

In this subsection, the relation of the proposed class of bounds in (41) to the Weiss–Weinstein class in (30) is discussed. First, we prove the following identity.

Proposition 1:

$$\begin{aligned}&{\rm E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{p}_{\mathcal{J}}\left(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi^{(\mathbf{H})}\right)\mathbf{p}_{\mathcal{J}}^H\left(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi^{(\mathbf{H})}\right)\right]\\&={\rm E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{g}(\boldsymbol{\theta})\mathbf{p}_{\mathcal{J}}^H\left(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi^{(\mathbf{H})}\right)\right]\\&={\rm E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{p}_{\mathcal{J}}\left(\mathbf{g}(\boldsymbol{\theta})\,|\,\mathcal{H}_\varphi^{(\mathbf{H})}\right)\mathbf{g}^T(\boldsymbol{\theta})\right].\end{aligned} \quad (47)$$

Proof: According to the Hilbert projection theorem stated in Appendix A

$$\left\langle p_{\mathcal{J}}\left([\mathbf{g}(\boldsymbol{\theta})]_k \,|\, \mathcal{H}_\varphi^{(\mathbf{H})}\right), \varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta})\right\rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})} = \langle [\mathbf{g}(\boldsymbol{\theta})]_k, \varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta})\rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})}, \quad \forall \varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{H}_\varphi^{(\mathbf{H})}. \tag{48}$$

Therefore, since

$$p_{\mathcal{J}}\left([\mathbf{g}(\boldsymbol{\theta})]_k \,|\, \mathcal{H}_\varphi^{(\mathbf{H})}\right) \in \mathcal{H}_\varphi^{(\mathbf{H})} \forall k = 0,\ldots,L-1$$

it is implied that

$$\langle p_{\mathcal{J}}\left([\mathbf{g}(\boldsymbol{\theta})]_k \,|\, \mathcal{H}_\varphi^{(\mathbf{H})}\right), p_{\mathcal{J}}\left([\mathbf{g}(\boldsymbol{\theta})]_l \,|\, \mathcal{H}_\varphi^{(\mathbf{H})}\right)\rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})} = \langle [\mathbf{g}(\boldsymbol{\theta})]_k, p_{\mathcal{J}}\left([\mathbf{g}(\boldsymbol{\theta})]_l \,|\, \mathcal{H}_\varphi^{(\mathbf{H})}\right)\rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})}. \tag{49}$$

$\forall k, l = 0,\ldots,L-1$. Rewriting (49) in a matrix form and using the definitions in (2) and (8) yields the desired identity in (47). ■

Second, by substituting $\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta}) = \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi^{(\mathbf{H})})$ in (30) and using the identity in (47), it is concluded from (41) that $\mathbf{C}_{\mathcal{H}_\varphi^{(\boldsymbol{\psi})}} = \mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H})}}$. Hence, the proposed class of bounds is a subclass of the Weiss–Weinstein class. In comparison to the Weiss–Weinstein class, instead of modifying $\boldsymbol{\psi}(\cdot,\cdot)$, we use the functions in $\mathcal{S}$ and modify only the kernel of the integral transform. Using this approach, a wide variety of bounds can be easily obtained. In Section V, it is shown that some well-known bounds in the Weiss–Weinstein class can be derived from $\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H})}}$, via specific choices of $\mathbf{H}(\cdot,\cdot)$.

V. Derivation of Existing Bounds From the Proposed Class of Bounds

The integral transform generalizes the *derivative* and *sampling* operators used for computation of some well-known lower bounds in the Weiss–Weinstein class. Hence, in this section it is shown that some well-known bounds can be derived from the proposed class in (42) by specific choices of the kernel, $\mathbf{H}(\cdot,\cdot)$. We begin with derivation of a new subclass of bounds from the proposed class in (42), using a class of kernel functions with a specific form. It is then shown that some well-known bounds are the limits of convergent sequences of bounds, which are obtained from the proposed subclass.

A. Subclass of Lower Bounds Using Structured Kernel Functions

In this subsection, a new subclass of Bayesian lower bounds is derived from the proposed class in (42) for the case where $\mathbf{H}(\cdot,\cdot)$ is of the form

$$\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau}) = \mathbf{U}(\boldsymbol{\tau}), \forall \boldsymbol{\alpha} \in \mathbf{V} \tag{50}$$

where $\mathbf{U}(\boldsymbol{\tau}) \in \mathcal{L}_1^{P\times 2}(\boldsymbol{\Lambda})$. Hence, according to (42), (43), and (50)

$$\mathbf{C}_\mathbf{U} \triangleq \mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{U})}} = \left(\int_{\boldsymbol{V}} \tilde{\mathbf{A}}^H(\boldsymbol{\alpha})d\boldsymbol{\alpha}\right)\mathbf{K}_\mathbf{U}\left(\int_{\boldsymbol{V}} \tilde{\mathbf{A}}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}'\right) \tag{51}$$

where

$$\mathbf{K}_\mathbf{U} \triangleq \int_{\boldsymbol{\Lambda}}\int_{\boldsymbol{\Lambda}} \mathbf{U}(\boldsymbol{\tau})\mathbf{K}_{\boldsymbol{\nu}}(\boldsymbol{\tau},\boldsymbol{\tau}')\mathbf{U}^H(\boldsymbol{\tau}')d\boldsymbol{\tau}'d\boldsymbol{\tau}. \tag{52}$$

Using (43), (45), and (50) it is implied that

$$\int_{\boldsymbol{V}} \tilde{\mathbf{A}}(\boldsymbol{\alpha})d\boldsymbol{\alpha} = \mathbf{K}_\mathbf{U}^{-1}\mathbf{S}_\mathbf{U} \tag{53}$$

where

$$\mathbf{S}_\mathbf{U} \triangleq \int_{\boldsymbol{\Lambda}} \mathbf{U}(\boldsymbol{\tau})\boldsymbol{\Gamma}(\boldsymbol{\tau})d\boldsymbol{\tau} \tag{54}$$

and it is assumed that $\mathbf{K}_\mathbf{U}$ is nonsingular. Therefore, substituting (53) into (51) yields the following subclass:

$$\mathbf{C}_\mathbf{U} = \mathbf{S}_\mathbf{U}^H\mathbf{K}_\mathbf{U}^{-1}\mathbf{S}_\mathbf{U}. \tag{55}$$

By modifying $\mathbf{U}(\cdot)$, a variety of bounds can be derived from (55).

B. Derivation of Existing Bayesian Bounds From the Subclass $\mathbf{C}_\mathbf{U}$

In this subsection, it is shown that some well-known Bayesian bounds are the limits of convergent sequences of bounds, which are obtained from the proposed subclass in (55). In a more detailed manner, derivation of each bound is carried out via the following procedure:

1) A sequence of functions, $\{\mathbf{U}_k(\cdot)\}$, is constructed using the following sequence of auxiliary "test-functions." Let $h : \mathbb{R}^M \to \mathbb{R}$ denote an infinitely differentiable, symmetric, and compactly supported "test-function," such that $\int_{\mathbb{R}^M} h(\mathbf{y})d\mathbf{y} = 1, \mathbf{y} \in \mathbb{R}^M$. A sequence of "test-functions" is given by

$$\{h_k(\mathbf{y})\} \triangleq \{k^M h(k\mathbf{y})\}, \quad k = 1,2,\ldots \tag{56}$$

We note that $\lim_{k\to\infty} h_k(\mathbf{y}) = \delta(\mathbf{y})$, where $\delta(\cdot)$ is the Dirac's delta function. For example, we can choose

$$h(\mathbf{y}) = \prod_{m=0}^{M-1} h'(y_m) \tag{57}$$

where $y_m \in \mathbb{R}, m = 0,\ldots,M-1$ denote the entries of $\mathbf{y}$, such that

$$h'(y) = \frac{h''(y)}{\int_{\mathbb{R}} h''(y)dy} \tag{58}$$

and

$$h''(y) \triangleq \begin{cases} \exp\left(-\frac{1}{1-y^2}\right), & |y| < 1 \\ 0, & \text{otherwise} \end{cases}. \tag{59}$$

For each bound, $\{\mathbf{U}_k(\cdot)\}$ is constructed using $\{h_k(\cdot)\}$ in a different manner, as will be detailed in the sequel.

2) Using $\{\mathbf{U}_k(\cdot)\}$, a sequence of bounds, $\{\mathbf{C}_{\mathbf{U}_k}\}$ is derived from (55). The desired bound is given by

$$\mathbf{C} \triangleq \lim_{k\to\infty} \mathbf{C}_{\mathbf{U}_k} = \lim_{k\to\infty} \mathbf{S}_{\mathbf{U}_k}^H (\lim_{k\to\infty} \mathbf{K}_{\mathbf{U}_k})^{-1} \lim_{k\to\infty} \mathbf{S}_{\mathbf{U}_k} \tag{60}$$

where it is assumed that $\{\mathbf{S}_{\mathbf{U}_k}\}$ and $\{\mathbf{K}_{\mathbf{U}_k}\}$ converge as $k \to \infty$, and the matrix $\lim_{k\to\infty} \mathbf{K}_{\mathbf{U}_k}$ is nonsingular. The second equality in (60) can be verified using basic properties of convergent sequence limits. In Appendix E, it is shown that

$$\lim_{k\to\infty} \mathbf{S}_{\mathbf{U}_k} = \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\gamma}(\mathbf{x},\boldsymbol{\theta})\mathbf{g}^T(\boldsymbol{\theta})] \quad (61)$$

and

$$\lim_{k\to\infty} \mathbf{K}_{\mathbf{U}_k} = \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\gamma}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\gamma}^H(\mathbf{x},\boldsymbol{\theta})] \quad (62)$$

where

$$\boldsymbol{\gamma}(\mathbf{x},\boldsymbol{\theta}) \triangleq \lim_{k\to\infty} \boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta}), \quad (63)$$

and

$$\boldsymbol{\eta}_{\mathbf{U}}(\mathbf{x},\boldsymbol{\theta}) \triangleq \int_{\boldsymbol{\Lambda}} \mathbf{U}(\boldsymbol{\tau})\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau}. \quad (64)$$

We note that the limit notation in (63) means that $\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta}) \to \boldsymbol{\gamma}(\mathbf{x},\boldsymbol{\theta})$ for a.e. $(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{X} \times \boldsymbol{\Theta}$.

Hence, substitution of (61) and (62) into (60) yields

$$\mathbf{C} = \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\mathbf{g}(\boldsymbol{\theta})\boldsymbol{\gamma}^H(\mathbf{x},\boldsymbol{\theta})]\mathrm{E}^{-1}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\gamma}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\gamma}^H(\mathbf{x},\boldsymbol{\theta})] \times \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\gamma}(\mathbf{x},\boldsymbol{\theta})\mathbf{g}^T(\boldsymbol{\theta})]. \quad (65)$$

By applying the approach described above, we show that some well-known bounds in the Weiss–Weinstein class [3] can be obtained from (65).

1) **The Bayesian Cramér–Rao bound** [10]: The BCR bound is obtained via the following steps:

a) Construction of $\{\mathbf{U}_{\mathrm{BCR},k}(\cdot)\}$: The kth member of $\{\mathbf{U}_{\mathrm{BCR},k}(\cdot)\}$ is given by

$$\mathbf{U}_{\mathrm{BCR},k}(\boldsymbol{\tau}) = \left[-\left(\frac{dh_k(-\boldsymbol{\tau})}{d\boldsymbol{\tau}}\right)^T \quad \mathbf{0}_{M\times 1}\right] \quad (66)$$

where $\mathbf{0}_{M\times 1}$ is an $M \times 1$ vector with zero entries.

b) Calculating the limit of $\{\mathbf{C}_{\mathbf{U}_{\mathrm{BCR},k}}\}$: According to (38), (64), and (66)

$$\boldsymbol{\eta}_{\mathbf{U}_{\mathrm{BCR},k}}(\mathbf{x},\boldsymbol{\theta}) \triangleq \int_{\boldsymbol{\Lambda}} -\left(\frac{dh_k(-\boldsymbol{\tau})}{d\boldsymbol{\tau}}\right)^T \nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau}. \quad (67)$$

Hence, assuming that $\mathrm{supp}\{h(\boldsymbol{\tau})\} \subset \boldsymbol{\Lambda}$, where $\mathrm{supp}\{\cdot\}$ denotes the support set, then applying integration by parts on the r.h.s. of (67) yields

$$\boldsymbol{\eta}_{\mathbf{U}_{\mathrm{BCR},k}}(\mathbf{x},\boldsymbol{\theta}) \triangleq \int_{\boldsymbol{\Lambda}} h_k(-\boldsymbol{\tau})\left(\frac{\partial \nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})}{\partial \boldsymbol{\tau}}\right)^T d\boldsymbol{\tau} = \left(h_k(\boldsymbol{\tau}) * \left(\frac{\partial \nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})}{\partial \boldsymbol{\tau}}\right)^T\right)(0) \quad (68)$$

where $(\cdot * \cdot)(\cdot)$ denotes an evaluation point of the convolution integral. Therefore, according to Theorem 7 in Appendix G, regarding the limit of the convolution integral, one obtains

$$\begin{aligned}\boldsymbol{\gamma}_{\mathrm{BCR}}(\mathbf{x},\boldsymbol{\theta}) &\triangleq \lim_{k\to\infty} \boldsymbol{\eta}_{\mathbf{U}_{\mathrm{BCR},k}}(\mathbf{x},\boldsymbol{\theta}) \\ &= \left(\left.\frac{\partial \nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})}{\partial \boldsymbol{\tau}}\right|_{\boldsymbol{\tau}=0}\right)^T \\ &= \left(\frac{\partial \log f(\mathbf{x},\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)^T. \end{aligned} \quad (69)$$

Therefore, substitution of (69) into (65) yields

$$\mathbf{C}_{\mathrm{BCR}} = \boldsymbol{\Phi}_{\mathrm{BCR}}\mathbf{I}^{-1}_{\mathrm{BFIM}}\boldsymbol{\Phi}^T_{\mathrm{BCR}} \quad (70)$$

where

$$\boldsymbol{\Phi}_{\mathrm{BCR}} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{g}(\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{BCR}}(\mathbf{x},\boldsymbol{\theta})\right] = -\mathrm{E}_{\boldsymbol{\theta}}\left[\frac{d\mathbf{g}(\boldsymbol{\theta})}{d\boldsymbol{\theta}}\right], \quad (71)$$

and

$$\mathbf{I}_{\mathrm{BFIM}} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\boldsymbol{\gamma}_{\mathrm{BCR}}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{BCR}}(\mathbf{x},\boldsymbol{\theta})\right]. \quad (72)$$

We note that the BCR bound is obtained subject to the following regularity conditions:

- $[\mathbf{g}(\boldsymbol{\theta})]_l, l = 0,\ldots,L-1$ are absolutely continuous in $\boldsymbol{\Theta}$.
- $f(\mathbf{x},\boldsymbol{\theta})$ is absolutely continuous in $\boldsymbol{\Theta}$ for a.e. $\mathbf{x} \in \mathcal{X}$.
- Let $\partial\boldsymbol{\Theta}$ denote the boundary set of $\boldsymbol{\Theta}$. Then for every $\boldsymbol{\theta} \in \partial\boldsymbol{\Theta}$ and for a.e. $\mathbf{x} \in \mathcal{X}$, $[\mathbf{g}(\boldsymbol{\theta})]_l f(\mathbf{x},\boldsymbol{\theta}) = 0, l = 0,\ldots,L-1$.
- $\mathbf{I}_{\mathrm{BFIM}}$ is nonsingular.

2) **The Bayesian Bhattacharayya bound** [3]: The Qth-order BBH bound is obtained via the following steps:

a) Construction of $\{\mathbf{U}^{(Q)}_{\mathrm{BBH},k}(\cdot)\}$: The kth member of $\left\{\mathbf{U}^{(Q)}_{\mathrm{BBH},k}(\cdot)\right\}$ is given by

$$\mathbf{U}^{(Q)}_{\mathrm{BBH},k}(\boldsymbol{\tau}) = \left[\left[\mathbf{U}^{(Q)}_{\mathrm{BBH},k}(\boldsymbol{\tau})\right]_1 \quad \left[\mathbf{U}^{(Q)}_{\mathrm{BBH},k}(\boldsymbol{\tau})\right]_2\right] \quad (73)$$

where

$$\begin{aligned}&\left[\mathbf{U}^{(Q)}_{\mathrm{BBH},k}(\boldsymbol{\tau})\right]_1 \\ &\quad = \underbrace{\left[-\frac{dh_k(-\boldsymbol{\tau})}{d\boldsymbol{\tau}},\ldots,(-1)^Q\frac{d^Q h_k(-\boldsymbol{\tau})}{d\boldsymbol{\tau}^{\otimes Q}}\right]^T}_{M(M^Q-1)/(M-1)\times 1}, \\ &\left[\mathbf{U}^{(Q)}_{\mathrm{BBH},k}(\boldsymbol{\tau})\right]_2 \\ &\quad = \mathbf{0}_{M(M^Q-1)/(M-1)\times 1} \end{aligned} \quad (74)$$

$\frac{d^Q}{d\boldsymbol{\tau}^{\otimes Q}}$ denotes the vector of derivatives, $\frac{\partial^Q}{\partial\tau_{i_1}\cdots\partial\tau_{i_Q}}$, $i_q = 0,\ldots,M-1$, and τ_m denotes the mth entry of $\boldsymbol{\tau}$.

b) Calculating the limit of $\{\mathbf{C}_{\mathbf{U}^{(Q)}_{\mathrm{BBH},k}}\}$: Using the same techniques described in (67)–(69) it can be shown that

$$\boldsymbol{\gamma}_{\mathrm{BBH}}(\mathbf{x},\boldsymbol{\theta}) \triangleq \lim_{k\to\infty} \boldsymbol{\eta}_{\mathbf{U}^{(Q)}_{\mathrm{BBH},k}}(\mathbf{x},\boldsymbol{\theta}) = \frac{1}{f(\mathbf{x},\boldsymbol{\theta})}\left[\frac{\partial f(\mathbf{x},\boldsymbol{\theta})}{\partial\boldsymbol{\theta}},\ldots,\frac{\partial^Q f(\mathbf{x},\boldsymbol{\theta})}{\partial\boldsymbol{\theta}^{\otimes Q}}\right]^T. \quad (75)$$

Thetefore, substitution of (75) into (65) yields

$$\mathbf{C}^{(Q)}_{\mathrm{BBH}} = \boldsymbol{\Phi}_{\mathrm{BBH}}\mathbf{K}^{-1}_{\mathrm{BBH}}\boldsymbol{\Phi}^T_{\mathrm{BBH}} \quad (76)$$

where

$$\boldsymbol{\Phi}_{\mathrm{BBH}} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{g}(\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{BBH}}(\mathbf{x},\boldsymbol{\theta})\right] = \left[\frac{d\mathbf{g}(\boldsymbol{\theta})}{d\boldsymbol{\theta}},\ldots,\frac{d^Q\mathbf{g}(\boldsymbol{\theta})}{d\boldsymbol{\theta}^{\otimes Q}}\right], \quad (77)$$

and

$$\mathbf{K}_{\mathrm{BBH}} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\boldsymbol{\gamma}_{\mathrm{BBH}}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\gamma}^T_{\mathrm{BBH}}(\mathbf{x},\boldsymbol{\theta})\right]. \quad (78)$$

We note that the BBH bound is obtained subject to the following regularity conditions.

- The derivatives $\frac{\partial^s[\mathbf{g}(\boldsymbol{\theta})]_l}{\partial\theta_{i_1}\partial\theta_{i_2}\cdots\partial\theta_{i_s}}$, $l=0,\ldots,L-1$, $s=0,\ldots,Q-1$, $i_s=0,\ldots,M-1$, are absolutely continuous in $\boldsymbol{\Theta}$.
- The derivatives $\frac{\partial^s f(\mathbf{x},\boldsymbol{\theta})}{\partial\theta_{i_1}\partial\theta_{i_2}\cdots\partial\theta_{i_s}}$, $s=0,\ldots,Q-1$, $i_s=0,\ldots,M-1$, are absolutely continuous in $\boldsymbol{\Theta}$, for a.e. $\mathbf{x}\in\mathcal{X}$.
- Let $\partial\boldsymbol{\Theta}$ denote the boundary set of $\boldsymbol{\Theta}$. Then for every $\boldsymbol{\theta}\in\partial\boldsymbol{\Theta}$ and for a.e. $\mathbf{x}\in\mathcal{X}$, $[\mathbf{g}(\boldsymbol{\theta})]_l\frac{\partial^s f(\mathbf{x},\boldsymbol{\theta})}{\partial\theta_{i_1}\partial\theta_{i_2}\cdots\partial\theta_{i_s}}=0$, $l=0,\ldots,L-1$, $s=0,\ldots,Q-1$, $i_s=0,\ldots,M-1$.
- The matrix $\mathbf{K}_{\mathrm{BBH}}$ is nonsingular.

3) **The Weiss–Weinstein bound** [13]: The Nth-order WW bound is obtained via the following steps:

a) Construction of $\{\mathbf{U}^{(Q)}_{\mathrm{WW},k}(\cdot)\}$: The kth member of $\{\mathbf{U}^{(Q)}_{\mathrm{WW},k}(\cdot)\}$ is given by

$$\mathbf{U}^{(Q)}_{\mathrm{WW},k}(\boldsymbol{\tau}) = [\mathbf{0}_{N\times 1},[h_k(\boldsymbol{\tau}_0-\boldsymbol{\tau}),\ldots,h_k(\boldsymbol{\tau}_{N-1}-\boldsymbol{\tau})]^T]. \quad (79)$$

b) Calculating the limit of $\{\mathbf{C}_{\mathbf{U}^{(Q)}_{\mathrm{WW},k}}\}$: According to (38), (64), and (79)

$$\begin{aligned}\boldsymbol{\eta}_{\mathbf{U}^{(Q)}_{\mathrm{WW},k}}(\mathbf{x},\boldsymbol{\theta}) &\triangleq \int_{\boldsymbol{\Lambda}}[h_k(\boldsymbol{\tau}_0-\boldsymbol{\tau}),\ldots,h_k(\boldsymbol{\tau}_{N-1}-\boldsymbol{\tau})]^T \nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau}\\ &= [(h_k(\boldsymbol{\tau})*\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}))(\boldsymbol{\tau}_0),\ldots,(h_k(\boldsymbol{\tau})*\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}))(\boldsymbol{\tau}_{N-1})]^T. \end{aligned} \quad (80)$$

Therefore, according to Theorem 7 in Appendix G, regarding the limit of the convolution integral, one obtains

$$\boldsymbol{\gamma}_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta}) \triangleq \lim_{k\to\infty}\boldsymbol{\eta}_{\mathbf{U}^{(Q)}_{\mathrm{WW},k}}(\mathbf{x},\boldsymbol{\theta}) = [\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_0),\ldots,\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_{N-1})]^T. \quad (81)$$

Therefore, substitution of (81) into (65) yields

$$\mathbf{C}^{(N)}_{\mathrm{WW}} = \boldsymbol{\Phi}_{\mathrm{WW}}\mathbf{K}^{-1}_{\mathrm{WW}}\boldsymbol{\Phi}^H_{\mathrm{WW}} \quad (82)$$

where

$$\boldsymbol{\Phi}_{\mathrm{WW}} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\mathbf{g}(\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta})] \quad (83)$$

and

$$\mathbf{K}_{\mathrm{WW}} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\gamma}_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta})]. \quad (84)$$

We note that the WW bound is obtained subject to the condition that $\mathbf{K}_{\mathrm{WW}}$ is nonsingular.

4) **The Reuven–Messer bound** [12]: The Nth-order RM bound is derived via the following steps:

a) Construction of $\{\mathbf{U}^{(N)}_{\mathrm{RM},k}(\cdot)\}$: The kth member of $\{\mathbf{U}^{(N)}_{\mathrm{RM},k}(\cdot)\}$ is given by

$$\mathbf{U}^{(N)}_{\mathrm{RM},k}(\boldsymbol{\tau}) = [[h_k(\boldsymbol{\tau}_0-\boldsymbol{\tau}),\ldots,h_k(\boldsymbol{\tau}_{N-1}-\boldsymbol{\tau})]^T,\mathbf{0}_{N\times 1}]. \quad (85)$$

b) Calculating the limit of $\{\mathbf{C}_{\mathbf{U}^{(N)}_{\mathrm{RM},k}}\}$: Using the same techniques described in (80) and (81) it can be shown that

$$\boldsymbol{\gamma}_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta}) \triangleq \lim_{k\to\infty}\boldsymbol{\eta}_{\mathbf{U}^{(N)}_{\mathrm{RM},k}}(\mathbf{x},\boldsymbol{\theta}) = [\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_0),\ldots,\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_{N-1})]^T. \quad (86)$$

Therefore, substitution of (86) into (65) yields

$$\mathbf{C}^{(N)}_{\mathrm{RM}} = \boldsymbol{\Phi}_{\mathrm{RM}}\mathbf{K}^{-1}_{\mathrm{RM}}\boldsymbol{\Phi}^H_{\mathrm{RM}} \quad (87)$$

where

$$\boldsymbol{\Phi}_{\mathrm{RM}} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{g}(\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta})\right] \quad (88)$$

and

$$\mathbf{K}_{\mathrm{RM}} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\boldsymbol{\gamma}_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta})\right]. \quad (89)$$

We note that the RM bound is obtained subject to the following regularity conditions.

- $\int_{\boldsymbol{\Theta}} f(\mathbf{x},\boldsymbol{\theta})d\boldsymbol{\theta} = \int_{\boldsymbol{\Theta}} f(\mathbf{x},\boldsymbol{\theta}+\boldsymbol{\tau}_n)d\boldsymbol{\theta}$, $n=0,\ldots,N-1$.
- The matrix $\mathbf{K}_{\mathrm{RM}}$ is nonsingular. One can notice that by choosing $N=1$, the Bobrovsky–Zakai bound [11] is obtained from (87).

5) **The Bayesian Abel bound** [14]}: The (Q,N)th-order BA bound is derived via the following steps:

a) Construction of $\{\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}(\cdot)\}$: The kth member of $\{\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}(\cdot)\}$ is given by

$$\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}(\boldsymbol{\tau}) = \begin{bmatrix}\left[\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}(\boldsymbol{\tau})\right]_{1,1} & \left[\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}(\boldsymbol{\tau})\right]_{1,2}\\ \left[\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}(\boldsymbol{\tau})\right]_{2,1} & \left[\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}(\boldsymbol{\tau})\right]_{2,2}\end{bmatrix} \quad (90)$$

where

$$\left[\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}(\boldsymbol{\tau})\right]_{1,1} = \underbrace{\left[-\frac{dh_k(-\boldsymbol{\tau})}{d\boldsymbol{\tau}},\ldots,(-1)^Q\frac{d^Q h_k(-\boldsymbol{\tau})}{d\boldsymbol{\tau}^{\otimes Q}}\right]^T}_{M(M^Q-1)/(M-1)\times 1},$$

$$\begin{aligned}&\left[\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}(\boldsymbol{\tau})\right]_{1,2}\\&\quad=\mathbf{0}_{M(M^Q-1)/(M-1)\times 1},\\&\left[\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}(\boldsymbol{\tau})\right]_{2,1}\\&\quad=\underbrace{[h_k(\boldsymbol{\tau}_0-\boldsymbol{\tau}),\ldots,h_k(\boldsymbol{\tau}_{N-1}-\boldsymbol{\tau})]^T}_{N\times 1},\\&\left[\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}(\boldsymbol{\tau})\right]_{2,2}\\&\quad=\mathbf{0}_{N\times 1}.\end{aligned}\tag{91}$$

b) Calculating the limit of $\{\mathbf{C}_{\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}}\}$: Using the same techniques described in (67)-(69) and in (80), (81), it can be shown that

$$\begin{aligned}\boldsymbol{\gamma}_{\mathrm{BA}}(\mathbf{x},\boldsymbol{\theta})&\triangleq\lim_{k\to\infty}\boldsymbol{\eta}_{\mathbf{U}^{(Q,N)}_{\mathrm{BA},k}}(\mathbf{x},\boldsymbol{\theta})\\&=\left[\boldsymbol{\gamma}^T_{\mathrm{BBH}}(\mathbf{x},\boldsymbol{\theta}),\boldsymbol{\gamma}^T_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta})\right]^T\end{aligned}\tag{92}$$

where $\boldsymbol{\gamma}_{\mathrm{BBH}}(\mathbf{x},\boldsymbol{\theta})$ and $\boldsymbol{\gamma}_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta})$ are defined in (75) and (86), respectively. Therefore, substitution of (92) into (65) yields

$$\mathbf{C}^{(Q,N)}_{\mathrm{BA}}=\boldsymbol{\Phi}_{\mathrm{BA}}\mathbf{K}^{-1}_{\mathrm{BA}}\boldsymbol{\Phi}^T_{\mathrm{BA}}\tag{93}$$

where

$$\boldsymbol{\Phi}_{\mathrm{BA}}\triangleq\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{g}(\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{BA}}(\mathbf{x},\boldsymbol{\theta})\right]\tag{94}$$

and

$$\mathbf{K}_{\mathrm{BA}}\triangleq\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\boldsymbol{\gamma}_{\mathrm{BA}}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{BA}}(\mathbf{x},\boldsymbol{\theta})\right].\tag{95}$$

We note that the BA bound is obtained subject to the regularity conditions of the Bayesian Bhattacharayya and Reuven–Messer bounds.

6) **The combined Cramér–Rao/Weiss–Weinstein bound** [15]: The Nth-order CRWW bound is obtained via the following steps:

a) Construction of $\{\mathbf{U}^{(N)}_{\mathrm{CRWW},k}(\cdot)\}$: The kth member of $\{\mathbf{U}^{(N)}_{\mathrm{CRWW},k}(\cdot)\}$ is given by

$$\mathbf{U}^{(N)}_{\mathrm{CRWW},k}(\boldsymbol{\tau})=\begin{bmatrix}\left[\mathbf{U}^{(N)}_{\mathrm{CRWW},k}(\boldsymbol{\tau})\right]_{1,1}&\left[\mathbf{U}^{(N)}_{\mathrm{CRWW},k}(\boldsymbol{\tau})\right]_{1,2}\\\left[\mathbf{U}^{(N)}_{\mathrm{CRWW},k}(\boldsymbol{\tau})\right]_{2,1}&\left[\mathbf{U}^{(N)}_{\mathrm{CRWW},k}(\boldsymbol{\tau})\right]_{2,2}\end{bmatrix}\tag{96}$$

where

$$\begin{aligned}&\left[\mathbf{U}^{(N)}_{\mathrm{CRWW},k}(\boldsymbol{\tau})\right]_{1,1}=-\underbrace{\left(\frac{dh_k(-\boldsymbol{\tau})}{d\boldsymbol{\tau}}\right)^T}_{M\times 1},\\&\left[\mathbf{U}^{(N)}_{\mathrm{CRWW},k}(\boldsymbol{\tau})\right]_{1,2}=\mathbf{0}_{M\times 1},\\&\left[\mathbf{U}^{(N)}_{\mathrm{CRWW},k}(\boldsymbol{\tau})\right]_{2,1}=\mathbf{0}_{N\times 1},\\&\left[\mathbf{U}^{(N)}_{\mathrm{CRWW},k}(\boldsymbol{\tau})\right]_{2,2}=\underbrace{[h_k(\boldsymbol{\tau}_0-\boldsymbol{\tau}),\ldots,h_k(\boldsymbol{\tau}_{N-1}-\boldsymbol{\tau})]^T}_{N\times 1}.\end{aligned}\tag{97}$$

b) Calculating the limit of $\{\mathbf{C}_{\mathbf{U}^{(N)}_{\mathrm{CRWW},k}}\}$: Using the same techniques described in (67)-(69) and in (80), (81), it can be shown that

$$\begin{aligned}\boldsymbol{\gamma}_{\mathrm{CRWW}}(\mathbf{x},\boldsymbol{\theta})&\triangleq\lim_{k\to\infty}\boldsymbol{\eta}_{\mathbf{U}^{(N)}_{\mathrm{CRWW},k}}(\mathbf{x},\boldsymbol{\theta})\\&=\left[\boldsymbol{\gamma}^T_{\mathrm{BCR}}(\mathbf{x},\boldsymbol{\theta}),\boldsymbol{\gamma}^T_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta})\right]^T.\end{aligned}\tag{98}$$

where $\boldsymbol{\gamma}_{\mathrm{BCR}}(\mathbf{x},\boldsymbol{\theta})$ and $\boldsymbol{\gamma}_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta})$ are defined in (69) and (81), respectively. Therefore, substitution of (99) into (65) yields

$$\mathbf{C}^{(N)}_{\mathrm{CRWW}}=\boldsymbol{\Phi}_{\mathrm{CRWW}}\mathbf{K}^{-1}_{\mathrm{CRWW}}\boldsymbol{\Phi}^T_{\mathrm{CRWW}}\tag{99}$$

where

$$\boldsymbol{\Phi}_{\mathrm{CRWW}}\triangleq\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{g}(\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{CRWW}}(\mathbf{x},\boldsymbol{\theta})\right]\tag{100}$$

and

$$\mathbf{K}_{\mathrm{CRWW}}\triangleq\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\boldsymbol{\gamma}_{\mathrm{CRWW}}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{CRWW}}(\mathbf{x},\boldsymbol{\theta})\right].\tag{101}$$

The CRWW bound is obtained subject to the regularity conditions of the BCR and WW bounds.

One can notice that any two bounds $\mathbf{C}_1$ and $\mathbf{C}_2$ obtained via the procedure described above, constitute the limits of two convergent sequences of bounds, $\{\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{U}_{1,k})}}\}$ and $\{\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{U}_{2,k})}}\}$, where $\{\mathbf{U}_{1,k}(\cdot)\}$ and $\{\mathbf{U}_{2,k}(\cdot)\}$ are two sequences of kernel functions in $\mathcal{L}_1^{P\times 2}(\boldsymbol{\Lambda})$. Hence, using the result of Theorem 5 in Appendix D it can be shown in similar to [31] that order relation between $\mathbf{C}_1$ and $\mathbf{C}_2$ can be obtained by comparing the limits of the Hilbert subspace sequences, $\{\mathcal{H}_\varphi^{(\mathbf{U}_{1,k})}\}$ and $\{\mathcal{H}_\varphi^{(\mathbf{U}_{2,k})}\}$, i.e., if $\lim_{k\to\infty}\mathcal{H}_\varphi^{(\mathbf{U}_{1,k})}\supset\lim_{k\to\infty}\mathcal{H}_\varphi^{(\mathbf{U}_{2,k})}$, then $\mathbf{C}_1\succeq\mathbf{C}_2$.

The disadvantage of bounds, such as the RM, WW, BA, and CRWW stems from the fact that in many cases, $\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\cdot)$ and $\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\cdot)$ should be evaluated at large amount of test points in order to obtain tight bounds. Since there is no analytical procedure for optimal selection of test points, numerical search methods, which become computationally cumbersome as the number of test points and the dimensionality of the parameter increase, are often used. In order to overcome this disadvantage, a new Bayesian bound is derived from (55), using the kernel of the Fourier transform.

VI. A New Bayesian Bound Using the Kernel of the Fourier Transform

In this section, a new lower bound is derived from (55) using the kernel function of the Fourier transform. We show that the proposed bound is computed by applying the discrete Fourier transform (DFT) of the sequence $\{\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_n)\}_{n=0}^{N-1}$. In cases where the DFT of this sequence is concentrated in few frequency components, a computationally manageable bound, which exploits all the information in $\{\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_n)\}_{n=0}^{N-1}$ is obtained. The proposed bound is derived via the following steps:

1) Construction of $\{\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}(\cdot)\}$:
The kth member of $\{\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}(\cdot)\}$ is given by

$$\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}(\boldsymbol{\tau}) = \begin{bmatrix} \left[\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}(\boldsymbol{\tau})\right]_{1,1} & \left[\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}(\boldsymbol{\tau})\right]_{1,2} \\ \left[\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}(\boldsymbol{\tau})\right]_{2,1} & \left[\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}(\boldsymbol{\tau})\right]_{2,2} \end{bmatrix} \tag{102}$$

where

$$\begin{aligned} \left[\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}(\boldsymbol{\tau})\right]_{1,1} &= -\underbrace{\left(\frac{dh_k(-\boldsymbol{\tau})}{d\boldsymbol{\tau}}\right)^T}_{M\times 1}, \\ \left[\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}(\boldsymbol{\tau})\right]_{1,2} &= \mathbf{0}_{M\times 1}, \\ \left[\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}(\boldsymbol{\tau})\right]_{2,1} &= \mathbf{0}_{N\times 1}, \end{aligned} \tag{103}$$

and

$$\left[\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}(\boldsymbol{\tau})\right]_{2,2} = \mathbf{W}\underbrace{\left[h_k(\boldsymbol{\tau}_0-\boldsymbol{\tau}),\ldots,h_k(\boldsymbol{\tau}_{N-1}-\boldsymbol{\tau})\right]^T}_{N\times 1}. \tag{104}$$

The set $\{\boldsymbol{\tau}_n\}_{n=0}^{N-1}$ contains N equally spaced test-points in $\boldsymbol{\Lambda}$, where $\boldsymbol{\tau}_n = [n_1\Delta\tau,\ldots,n_M\Delta\tau]^T$, $\Delta\tau$ is a sampling interval, $n_m \in \{0,\ldots,N_m-1\}$ denotes a test-point index in the mth dimension of $\boldsymbol{\Lambda}$, the index n is a unique combination of $\{n_1,\ldots,n_M\}$, i.e., $\{n_1,\ldots,n_M\}\leftrightarrow n$, and $N=\prod_{m=1}^{M}N_m$. The M-dimensional DFT matrix with $J<N$ frequency components is denoted by $\mathbf{W}\in\mathbb{C}^{J\times N}$, such that

$$[\mathbf{W}]_{j,n} \triangleq \exp\left(-i\boldsymbol{\Omega}_j^T\boldsymbol{\tau}_n\right), j\in\mathcal{J}, n=0,\ldots,N-1. \tag{105}$$

$\boldsymbol{\Omega}_j = [\frac{2\pi j_1}{\Delta\tau N_1},\ldots,\frac{2\pi j_M}{\Delta\tau N_M}]^T$ is a frequency test-bin, $j_m \in \{0,\ldots,N_m-1\}$ denotes a test-bin index in the mth dimension of the frequency domain, $\mathbf{V}$, the index j is a unique combination of $\{j_1,\ldots,j_M\}$, i.e., $\{j_1,\ldots,j_M\}\leftrightarrow j$, and $\mathcal{J}\subset\{0,\ldots,N-1\}$ denotes an index set of the selected frequency test-bins with cardinality $|\mathcal{J}|=J$.

2) Calculating the limit of $\{\mathbf{C}_{\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}}\}$:
Using the same techniques described in (67)-(69) and in (80), (81), it can be shown that

$$\begin{aligned} \boldsymbol{\gamma}_{\mathrm{CRF}}(\mathbf{x},\boldsymbol{\theta}) &\triangleq \lim_{k\to\infty}\boldsymbol{\eta}_{\mathbf{U}^{(J,N)}_{\mathrm{CRF},k}}(\mathbf{x},\boldsymbol{\theta}) \\ &= \left[\boldsymbol{\gamma}^T_{\mathrm{BCR}}(\mathbf{x},\boldsymbol{\theta}),\boldsymbol{\gamma}^T_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta})\mathbf{W}^T\right]^T. \end{aligned} \tag{106}$$

where $\boldsymbol{\gamma}_{\mathrm{BCR}}(\mathbf{x},\boldsymbol{\theta})$ and $\boldsymbol{\gamma}_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta})$ are defined in (69) and (81), respectively. Therefore, substitution of (106) into (65) yields

$$\mathbf{C}^{(J,N)}_{\mathrm{CRF}} = \boldsymbol{\Phi}_{\mathrm{CRF}}\mathbf{K}^{-1}_{\mathrm{CRF}}\boldsymbol{\Phi}^T_{\mathrm{CRF}} \tag{107}$$

where

$$\boldsymbol{\Phi}_{\mathrm{CRF}} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{g}(\boldsymbol{\theta})\boldsymbol{\gamma}^H_{\mathrm{CRF}}(\mathbf{x},\boldsymbol{\theta})\right] \tag{108}$$

and

$$\mathbf{K}_{\mathrm{CRF}} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\boldsymbol{\gamma}_{\mathrm{CRF}}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\gamma}^T_{\mathrm{CRF}}(\mathbf{x},\boldsymbol{\theta})\right]. \tag{109}$$

In Appendix H, it is shown that substitution of $\boldsymbol{\gamma}_{\mathrm{CRF}}(\mathbf{x},\boldsymbol{\theta})$ into (107), yields the following bound:

$$\mathbf{C}_{\mathrm{CRF}} = \boldsymbol{\Phi}_{\mathrm{BCR}}\mathbf{I}^{-1}_{\mathrm{BFIM}}\boldsymbol{\Phi}^T_{\mathrm{BCR}} + \mathbf{Q}\mathbf{W}^H(\mathbf{W}\mathbf{R}\mathbf{W}^H)^{-1}\mathbf{W}\mathbf{Q}^T \tag{110}$$

where $\boldsymbol{\Phi}_{\mathrm{BCR}}$ and $\mathbf{I}_{\mathrm{BFIM}}$ are defined in (71) and (72), respectively,

$$\mathbf{Q} \triangleq \boldsymbol{\Phi}_{\mathrm{BCR}}\mathbf{I}^{-1}_{\mathrm{BFIM}}\mathbf{G} - \boldsymbol{\Phi}_{\mathrm{WW}}, \tag{111}$$

$$\mathbf{G} \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\left(\frac{\partial\log f(\mathbf{x},\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right)^T\boldsymbol{\gamma}^T_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta})\right], \tag{112}$$

and

$$\mathbf{R} \triangleq \mathbf{K}_{\mathrm{WW}} - \mathbf{G}^T\mathbf{I}^{-1}_{\mathrm{BFIM}}\mathbf{G}. \tag{113}$$

The matrices $\boldsymbol{\Phi}_{\mathrm{WW}}$ and $\mathbf{K}_{\mathrm{WW}}$ are defined in (83) and (84), respectively.

The bound in (110) is composed of the BCR bound, supplemented by a positive-semidefinite term. Therefore, the regularity conditions for the BCR bound are required also here. In cases where these conditions are not satisfied, $\mathbf{U}^{(J,N)}_{\mathrm{F},k}(\boldsymbol{\tau}) = [\mathbf{0}_{N\times 1}\ \ \mathbf{W}[h_k(\boldsymbol{\tau}_0-\boldsymbol{\tau}),\ldots,h_k(\boldsymbol{\tau}_{N-1}-\boldsymbol{\tau})]^T]$ may be chosen and the bound in (110) becomes

$$\mathbf{C}^{(J,N)}_{\mathrm{F}} = \boldsymbol{\Phi}_{\mathrm{WW}}\mathbf{W}^H(\mathbf{W}\mathbf{K}_{\mathrm{WW}}\mathbf{W}^H)^{-1}\mathbf{W}\boldsymbol{\Phi}^T_{\mathrm{WW}}. \tag{114}$$

Observing (110), one can notice that $[\mathbf{W}\mathbf{Q}^T]_{j,l}$, $j=0,\ldots,J-1$, $l=0,\ldots,L-1$ is the DFT of the lth column of $\mathbf{Q}^T$ evaluated at $\boldsymbol{\Omega}_j$, and that $[\mathbf{W}\mathbf{R}\mathbf{W}^H]_{j,j'}$, $j,j'=0,\ldots,J-1$ is the two-dimensional DFT of $\mathbf{R}$, evaluated at $(\boldsymbol{\Omega}_j,-\boldsymbol{\Omega}_{j'})$. Hence, implementation of the bound can be easily performed using the fast Fourier transform (FFT).

The bound in (110) is computed using N equally spaced test-points in $\boldsymbol{\Lambda}$ and $J<N$ frequency test-bins in $\mathbf{V}$. For a given scenario, the frequency test-bins are selected such that the bound is maximized. According to (105), (106), and the definition of $\boldsymbol{\gamma}_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta})$ in (81), one can notice that the transform $\mathbf{W}\boldsymbol{\gamma}_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta})$ in (106) is actually the M-dimensional DFT of the sequence $\{\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_n)\}_{n=0}^{N-1}$, evaluated at $J<N$ frequency test-bins. Hence, let

$$\hat{\nu}_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\Omega}_j) \triangleq \sum_{n=0}^{N-1}\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_n)\exp(-i\boldsymbol{\Omega}_j^T\boldsymbol{\theta}_n). \tag{115}$$

$j=0,\ldots,N-1$ denote the N-points M-dimensional-DFT of the sequence $\{\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}_n)\}_{n=0}^{N-1}$, and let $\mathcal{J}\subset\{0,\ldots,N-1\}$ denote the index set of the selected frequency test-bins with cardinality $|\mathcal{J}|=J<N$. If

$$\hat{\nu}_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\Omega}_j) = 0 \tag{116}$$

for a.e. $(\mathbf{x}, \boldsymbol{\theta}) \in \mathcal{X} \times \boldsymbol{\Theta}$ and $\forall j \in \{0, \ldots, N-1\} \backslash \mathcal{J}$, such that $J \ll N$, then a computationally manageable bound, which exploits all the information in $\{\nu_{\mathrm{WW}}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}_n)\}_{n=0}^{N-1}$ is obtained. In this case, the computational complexity of the proposed bound is significantly lower in comparison to the RM, WW, BA, and CRWW bounds in (82), (87), (93), and (99), respectively, in which inversion of $N \times N$, $N > J$, matrices is carried out, and maximization w.r.t. $N > J$ test-points in $\boldsymbol{\Lambda}$ is required in order to obtain tight bounds. Since by the norm properties [26] $\hat{\nu}_{\mathrm{WW}}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Omega}_j) = 0$ for a.e. $(\mathbf{x}, \boldsymbol{\theta}) \in \mathcal{X} \times \boldsymbol{\Theta} \Leftrightarrow \|\hat{\nu}_{\mathrm{WW}}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Omega}_j)\|_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} = 0$, the condition in (116) is equivalent to

$$\|\hat{\nu}_{\mathrm{WW}}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Omega}_j)\|_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} = 0 \forall j \in \{0, \ldots, N-1\} \backslash \mathcal{J} \tag{117}$$

where $\|\hat{\nu}(\mathbf{x}, \boldsymbol{\Omega}_j)\|_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})}$ is the $\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})$-norm of the spectrum of $\{\nu_{\mathrm{WW}}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}_n)\}_{n=0}^{N-1}$ at $\boldsymbol{\Omega}_j$, which is given by $|\hat{\nu}_{\mathrm{WW}}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\Omega}_j)|$. Hence, similar to [31, Prop. 4], it can be shown that if the condition in (117) is satisfied, then $\mathbf{C}_{\mathrm{CRF}}^{(J,N)} = \mathbf{C}_{\mathrm{CRWW}}^{(N)}$. Therefore, under this condition, if the CRWW bound is computed with $N' < N$ test-points, then $\mathbf{C}_{\mathrm{CRF}}^{(J,N)} \succeq \mathbf{C}_{\mathrm{CRWW}}^{(N')}$.

VII. Example

In this section, the proposed bound is compared with the WW [3], BCR [10], BA [14], and CRWW [15] bounds, in the problem of frequency estimation with zero-mean additive white circular complex Gaussian noise. The comparison criterion is prediction of the threshold SNR region exhibited by the MAP and MMSE estimators. The observation model is given by

$$\mathbf{x} = s\mathbf{b}(\theta) + \mathbf{n} \tag{118}$$

where $\mathbf{x}$ denotes a $K \times 1$ observation vector, $s \in \mathbb{C}$ is a known complex amplitude

$$\mathbf{b}(\theta) = [1, \exp(i\theta), \ldots, \exp(i(K-1)\theta)]^T \tag{119}$$

is the normalized sinusoid signal, $\mathbf{n}$ denotes a $K \times 1$ complex circular Gaussian noise vector, with zero-mean and known covariance $\mathbf{C}_{\mathbf{n}} = \sigma_n^2 \mathbf{I}_K$, and $\theta \in \boldsymbol{\Theta} = (-\infty, \infty)$ is the parameter of interest, i.e., $g(\theta) = \theta$. The *a priori* distribution of θ is zero-mean Gaussian with variance $\sigma_\theta^2 = \frac{1}{2}$, such that the tails of the PDF for $|\theta| > \pi$ are negligible. Moreover, it is assumed that θ and $\mathbf{n}$ are statistically independent.

Hence, by choosing in (33), $\beta(\boldsymbol{\tau}) = \frac{1}{2}$, it can be shown that the terms comprising (110) are given by

$$\boldsymbol{\Phi}_{\mathrm{CRB}} = -1, \tag{120}$$

$$\mathbf{I}_{\mathrm{BFIM}} = \frac{1}{3}\mathrm{SNR} \cdot K(K-1)(2K-1) + \frac{1}{\sigma_\theta^2} \tag{121}$$

where $\mathrm{SNR} = \frac{|s|^2}{\sigma_n^2}$

$$\mathbf{G} = [G(\tau_0), \ldots, G(\tau_{N-1})], \tag{122}$$

$$G(\tau_n) = 2\exp(-\mu(\tau_n, 0)) \times \eta(\tau_n), \tag{123}$$

$$\mu(\tau_n, \tau_m) = \frac{(\tau_n - \tau_m)^2}{8\sigma_\theta^2} - \frac{1}{2}\mathrm{SNR} \times \left(\frac{\sin\left(\frac{(\tau_n - \tau_m)K}{2}\right) \cdot \cos\left(\frac{(\tau_n - \tau_m)(K-1)}{2}\right)}{\sin\left(\frac{\tau_n - \tau_m}{2}\right)} - K \right), \tag{124}$$

$$\eta(\tau_n) = \frac{\tau_n}{2\sigma_\theta^2} + \frac{\mathrm{SNR}}{4\sin\left(\frac{\tau_n}{2}\right)} \times \left(\frac{\sin\left(\tau_n\left(K - \frac{1}{2}\right)\right)}{\tan\left(\frac{\tau_n}{2}\right)} - (2K-1) \cdot \cos\left(\tau_n\left(K - \frac{1}{2}\right)\right)\right), \tag{125}$$

$$\boldsymbol{\Phi}_{\mathrm{WW}} \triangleq [\phi(\tau_0), \ldots, \phi(\tau_{N-1})], \tag{126}$$

$$\phi(\tau_n) = -\tau_n \exp(-\mu(\tau_n, 0)), \quad n = 0, \ldots, N-1, \tag{127}$$

$$[\mathbf{K}_{\mathrm{WW}}]_{m,n} = 2(\exp(-\mu(\tau_n, \tau_m)) - \exp(-\mu(\tau_n, -\tau_m))) \tag{128}$$

for $m, n = 0, \ldots, N-1$, and

$$[\mathbf{W}]_{j,n} = \exp(-i\omega_j n) \tag{129}$$

where $\omega_j = \boldsymbol{\Omega}_j \Delta\tau \in [-\pi, \pi)$.

The comparison was carried out under the following conditions. The number of observations was set to $K = 2^7$. The proposed bound was computed using a set of $N = 2^8$ equally spaced test points in $\boldsymbol{\Lambda} = [-\pi, \pi)$, given by $\{\tau_n = \frac{2\pi n}{N} - \pi\}_{n=0}^{N-1}$ and $J = 1$ frequency test-bin, denoted by ω. For each SNR, the proposed bound was maximized w.r.t. $\omega \in \{\frac{2\pi k}{N} - \pi\}_{k=0}^{N-1}$. All other compared bounds, except the BCR, were computed using a single test point in $\boldsymbol{\Lambda}$, denoted by τ. For each SNR, these bounds were maximized w.r.t. $\tau \in \{\frac{2\pi n}{N} - \pi\}_{n=0}^{N-1}$.

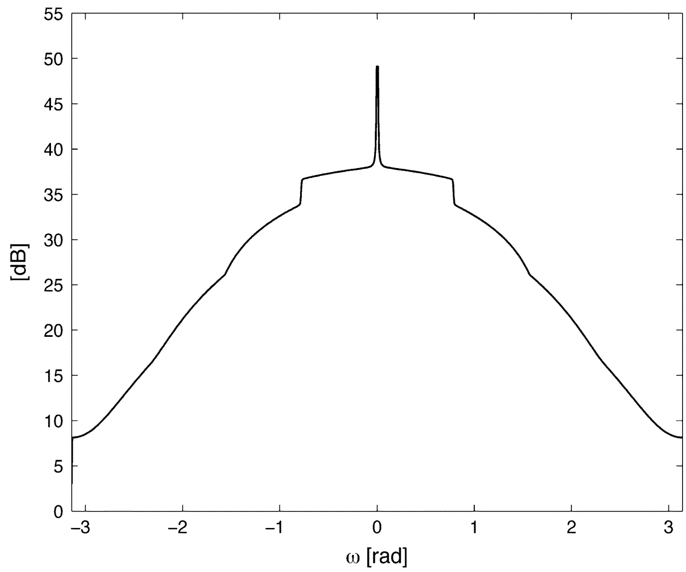

Fig. 2. The squared $\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})$-norm of the spectrum of $\{\nu_{\mathrm{WW}}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}_n)\}_{n=0}^{N-1}$, in the scenario of Bayesian frequency estimation, where the number of observations, $K = 2^7$, the number of test points, $N = 2^{10}$ and $\mathrm{SNR} = -15$ dB.

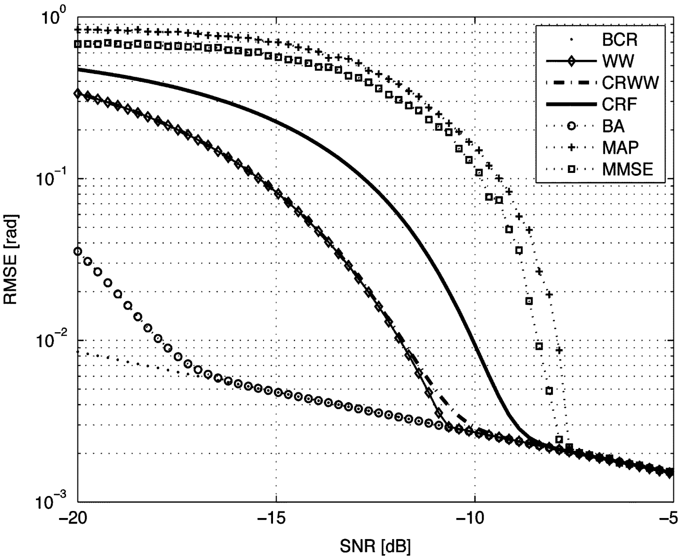

Fig. 3. Comparison of Bayesian lower bounds versus SNR. The comparison criterion is prediction of the SNR threshold regions exhibited by the MAP and MMSE estimators.

Fig. 2 depicts $\|\hat{\nu}_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\omega)\|^2_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})}$ for SNR of -15 dB, where $\hat{\nu}_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\omega) = \sum_{n=0}^{N-1}\nu_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\tau_n)\exp(-i\omega n)$ is the DFT of $\{\nu_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\tau_n)\}_{n=0}^{N-1}$ at $\omega \in \frac{2\pi k}{N}-\pi$, $k=0,\ldots,N-1$. We note that $\|\hat{\nu}_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\omega)\|^2_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})}$ is the squared $\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$-norm of the spectrum of the sequence $\{\nu_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\tau_n)\}_{n=0}^{N-1}$ at ω, and it is obtained by evaluating the two-dimensional DFT of the sequence $\{[\mathbf{K}_{\rm WW}]_{m,n}\}_{m,n=1}^{N}$ at $(\omega,-\omega)$. One can notice that the squared $\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$-norm of the spectrum is concentrated in low frequencies. Therefore, the sequence $\{\nu_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\tau_n)\}_{n=0}^{N-1}$ can be "compressed" into a few low frequency components and the use of the proposed bound is suitable for this scenario. Fig. 3 depicts the compared bounds on the root MSE (RMSE) as a function of SNR. The RMSE of the MAP and MMSE estimators are depicted as well in order to compare the SNR threshold values predicted by the compared bounds. According to Fig. 3, the proposed bound in (110) is the tightest and allows better prediction of the SNR threshold region.

VIII. Conclusion

In this paper, a new class of Bayesian lower bounds was derived by projecting each entry of the vector-function to be estimated on a Hilbert subspace of $\mathcal{L}_2$. This Hilbert subspace contains linear transformations of elements in the domain of an integral transform, applied on functions, which are used in computations of bounds in the Weiss–Weinstein class. The integral transform generalizes the traditional derivative and sampling operators, applied for computation of existing performance lower bounds. Hence, it was shown that some well-known Bayesian bounds can be derived from this class via specific choices of the integral transform kernel. A new lower bound was derived from the proposed class using the kernel of the Fourier transform. The bound was shown to be computationally manageable and in comparison with other existing bounds, provided better prediction of the SNR threshold region, exhibited by the MAP and MMSE estimators, in the problem of frequency estimation. Examining some other integral transforms, for which new computationally manageable and tight lower bounds will be derived from the proposed class, is a topic for future research.

Appendix A

In this Appendix the Hilbert projection theorem is stated.

Theorem 3: Let $\mathcal{V}$ denote an abstract Hilbert space, $\mathcal{U}$ be a closed subspace of $\mathcal{V}$, and v be an element in $\mathcal{V}$. Then there exists a unique element in $\mathcal{U}$, denoted by $p_{\mathcal{J}}({\rm v}\,|\,\mathcal{U})$, and termed as the *projection* of v on $\mathcal{U}$, which satisfies the following equivalent conditions:

$$\|{\rm v}-p_{\mathcal{J}}({\rm v}\,|\,\mathcal{U})\|_{\mathcal{V}} = \min_{{\rm u}\in\mathcal{U}}\|{\rm v}-{\rm u}\|_{\mathcal{V}}, \tag{130}$$

$$\langle {\rm v}-p_{\mathcal{J}}({\rm v}\,|\,\mathcal{U}),{\rm u}\rangle_{\mathcal{V}} = 0, \quad \forall {\rm u}\in\mathcal{U}. \tag{131}$$

The proof can be found in [28].

Appendix B

In this Appendix, a sufficient condition, according to which $\mathcal{H}_{\varphi}^{(\mathbf{H})}\subset\mathcal{H}_{\phi}$ is derived.

Theorem 4: If $\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\in\mathcal{L}_1^{P\times 2}(\boldsymbol{\Lambda})\triangleq\mathcal{L}_1^{P}(\boldsymbol{\Lambda})\times\mathcal{L}_1^{P}(\boldsymbol{\Lambda})$, $\forall\boldsymbol{\alpha}\in\mathbf{V}$, then $\mathcal{H}_{\varphi}^{(\mathbf{H})}\subset\mathcal{H}_{\phi}$.

Proof: According to the definitions of $\mathcal{H}_{\phi}$ and $\mathcal{H}_{\varphi}^{(\mathbf{H})}$ in (6) and (40), respectively, $\mathcal{H}_{\varphi}^{(\mathbf{H})}\subset\mathcal{H}_{\phi}$ if $\forall\mathbf{a}(\boldsymbol{\tau})\in\mathcal{L}_1^{P}(\mathbf{V})$, the following conditions are satisfied:

1) $\varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta})\in\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$, and
2) $\int_{\boldsymbol{\Theta}}\varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta})f(\mathbf{x},\boldsymbol{\theta})d\boldsymbol{\theta}=0$ for a.e. $\mathbf{x}\in\mathcal{X}$.

Proof of Condition 1: Given any $\mathbf{a}(\boldsymbol{\tau})\in\mathcal{L}_1^{P}(\mathbf{V})$, then

$$\begin{aligned}
&\|\varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta})\|^2_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})}\\
&= \int_{\mathcal{X}\times\boldsymbol{\Theta}}\left|\int_{\boldsymbol{V}}\mathbf{a}^H(\boldsymbol{\alpha})\int_{\boldsymbol{\Lambda}}\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau}d\boldsymbol{\alpha}\right|^2\\
&\quad\times d\mathcal{P}(\mathbf{x},\boldsymbol{\theta})\\
&\le \sum_{k=0}^{P-1}\sum_{l=0}^{1}\sum_{m=0}^{P-1}\sum_{n=0}^{1}\int_{\mathcal{X}\times\boldsymbol{\Theta}}\int_{\boldsymbol{V}}|[\mathbf{a}(\boldsymbol{\alpha})]_k|\int_{\boldsymbol{\Lambda}}|[\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})]_{k,l}|\\
&\quad\times|[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})]_l|d\boldsymbol{\tau}d\boldsymbol{\alpha}\int_{\boldsymbol{V}}|[\mathbf{a}(\boldsymbol{\alpha}')]_m|\\
&\quad\times\int_{\boldsymbol{\Lambda}}|[\mathbf{H}(\boldsymbol{\alpha}',\boldsymbol{\tau}')]_{m,n}||[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}')]_n|d\boldsymbol{\tau}'d\boldsymbol{\alpha}'d\mathcal{P}(\mathbf{x},\boldsymbol{\theta})\\
&= \sum_{k=0}^{P-1}\sum_{l=0}^{1}\sum_{m=0}^{P-1}\sum_{n=0}^{1}\int_{\boldsymbol{V}\times\boldsymbol{\Lambda}\times\boldsymbol{V}\times\boldsymbol{\Lambda}}|[\mathbf{a}(\boldsymbol{\alpha})]_k||[\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})]_{k,l}|\\
&\quad\times\left(\int_{\mathcal{X}\times\boldsymbol{\Theta}}|[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})]_l[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}')]_n|d\mathcal{P}(\mathbf{x},\boldsymbol{\theta})\right)\\
&\quad\times|[\mathbf{a}(\boldsymbol{\alpha}')]_m||[\mathbf{H}(\boldsymbol{\alpha}',\boldsymbol{\tau}')]_{m,n}|d(\boldsymbol{\tau}\times\boldsymbol{\alpha}\times\boldsymbol{\tau}'\times\boldsymbol{\alpha}')
\end{aligned}\tag{132}$$

where the first equality stems from (2), (3), (39), and the definition of $\varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta})$ in (40), the following inequality stems from the triangle inequality and the properties of the Lebesgue integral, and the last equality stems from Tonelli's Theorem [26].

Observing (132), one can notice that

$$\int_{\mathcal{X}\times\boldsymbol{\Theta}} |[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})]_l[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}')]_n| d\mathcal{P}(\mathbf{x},\boldsymbol{\theta}) \leq \|[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})]_l\|_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})} \times \|[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}')]_n\|_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})} < \infty \quad (133)$$

where the first inequality stems from Cauchy–Schwartz inequality [26], and the second one stems from the assumption that $\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \in \mathcal{L}_2^2(\mathcal{X}\times\boldsymbol{\Theta})$, $\forall\boldsymbol{\tau}\in\boldsymbol{\Lambda}$. Hence, it is concluded from (132) and (133) that there exists a positive constant, $c \in \mathbb{R}$, such that

$$\|\varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta})\|^2_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})} \leq c\sum_{k=0}^{P-1}\sum_{l=0}^{1}\sum_{m=0}^{P-1}\sum_{n=0}^{1}\int_{\boldsymbol{V}\times\boldsymbol{\Lambda}\times\boldsymbol{V}\times\boldsymbol{\Lambda}} |[\mathbf{a}(\boldsymbol{\alpha})]_k| \times |[\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})]_{k,l}||[\mathbf{a}(\boldsymbol{\alpha}')]_m| \times |[\mathbf{H}(\boldsymbol{\alpha}',\boldsymbol{\tau}')]_{m,n}| d(\boldsymbol{\tau}\times\boldsymbol{\alpha}\times\boldsymbol{\tau}'\times\boldsymbol{\alpha}'). \quad (134)$$

The integral term in (134) is bounded since

$$\begin{aligned}&\int_{\boldsymbol{V}\times\boldsymbol{\Lambda}\times\boldsymbol{V}\times\boldsymbol{\Lambda}} |[\mathbf{a}(\boldsymbol{\alpha})]_k||[\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})]_{k,l}| \times |[\mathbf{a}(\boldsymbol{\alpha}')]_m||[\mathbf{H}(\boldsymbol{\alpha}',\boldsymbol{\tau}')]_{m,n}| d(\boldsymbol{\tau}\times\boldsymbol{\alpha}\times\boldsymbol{\tau}'\times\boldsymbol{\alpha}')\\ &= \int_{\boldsymbol{V}} |[\mathbf{a}(\boldsymbol{\alpha})]_k| \left(\int_{\boldsymbol{\Lambda}} |[\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})]_{k,l}| d\boldsymbol{\tau}\right) d\boldsymbol{\alpha} \times \int_{\boldsymbol{V}} |[\mathbf{a}(\boldsymbol{\alpha}')]_m| \left(\int_{\boldsymbol{\Lambda}} |[\mathbf{H}(\boldsymbol{\alpha}',\boldsymbol{\tau}')]_{m,n}| d\boldsymbol{\tau}'\right) d\boldsymbol{\alpha}' < \infty\end{aligned} \quad (135)$$

where the first equality stems from Tonelli's Theorem [26], and the following inequality stems from the assumptions that $\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau}) \in \mathcal{L}_1^{P\times 2}(\boldsymbol{\Lambda})$, $\forall\boldsymbol{\alpha}\in\mathbf{V}$, and $\mathbf{a}(\boldsymbol{\alpha}) \in \mathcal{L}_1^P(\mathbf{V})$.

Therefore, it is concluded from (134) and (135) that $\forall \mathbf{a}(\boldsymbol{\alpha}) \in \mathcal{L}_1^P(\mathbf{V})$

$$\|\varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta})\|^2_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})} < \infty\square \quad (136)$$

Proof of Condition 2: Given any $\mathbf{a}(\boldsymbol{\tau}) \in \mathcal{L}_1^P(\mathbf{V})$, according to (39) and the definition of $\varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta})$ in (40)

$$\int_{\boldsymbol{\Theta}} \varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta}) f(\mathbf{x},\boldsymbol{\theta}) d\boldsymbol{\theta} = \int_{\boldsymbol{\Theta}}\int_{\boldsymbol{V}} \mathbf{a}^H(\boldsymbol{\alpha}) \int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \times f(\mathbf{x},\boldsymbol{\theta}) d\boldsymbol{\tau} d\boldsymbol{\alpha} d\boldsymbol{\theta}. \quad (137)$$

In the following, it is shown that under the assumptions that $\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau}) \in \mathcal{L}_1^{P\times 2}(\boldsymbol{\Lambda})$, $\forall\boldsymbol{\alpha}\in\mathbf{V}$, and $\mathbf{a}(\boldsymbol{\alpha}) \in \mathcal{L}_1^P(\mathbf{V})$, integration order in the r.h.s. of (137) can be interchanged.

We first show that

$$\xi(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\alpha},\boldsymbol{\tau}) \triangleq \mathbf{a}^H(\boldsymbol{\alpha})\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) f(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{L}_1(\mathcal{X}\times\boldsymbol{\Theta}\times\mathbf{V}\times\boldsymbol{\Lambda}). \quad (138)$$

Hence

$$\begin{aligned}&\int_{\mathcal{X}\times\boldsymbol{\Theta}\times\boldsymbol{V}\times\boldsymbol{\Lambda}} |\mathbf{a}^H(\boldsymbol{\alpha})\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) f(\mathbf{x},\boldsymbol{\theta})| \times d(\mu(\mathbf{x})\times\boldsymbol{\theta}\times\boldsymbol{\alpha}\times\boldsymbol{\tau})\\ &= \int_{\mathcal{X}\times\boldsymbol{\Theta}\times\boldsymbol{V}\times\boldsymbol{\Lambda}} |\mathbf{a}^H(\boldsymbol{\alpha})\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})| \times d(\mathcal{P}(\mathbf{x},\boldsymbol{\theta})\times\boldsymbol{\alpha}\times\boldsymbol{\tau})\\ &\leq \sum_{m=0}^{P-1}\sum_{n=0}^{1}\int_{\mathcal{X}\times\boldsymbol{\Theta}\times\boldsymbol{V}\times\boldsymbol{\Lambda}} |[\mathbf{a}(\boldsymbol{\alpha})]_m[\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})]_{m,n} \times [\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})]_n| d(\mathcal{P}(\mathbf{x},\boldsymbol{\theta})\times\boldsymbol{\alpha}\times\boldsymbol{\tau})\end{aligned} \quad (139)$$

where the first equality stems from (1) and the following inequality stems from the triangle inequality and the properties of the Lebesgue integral. By Tonelli's Theorem, one obtains

$$\begin{aligned}\gamma_{m,n} &\triangleq \int_{\mathcal{X}\times\boldsymbol{\Theta}\times\boldsymbol{V}\times\boldsymbol{\Lambda}} |[\mathbf{a}(\boldsymbol{\alpha})]_m[\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})]_{m,n}[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})]_n| \times d(\mathcal{P}(\mathbf{x},\boldsymbol{\theta})\times\boldsymbol{\alpha}\times\boldsymbol{\tau})\\ &= \int_{\boldsymbol{V}} |[\mathbf{a}(\boldsymbol{\alpha})]_m| \int_{\boldsymbol{\Lambda}} |[\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})]_{m,n}| \times \left(\int_{\mathcal{X}\times\boldsymbol{\Theta}} |[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})]_n| d\mathcal{P}(\mathbf{x},\boldsymbol{\theta})\right) d\boldsymbol{\tau} d\boldsymbol{\alpha}.\end{aligned} \quad (140)$$

Observing (140), one can notice that

$$\int_{\mathcal{X}\times\boldsymbol{\Theta}} |[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})]_n| d\mathcal{P}(\mathbf{x},\boldsymbol{\theta}) \leq \|[\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})]_n\|_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})} < \infty \quad (141)$$

where the first inequality stems from Hölder's inequality [26], and the second one, stems from the assumption that $\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \in \mathcal{L}_2^2(\mathcal{X}\times\boldsymbol{\Theta})$, $\forall\boldsymbol{\tau}\in\boldsymbol{\Lambda}$. Hence, there exists a constant $c \in \mathbb{R}$, such that

$$\gamma_{m,n} \leq c\int_{\boldsymbol{V}} |[\mathbf{a}(\boldsymbol{\alpha})]_m| \left(\int_{\boldsymbol{\Lambda}} |[\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})]_{m,n}| d\boldsymbol{\tau}\right) d\boldsymbol{\alpha}. \quad (142)$$

Since $\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau}) \in \mathcal{L}_1^{P\times 2}(\boldsymbol{\Lambda})$, $\forall\boldsymbol{\alpha}\in\mathbf{V}$, and $\mathbf{a}(\boldsymbol{\alpha}) \in \mathcal{L}_1^P(\mathbf{V})$, it is concluded from (142) that $\gamma_{m,n} < \infty$. Therefore, according to (138)–(140), $\xi(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\alpha},\boldsymbol{\tau}) \in \mathcal{L}_1(\mathcal{X}\times\boldsymbol{\Theta}\times\mathbf{V}\times\boldsymbol{\Lambda})$.

Thus, by Fubini's Theorem [26], for a.e. $\mathbf{x} \in \mathcal{X}$, $\xi(\boldsymbol{\alpha}, \boldsymbol{\tau}, \mathbf{x}, \boldsymbol{\theta}) \in \mathcal{L}_1(\boldsymbol{\Theta} \times \mathbf{V} \times \boldsymbol{\Lambda})$, and (137) can be written as

$$\int_{\boldsymbol{\Theta}} \int_{\boldsymbol{V}} \mathbf{a}^H(\boldsymbol{\alpha}) \int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) \boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) f(\mathbf{x}, \boldsymbol{\theta}) d\boldsymbol{\tau} d\boldsymbol{\alpha} d\boldsymbol{\theta} = \int_{\boldsymbol{V}} \mathbf{a}^H(\boldsymbol{\alpha}) \int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) \times \left(\int_{\boldsymbol{\Theta}} \boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) f(\mathbf{x}, \boldsymbol{\theta}) d\boldsymbol{\theta} \right) d\boldsymbol{\tau} d\boldsymbol{\alpha}. \tag{143}$$

Therefore, due to the fact that $\int_{\boldsymbol{\Theta}} \boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) f(\mathbf{x}, \boldsymbol{\theta}) d\boldsymbol{\theta} = 0$ for a.e. $\mathbf{x} \in \mathcal{X}$, and $\forall \boldsymbol{\tau} \in \boldsymbol{\Lambda}$, it is concluded from (137) and (143), that $\forall \mathbf{a}(\boldsymbol{\alpha}) \in \mathcal{L}_1^P(\mathbf{V})$

$$\int_{\boldsymbol{\Theta}} \varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x}, \boldsymbol{\theta}) f(\mathbf{x}, \boldsymbol{\theta}) d\boldsymbol{\theta} = 0, \text{for a.e. } \mathbf{x} \in \mathcal{X} \square \tag{144}$$

■

APPENDIX C

In this Appendix, the closed form expression of $\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H})}}$ defined in (42) is derived. Let

$$\varphi_{\tilde{\mathbf{a}}_l,\mathbf{H}}(\mathbf{x}, \boldsymbol{\theta}) \triangleq p_{\mathcal{J}} \left([\mathbf{g}(\boldsymbol{\theta})]_l \,|\, \mathcal{H}_\varphi^{(\mathbf{H})} \right) = \int_{\boldsymbol{V}} \tilde{\mathbf{a}}_l^H(\boldsymbol{\alpha}) \int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) \boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) d\boldsymbol{\tau} d\boldsymbol{\alpha} \tag{145}$$

where $p_{\mathcal{J}}([\mathbf{g}(\boldsymbol{\theta})]_l \,|\, \mathcal{H}_\varphi^{(\mathbf{H})})$ denotes the projection of $[\mathbf{g}(\boldsymbol{\theta})]_l$, $l = 0, \ldots, L-1$, on $\mathcal{H}_\varphi^{(\mathbf{H})}$ and the last equality in (145) stems from the fact that $\varphi_{\tilde{\mathbf{a}}_l,\mathbf{H}}(\mathbf{x}, \boldsymbol{\theta}) \in \mathcal{H}_\varphi^{(\mathbf{H})}$ and from (39) and (40). According to (2), (41) and (145)

$$\left[\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H})}}\right]_{k,l} = \langle \varphi_{\tilde{\mathbf{a}}_k,\mathbf{H}}(\mathbf{x}, \boldsymbol{\theta}), \varphi_{\tilde{\mathbf{a}}_l,\mathbf{H}}(\mathbf{x}, \boldsymbol{\theta}) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} = \int_{\mathcal{X} \times \boldsymbol{\Theta}} \int_{\boldsymbol{V}} \int_{\boldsymbol{\Lambda}} \phi_k(\boldsymbol{\alpha}, \boldsymbol{\tau}, \mathbf{x}, \boldsymbol{\theta}) d\boldsymbol{\tau} d\boldsymbol{\alpha} \times \int_{\boldsymbol{V}} \int_{\boldsymbol{\Lambda}} \phi_l^*(\boldsymbol{\alpha}', \boldsymbol{\tau}', \mathbf{x}, \boldsymbol{\theta}) d\boldsymbol{\tau}' d\boldsymbol{\alpha}' d\mathcal{P}(\mathbf{x}, \boldsymbol{\theta}) \tag{146}$$

where

$$\phi_k(\boldsymbol{\alpha}, \boldsymbol{\tau}, \mathbf{x}, \boldsymbol{\theta}) \triangleq \tilde{\mathbf{a}}_k^H(\boldsymbol{\alpha}) \mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) \boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) \tag{147}$$

$k = 0, \ldots, L-1$.

Subject to:

1) $\mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) \in \mathcal{L}_1^{P \times 2}(\boldsymbol{\Lambda}), \forall \boldsymbol{\alpha} \in \mathbf{V}$;
2) $\tilde{\mathbf{a}}_k(\boldsymbol{\alpha}) \in \mathcal{L}_1^P(\mathbf{V}), \forall k = 0, \ldots, L-1$; and
3) $\boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) \in \mathcal{L}_2^2(\mathcal{X} \times \boldsymbol{\Theta}), \forall \boldsymbol{\tau} \in \boldsymbol{\Lambda}$,

it can be shown using the Tonelli Theorem [26], and the Cauchy–Schwartz inequality (in similar to the proof of the first part of Theorem 4), that the term $\phi_k(\boldsymbol{\alpha}, \boldsymbol{\tau}, \mathbf{x}, \boldsymbol{\theta}) \phi_l^*(\boldsymbol{\alpha}', \boldsymbol{\tau}', \mathbf{x}, \boldsymbol{\theta})$ is absolutely integrable in $\mathcal{X} \times \boldsymbol{\Theta} \times \mathbf{V} \times \boldsymbol{\Lambda} \times \mathbf{V} \times \boldsymbol{\Lambda}$. Thus, by the Fubini Theorem [26], integration order in the r.h.s. of (146) can be interchanged and hence

$$\left[\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H})}}\right]_{k,l} = \int_{\boldsymbol{V}} \int_{\boldsymbol{V}} \tilde{\mathbf{a}}_k^H(\boldsymbol{\alpha}) \int_{\boldsymbol{\Lambda}} \int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) \times \left(\int_{\mathcal{X} \times \boldsymbol{\Theta}} \boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) \boldsymbol{\nu}^T(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}') d\mathcal{P}(\mathbf{x}, \boldsymbol{\theta}) \right) \times \mathbf{H}^H(\boldsymbol{\alpha}', \boldsymbol{\tau}') d\boldsymbol{\tau}' d\boldsymbol{\tau} \tilde{\mathbf{a}}_l(\boldsymbol{\alpha}') d\boldsymbol{\alpha}' d\boldsymbol{\alpha}. \tag{148}$$

Let

$$\mathbf{K}_{\boldsymbol{\nu}}(\boldsymbol{\tau}, \boldsymbol{\tau}') \triangleq \int_{\mathcal{X} \times \boldsymbol{\Theta}} \boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) \boldsymbol{\nu}^T(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}') d\mathcal{P}(\mathbf{x}, \boldsymbol{\theta}) = \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}} \left[\boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) \boldsymbol{\nu}^T(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) \right] \tag{149}$$

denote the autocorrelation kernel of $\boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau})$, and let

$$\mathbf{K}_{\mathbf{H}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}') \triangleq \int_{\boldsymbol{\Lambda}} \int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) \mathbf{K}_{\boldsymbol{\nu}}(\boldsymbol{\tau}, \boldsymbol{\tau}') \times \mathbf{H}^H(\boldsymbol{\alpha}', \boldsymbol{\tau}') d\boldsymbol{\tau}' d\boldsymbol{\tau} \tag{150}$$

denote the transformed autocorrelation kernel of $\boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau})$. Therefore, according to (148)–(150)

$$\left[\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H})}}\right]_{k,l} = \int_{\boldsymbol{V}} \int_{\boldsymbol{V}} \tilde{\mathbf{a}}_k^H(\boldsymbol{\alpha}) \mathbf{K}_{\mathbf{H}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}') \tilde{\mathbf{a}}_l(\boldsymbol{\alpha}') d\boldsymbol{\alpha}' d\boldsymbol{\alpha}. \tag{151}$$

Let $u_k(\mathbf{x}) = [\mathbf{g}(\boldsymbol{\theta})]_k - \varphi_{\tilde{\mathbf{a}}_k,\mathbf{H}}(\mathbf{x})$ denote the projection error of $[\mathbf{g}(\boldsymbol{\theta})]_k$ on $\mathcal{H}_\varphi^{(\mathbf{H})}$. Since $\mathcal{H}_\varphi^{(\mathbf{H})}$ is closed, then by the Hilbert projection theorem, stated in Appendix A, $\varphi_{\tilde{\mathbf{a}}_k,\mathbf{H}}(\mathbf{x})$ is unique and $u_k(\mathbf{x}) \perp \mathcal{H}_\varphi^{(\mathbf{H})}$. Therefore, $\varphi_{\tilde{\mathbf{a}}_k,\mathbf{H}}(\mathbf{x})$ is the unique solution of the following system of equations:

$$\langle \varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x}), \varphi_{\tilde{\mathbf{a}}_k,\mathbf{H}}(\mathbf{x}, \boldsymbol{\theta}) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} = \langle \varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x}), [\mathbf{g}(\boldsymbol{\theta})]_k \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \quad \forall \varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x}) \in \mathcal{H}_\varphi^{(\mathbf{H})}. \tag{152}$$

Similar to the derivation of (151), it can be shown that

$$\langle \varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x}), \varphi_{\tilde{\mathbf{a}}_k,\mathbf{H}}(\mathbf{x}) \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} = \int_{\boldsymbol{V}} \int_{\boldsymbol{V}} \mathbf{a}^H(\boldsymbol{\alpha}) \mathbf{K}_{\mathbf{H}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}') \tilde{\mathbf{a}}_k(\boldsymbol{\alpha}') d\boldsymbol{\alpha}' d\boldsymbol{\alpha} \tag{153}$$

and

$$\langle \varphi_{\mathbf{a},\mathbf{H}}(\mathbf{x}), [\mathbf{g}(\boldsymbol{\theta})]_k \rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} = \int_{\boldsymbol{V}} \mathbf{a}^H(\boldsymbol{\alpha}) \int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}} [\boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) [\mathbf{g}(\boldsymbol{\theta})]_k] d\boldsymbol{\tau} d\boldsymbol{\alpha}. \tag{154}$$

Hence, substitution of (153) and (154) into (152) implies that $\forall \mathbf{a}(\boldsymbol{\tau}) \in \mathcal{L}_1^P(\boldsymbol{\Lambda})$

$$\int_{\boldsymbol{V}} \mathbf{a}^H(\boldsymbol{\alpha}) \int_{\boldsymbol{V}} \mathbf{K}_{\mathbf{H}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}') \tilde{\mathbf{a}}_k(\boldsymbol{\alpha}') d\boldsymbol{\alpha}' d\boldsymbol{\alpha} = \int_{\boldsymbol{V}} \mathbf{a}^H(\boldsymbol{\alpha}) \int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau})[\mathbf{g}(\boldsymbol{\theta})]_k] d\boldsymbol{\tau} d\boldsymbol{\alpha} \tag{155}$$

and thus, $\tilde{\mathbf{a}}_k(\boldsymbol{\alpha})$ is the solution of the following integral equation:

$$\int_{\boldsymbol{V}} \mathbf{K}_{\mathbf{H}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}') \tilde{\mathbf{a}}_k(\boldsymbol{\alpha}') d\boldsymbol{\alpha}' = \int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau})[\mathbf{g}(\boldsymbol{\theta})]_k] d\boldsymbol{\tau}. \tag{156}$$

Finally, the closed form expression of $\mathbf{C}_{\mathcal{H}_\varphi^{(\mathbf{H})}}$ in (42) is obtained by rewriting (151) and (156) in a matrix form, where $\tilde{\mathbf{A}}(\boldsymbol{\alpha}) \triangleq [\tilde{\mathbf{a}}_0(\boldsymbol{\alpha}), \ldots, \tilde{\mathbf{a}}_{L-1}(\boldsymbol{\alpha})]$, and $\boldsymbol{\Gamma}(\boldsymbol{\tau}) \triangleq \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) \mathbf{g}^T(\boldsymbol{\theta})]$.

Appendix D

Theorem 5: Let $\mathcal{H}_\varphi$ and $\mathcal{H}'_\varphi$ denote closed subspaces of $\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})$. If $\mathcal{H}_\varphi \supset \mathcal{H}'_\varphi$ then $\mathbf{C}_{\mathcal{H}_\varphi} \succeq \mathbf{C}_{\mathcal{H}'_\varphi}$.

Proof: Let

$$\mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \triangleq \mathbf{g}(\boldsymbol{\theta}) - \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi), \tag{157}$$

and

$$\mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi) \triangleq \mathbf{g}(\boldsymbol{\theta}) - \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi) \tag{158}$$

denote the projection-errors for $\mathcal{H}_\varphi$ and $\mathcal{H}'_\varphi$, respectively, where $\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)$ is defined in (8). According to (157) and (158)

$$\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) = \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi) + (\mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi) - \mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)). \tag{159}$$

By the Hilbert projection theorem, stated in Appendix A, it is implied that

$$[\mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)]_l \perp \mathcal{H}_\varphi, \quad \forall l = 0, \ldots, L-1 \tag{160}$$

and

$$[\mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi)]_l \perp \mathcal{H}'_\varphi, \quad \forall l = 0, \ldots, L-1. \tag{161}$$

Therefore, since $\mathcal{H}_\varphi \supset \mathcal{H}'_\varphi$ it is implied by (160) that

$$[\mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)]_l \perp \mathcal{H}'_\varphi, \quad \forall l = 0, \ldots, L-1. \tag{162}$$

Thus, due to the fact that $[\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi)]_l \in \mathcal{H}'_\varphi$, $\forall l = 0, \ldots, L-1$, then by (161) and (162)

$$\mathbf{r}^H \left(\mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi) - \mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)\right) \perp \mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi), \forall \mathbf{r} \in \mathbb{C}^L. \tag{163}$$

Hence, according to (159) and the Pythagorean theorem [26]

$$\begin{aligned} \|\mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} &= \|\mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi)\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \\ &\quad + \|\mathbf{r}^H (\mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi) - \mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi))\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})}. \end{aligned} \tag{164}$$

Therefore, due to the fact that

$$0 \le \|\mathbf{r}^H (\mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi) - \mathbf{u}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi))\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} < \infty$$

then

$$\|\mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \ge \|\mathbf{r}^H \mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi)\|^2_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})} \quad \forall \mathbf{r} \in \mathbb{C}^L. \tag{165}$$

Hence, according to (2) and the definition of $\mathbf{C}_{\mathcal{H}_\varphi}$ in (9), $\forall \mathbf{r} \in \mathbb{C}^L$

$$\begin{aligned} \mathbf{r}^H \mathbf{C}_{\mathcal{H}_\varphi} \mathbf{r} &= \mathbf{r}^H \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}_t}[\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi) \mathbf{p}^H_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}_\varphi)] \mathbf{r} \\ &\ge \mathbf{r}^H \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}_t}[\mathbf{p}_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi) \mathbf{p}^H_{\mathcal{J}}(\mathbf{g}(\boldsymbol{\theta}) \,|\, \mathcal{H}'_\varphi)] \mathbf{r} \\ &= \mathbf{r}^H \mathbf{C}_{\mathcal{H}'_\varphi} \mathbf{r}. \end{aligned} \tag{166}$$

Thus, since $\mathbf{C}_{\mathcal{H}_\varphi}$ and $\mathbf{C}_{\mathcal{H}'_\varphi}$ are Hermitian matrices, it is implied by (166) that $\mathbf{C}_{\mathcal{H}_\varphi} \succeq \mathbf{C}_{\mathcal{H}'_\varphi}$. ■

Appendix E

In this Appendix, the identities in (61) and (62) are proved. According to (2), (46), and (54)

$$\mathbf{S}_{\mathbf{U}} = \int_{\boldsymbol{\Lambda}} \mathbf{U}(\boldsymbol{\tau}) \left(\int_{\mathcal{X} \times \boldsymbol{\Theta}} \boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) \mathbf{g}^T(\boldsymbol{\theta}) d\mathcal{P}(\mathbf{x}, \boldsymbol{\theta}) \right) d\boldsymbol{\tau}. \tag{167}$$

Since $\mathbf{U}(\boldsymbol{\tau}) \in \mathcal{L}_1^{P \times 2}(\boldsymbol{\Lambda})$, $\boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) \in \mathcal{L}_2^2(\mathcal{X} \times \boldsymbol{\Theta})$, $\forall \boldsymbol{\tau} \in \boldsymbol{\Lambda}$, and $\mathbf{g}(\boldsymbol{\theta}) \in \mathcal{L}_2^L(\mathcal{X} \times \boldsymbol{\Theta})$, it can be shown using the Tonelli theorem and the Cauchy–Schwartz inequality [26] that $\mathbf{U}(\boldsymbol{\tau}) \boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) \mathbf{g}^T(\boldsymbol{\theta}) \in \mathcal{L}_2^{P \times L}(\boldsymbol{\Lambda} \times \mathcal{X} \times \boldsymbol{\Theta})$. Hence, by the Fubini theorem [26] integration order in the r.h.s. of (167) can be interchanged and

$$\begin{aligned} \mathbf{S}_{\mathbf{U}} &= \int_{\mathcal{X} \times \boldsymbol{\Theta}} \left(\int_{\boldsymbol{\Lambda}} \mathbf{U}(\boldsymbol{\tau}) \boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) d\boldsymbol{\tau} \right) \mathbf{g}^T(\boldsymbol{\theta}) d\mathcal{P}(\mathbf{x}, \boldsymbol{\theta}) \\ &= \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\boldsymbol{\eta}_{\mathbf{U}}(\mathbf{x}, \boldsymbol{\theta}) \mathbf{g}^T(\boldsymbol{\theta})] \end{aligned} \tag{168}$$

where the second equality in (168) stems from (2) and $\boldsymbol{\eta}_{\mathbf{U}}(\mathbf{x}, \boldsymbol{\theta}) \triangleq \int_{\boldsymbol{\Lambda}} \mathbf{U}(\boldsymbol{\tau}) \boldsymbol{\nu}(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\tau}) d\boldsymbol{\tau}$. Therefore, according to (2) and (168)

$$[\mathbf{S}_{\mathbf{U}_k}]_{m,n} = \left\langle [\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x}, \boldsymbol{\theta})]_m, [\mathbf{g}(\boldsymbol{\theta})]_n \right\rangle_{\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta})}. \tag{169}$$

Under the assumption that $[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})]_m \rightarrow \gamma_m(\mathbf{x},\boldsymbol{\theta})$ for a.e. $(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{X}\times\boldsymbol{\Theta}$, as $k \rightarrow \infty$, it is concluded from Theorem 6 in Appendix F that $\gamma_m(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$ and $[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})]_m \rightarrow \gamma_m(\mathbf{x},\boldsymbol{\theta})$ in the $\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$-norm. Therefore, according to proposition (5.21) in [26], regarding the continuity of the inner-product operator, it is implied that

$$\begin{aligned}\lim_{k\rightarrow\infty}&[\mathbf{S}_{\mathbf{U}_k}]_{m,n}\\ &= \lim_{k\rightarrow\infty}\left\langle\left[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})\right]_m, [\mathbf{g}(\boldsymbol{\theta})]_n\right\rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})}\\ &= \left\langle\lim_{k\rightarrow\infty}\left[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})\right]_m, [\mathbf{g}(\boldsymbol{\theta})]_n\right\rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})}\\ &= \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\lim_{k\rightarrow\infty}\left[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})\right]_m[\mathbf{g}(\boldsymbol{\theta})]_n\right]. \end{aligned}\tag{170}$$

Therefore, by rewriting (170) in a matrix form, the equality in (61) is obtained.

According to (2), (44), and (52)

$$\begin{aligned}\mathbf{K}_{\mathbf{U}} = \int_{\boldsymbol{\Lambda}}\int_{\boldsymbol{\Lambda}} &\mathbf{U}(\boldsymbol{\tau})\\ &\times\left(\int_{\mathcal{X}\times\boldsymbol{\Theta}}\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})\boldsymbol{\nu}^T(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}')d\mathcal{P}(\mathbf{x},\boldsymbol{\theta})\right)\\ &\times\mathbf{U}^H(\boldsymbol{\tau}')d\boldsymbol{\tau}'d\boldsymbol{\tau}.\end{aligned}\tag{171}$$

Since $\mathbf{U}(\boldsymbol{\tau}) \in \mathcal{L}_1^{P\times 2}(\boldsymbol{\Lambda})$ and $\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \in \mathcal{L}_2^2(\mathcal{X}\times\boldsymbol{\Theta})$, $\forall\boldsymbol{\tau} \in \boldsymbol{\Lambda}$, it can be shown using the Tonelli theorem and the Cauchy–Schwartz inequality [26] that the term $\mathbf{U}(\boldsymbol{\tau})\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})\boldsymbol{\nu}^T(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}')(\boldsymbol{\theta})\mathbf{U}(\boldsymbol{\tau}') \in \mathcal{L}_2^{P\times P}(\boldsymbol{\Lambda}\times\boldsymbol{\Lambda}\times\mathcal{X}\times\boldsymbol{\Theta})$. Hence, by the Fubini theorem [26] integration order in the r.h.s. of (171) can be interchanged and

$$\begin{aligned}\mathbf{K}_{\mathbf{U}} = \int_{\mathcal{X}\times\boldsymbol{\Theta}}&\left(\int_{\boldsymbol{\Lambda}}\mathbf{U}(\boldsymbol{\tau})\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau}\right)\\ &\times\left(\int_{\boldsymbol{\Lambda}}\boldsymbol{\nu}^T(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}')\mathbf{U}^H(\boldsymbol{\tau}')d\boldsymbol{\tau}'\right)d\mathcal{P}(\mathbf{x},\boldsymbol{\theta})\\ =& \,\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\boldsymbol{\eta}_{\mathbf{U}}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\eta}_{\mathbf{U}}^H(\mathbf{x},\boldsymbol{\theta})\right]\end{aligned}\tag{172}$$

where the second equality in (172) stems from (2) and the definition of $\boldsymbol{\eta}_{\mathbf{U}}(\mathbf{x},\boldsymbol{\theta})$ below (168). Therefore, according to (2) and (172)

$$[\mathbf{K}_{\mathbf{U}_k}]_{m,n} = \left\langle\left[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})\right]_m, \left[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})\right]_n\right\rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})}. \tag{173}$$

As shown above, $[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})]_m \rightarrow \gamma_m(\mathbf{x},\boldsymbol{\theta})$ for a.e. $(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{X}\times\boldsymbol{\Theta} \Rightarrow [\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})]_m \rightarrow \gamma_m(\mathbf{x},\boldsymbol{\theta})$ in the $\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$-norm. Therefore, according to in [26, Prop. (5.21)], regarding the continuity of the inner-product operator, it is implied that

$$\begin{aligned}\lim_{k\rightarrow\infty}&[\mathbf{K}_{\mathbf{U}_k}]_{m,n}\\ &= \lim_{k\rightarrow\infty}\left\langle\left[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})\right]_m, \left[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})\right]_n\right\rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})}\\ &= \left\langle\lim_{k\rightarrow\infty}\left[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})\right]_m, \lim_{k\rightarrow\infty}\left[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})\right]_n\right\rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})}\\ &= \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\lim_{k\rightarrow\infty}\left[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})\right]_m\lim_{k\rightarrow\infty}\left[\boldsymbol{\eta}_{\mathbf{U}_k}(\mathbf{x},\boldsymbol{\theta})\right]_n^*\right]. \end{aligned}\tag{174}$$

Therefore, by rewriting (174) in a matrix form, the equality in (62) is obtained.

APPENDIX F

Theorem 6: let $\phi_k(\mathbf{x},\boldsymbol{\theta}) \triangleq \int_{\boldsymbol{\Lambda}}\mathbf{u}_k^H(\boldsymbol{\tau})\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau}$, $k \in \mathbb{N}$ denote a sequence in $\mathcal{H}_\phi$, where $\mathbf{u}_k(\boldsymbol{\tau}) \in \mathcal{L}_1^2(\boldsymbol{\Lambda})\forall k \in \mathbb{N}$, $\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})$ is defined in (38), and $\mathcal{H}_\phi$ defined in (6) constitutes a closed subspace of $\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$. If $\phi_k(\mathbf{x},\boldsymbol{\theta}) \rightarrow \phi(\mathbf{x},\boldsymbol{\theta})$ for a.e. $(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{X}\times\boldsymbol{\Theta}$ as $k \rightarrow \infty$, then $\phi(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{H}_\phi$ and $\phi_k(\mathbf{x},\boldsymbol{\theta}) \rightarrow \phi(\mathbf{x},\boldsymbol{\theta})$ in the $\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$-norm as $k \rightarrow \infty$.

Proof: First, we find a function in $\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$, which dominates each element of $\{\phi_k(\mathbf{x},\boldsymbol{\theta})\}$. Let $[\mathbf{u}(\boldsymbol{\tau})]_n \triangleq u_n(\boldsymbol{\tau})$, $n = 1, 2$, then by the definition of $\{\phi_k(\mathbf{x},\boldsymbol{\theta})\}$ above, it is implied that

$$\begin{aligned}|\phi_k(\mathbf{x},\boldsymbol{\theta})| &= \left|\int_{\boldsymbol{\Lambda}}\mathbf{u}_k^H(\boldsymbol{\tau})\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau}\right|\\ &= \left|\int_{\boldsymbol{\Lambda}}u_{1,k}^*(\boldsymbol{\tau})\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau} + \int_{\boldsymbol{\Lambda}}u_{2,k}^*(\boldsymbol{\tau})\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau}\right|\\ &\le \left|\int_{\boldsymbol{\Lambda}}u_{1,k}^*(\boldsymbol{\tau})\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau}\right| + \left|\int_{\boldsymbol{\Lambda}}u_{2,k}^*(\boldsymbol{\tau})\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})d\boldsymbol{\tau}\right|\\ &\le \int_{\boldsymbol{\Lambda}}|u_{1,k}(\boldsymbol{\tau})||\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})|d\boldsymbol{\tau} + \int_{\boldsymbol{\Lambda}}|u_{2,k}(\boldsymbol{\tau})||\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})|d\boldsymbol{\tau}\\ &\le \int_{\boldsymbol{\Lambda}}|u_{1,k}(\boldsymbol{\tau})|d\boldsymbol{\tau}\cdot\max_{\boldsymbol{\tau}\in\boldsymbol{\Lambda}}|\nu_{\mathrm{RM}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})| + \int_{\boldsymbol{\Lambda}}|u_{2,k}(\boldsymbol{\tau})|d\boldsymbol{\tau}\cdot\max_{\boldsymbol{\tau}\in\boldsymbol{\Lambda}}|\nu_{\mathrm{WW}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})|\end{aligned}\tag{175}$$

where the second equality stems from the definition of $\boldsymbol{\nu}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})$ in (38), the first inequality stems from the triangle inequality, the second inequality stems from the properties of the Lebesgue integral, and the third inequality stems from the fact that $|\nu_{\rm RM}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})| \leq \max_{\boldsymbol{\tau}\in\boldsymbol{\Lambda}} |\nu_{\rm RM}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})|$ and $|\nu_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})| \leq \max_{\boldsymbol{\tau}\in\boldsymbol{\Lambda}} |\nu_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})| \forall(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{X}\times\boldsymbol{\Theta}$.

Since $u_{n,k}(\boldsymbol{\tau}) \in \mathcal{L}_1(\boldsymbol{\Lambda}) \forall k \in \mathbb{N}$, $n = 1, 2$, then there exist positive constants c_1, c_2, such that $\int_{\boldsymbol{\Theta}} |u_{n,k}(\boldsymbol{\theta})| d\boldsymbol{\theta} \leq c_n \forall k \in \mathbb{N}$, $n = 1, 2$. Hence, by (175) it is implied that

$$|\phi_k(\mathbf{x},\boldsymbol{\theta})| \leq c_1 \cdot \max_{\boldsymbol{\tau}\in\boldsymbol{\Lambda}} |\nu_{\rm RM}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})| + c_2 \cdot \max_{\boldsymbol{\tau}\in\boldsymbol{\Lambda}} |\nu_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})| \quad \forall(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{X}\times\boldsymbol{\Theta}. \quad (176)$$

Moreover, let

$$\phi_{\rm Dom}(\mathbf{x},\boldsymbol{\theta}) \triangleq c_1 \cdot \max_{\boldsymbol{\tau}\in\boldsymbol{\Lambda}} |\nu_{\rm RM}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})| + c_2 \cdot \max_{\boldsymbol{\tau}\in\boldsymbol{\Lambda}} |\nu_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau})|.$$

Since the functions $\nu_{\rm RM}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}), \nu_{\rm WW}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\tau}) \in \mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta}) \forall\boldsymbol{\tau}\in\boldsymbol{\Lambda}$, then $\phi_{\rm Dom}(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$ as well. Therefore, each element of $\{\phi_k(\mathbf{x},\boldsymbol{\theta})\}$ is dominated by $\phi_{\rm Dom}(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$. Hence, by the dominated convergence theorem in $\mathcal{L}_p$ spaces, stated in Theorem 5.2.2 in [27], it is concluded that $\phi(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$ and $\phi_k(\mathbf{x},\boldsymbol{\theta}) \to \phi(\mathbf{x},\boldsymbol{\theta})$ in the $\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta})$-norm, as $k\to\infty$. Since $\mathcal{H}_\phi$ is closed, the limit of any convergent sequence in $\mathcal{H}_\phi$ is contained in $\mathcal{H}_\phi$. Thus, $\phi(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{H}_\phi$. ■

APPENDIX G

Theorem 7: ([26, Th. (8.15)]) Suppose $h(\mathbf{y}) \in \mathcal{L}_1(\mathbb{R}^M)$, $\int_{\mathbb{R}^M} h(\mathbf{y})d\mathbf{y} = 1$, and $\{h_k(\mathbf{y})\} \triangleq \{k^M h(k\mathbf{y})\}$, $k = 1, 2, \ldots$. If $\phi \in \mathcal{L}_p(\mathbb{R}^M)(1 \leq p < \infty)$, then $\lim_{k\to\infty}(h_k(\mathbf{y}) * \phi(\mathbf{y}))(\mathbf{y}') = \phi(\mathbf{y}')$, for every $\mathbf{y}'$ in the Lebesgue set of $\phi(\cdot)$-in particular, for almost every $\mathbf{y}' \in \mathbb{R}^M$, and for every $\mathbf{y}' \in \mathbb{R}^M$ at which $\phi(\cdot)$ is continuous.

The proof can be found in [26].

APPENDIX H

In this Appendix, the proposed bound in (110) is derived via the following steps. First, substitution of $\boldsymbol{\gamma}_{\rm CRF}(\mathbf{x},\boldsymbol{\theta})$ in (106), into (107) yields

$$\mathbf{C}_{\rm CRF} = [\boldsymbol{\Phi}_{\rm BCR}, \boldsymbol{\Phi}_{\rm WW}\mathbf{W}^H]\mathbf{K}_{\rm CRF}^{-1} \times [\boldsymbol{\Phi}_{\rm BCR}, \boldsymbol{\Phi}_{\rm WW}\mathbf{W}^H]^H \quad (177)$$

where $\boldsymbol{\Phi}_{\rm BCR}$ and $\boldsymbol{\Phi}_{\rm WW}$ are defined in (71) and (83), respectively,

$$\mathbf{K}_{\rm CRF} \triangleq {\rm E}_{\mathbf{x},\boldsymbol{\theta}}\left[\boldsymbol{\gamma}_{\rm CRF}(\mathbf{x},\boldsymbol{\theta})\boldsymbol{\gamma}_{\rm CRF}^H(\mathbf{x},\boldsymbol{\theta})\right] = \begin{bmatrix} \mathbf{I}_{\rm BFIM} & \mathbf{G}\mathbf{W}^H \\ \mathbf{W}\mathbf{G}^T & \mathbf{W}\mathbf{K}_{\rm WW}\mathbf{W}^H \end{bmatrix}. \quad (178)$$

$\mathbf{I}_{\rm BFIM}$, defined in (72), is the Bayesian Fisher information matrix

$$\mathbf{G} \triangleq {\rm E}_{\mathbf{x},\boldsymbol{\theta}}\left[\left(\frac{\partial \log f(\mathbf{x},\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right)^T \gamma_{\rm WW}^T(\mathbf{x},\boldsymbol{\theta})\right] \quad (179)$$

and $\mathbf{K}_{\rm WW}$ is defined in (84).

Second, the inverse of the matrix $\mathbf{K}_{\rm CRF}$, in (178) is derived. Assuming that $\mathbf{K}_{\rm CRF}$ is positive-definite, then according to formula (7.7.5) in [29]

$$\begin{aligned}\left[\mathbf{K}_{\rm CRF}^{-1}\right]_{1,1} &= (\mathbf{I}_{\rm BFIM} - \mathbf{G}\mathbf{W}^H(\mathbf{W}\mathbf{K}_{\rm WW}\mathbf{W}^H)^{-1}\mathbf{W}\mathbf{G}^T)^{-1} \\ &= \mathbf{I}_{\rm BFIM}^{-1} + \mathbf{I}_{\rm BFIM}^{-1}\mathbf{G}\mathbf{W}^H \\ &\quad \times \left(\mathbf{W}\left(\mathbf{K}_{\rm WW} - \mathbf{G}^T\mathbf{I}_{\rm BFIM}^{-1}\mathbf{G}\right)\mathbf{W}^H\right)^{-1} \\ &\quad \times \mathbf{W}\mathbf{G}^T\mathbf{I}_{\rm BFIM}^{-1} \end{aligned} \quad (180)$$

where the second equality in (180) stems from the Sherman–Morrison–Woodbury formula [30]

$$\left[\mathbf{K}_{\rm CRF}^{-1}\right]_{1,2} = \mathbf{I}_{\rm BFIM}^{-1}\mathbf{G}\mathbf{W}^H \times \left(\mathbf{W}\left(\mathbf{G}^T\mathbf{I}_{\rm BFIM}^{-1}\mathbf{G} - \mathbf{K}_{\rm WW}\right)\mathbf{W}^H\right)^{-1}, \quad (181)$$

$$\left[\mathbf{K}_{\rm CRF}^{-1}\right]_{2,1} = \left[\mathbf{K}_{\rm CRF}^{-1}\right]_{1,2}^H, \quad (182)$$

$$\left[\mathbf{K}_{\rm CRF}^{-1}\right]_{2,2} = \left(\mathbf{W}\left(\mathbf{K}_{\rm WW} - \mathbf{D}^T\mathbf{I}_{\rm BFIM}^{-1}\mathbf{G}\right)\mathbf{W}^H\right)^{-1}. \quad (183)$$

We note that by [29, Th. (7.7.6)], (183) is positive-definite.

Third, substitution of (180)–(183) into (177) yields

$$\mathbf{C}_{\rm CRF} = \boldsymbol{\Phi}_{\rm BCR}\mathbf{I}_{\rm BFIM}^{-1}\boldsymbol{\Phi}_{\rm BCR}^T + \mathbf{Q}\mathbf{W}^H(\mathbf{W}\mathbf{R}\mathbf{W}^H)^{-1}\mathbf{W}\mathbf{Q}^T \quad (184)$$

where $\mathbf{R} \triangleq \mathbf{K}_{\rm WW} - \mathbf{G}^T\mathbf{I}_{\rm BFIM}^{-1}\mathbf{G}$, and $\mathbf{Q} \triangleq \boldsymbol{\Phi}_{\rm BCR}\mathbf{I}_{\rm BFIM}^{-1}\mathbf{G} - \boldsymbol{\Phi}_{\rm WW}$

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