Performance Analysis of Post Detection Integration Techniques in the Presence of Noise Uncertainty

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Outline

Introduction

- The detection problem in a typical DS-CDMA (GNSS) system is a composite hypothesis testing problem,
  - Under H_1 , parameter space is characterized by time, frequency and noise variance.
  - Under H_0, it is parameterized by the noise variance.
- The test statistic is a combination of coherent and post-coherent technique (known as Post Detection Integration - PDI).
- The coherent integration exploits the deterministic part of the signal and is similar to matched filtering.
- Coherent integration duration is limited due to,
  - Frequency Uncertainty.
  - Navigation data bits.

Introduction

- The limitation results in a block PDI technique, and the test statistic is expressed as,

$$T(Y) = \sum_{k=1}^{M} f\left(Y[k]\right) \tag{1}$$

  where $Y[k]$ is the coherent integration output.
- The final test statistic is a function of the coherent integration output and typically is of a quadratic form.
- The required dwell time (or the sample complexity) is determined from the Neyman-Pearson criterion.
- Further, in order to detect weak signals with a minimum guaranteed performance i.e, (P_{fa}, P_d), longer dwell times are required.

Signal Model

- The distribution of $Y[k]$ is

$$Y[k] \sim \begin{cases} CN(0, \sigma^2), & Under\ \ H_0 \\ CN(\mu_Y, \sigma^2), & Under\ \ H_1 \end{cases}$$

- $\mu_Y = \sqrt{P} N_{coh} \frac{\sin(\pi \Delta f T_{coh})}{\pi \Delta f T_{coh}} R(\Delta\tau) \exp(j\Delta\theta)$.
- P, Δf, $\Delta\tau$, and $\Delta\theta$ are the signal power, offset in time, frequency and phase, respectively.
- The null hypothesis (H_0), is that the estimated doppler and code phase are not close to the true values
- The alternate hypothesis (H_1), is that the estimated doppler and code-phase match the true values.
- Mathematically, $H_0 \in (f \neq \hat{f}, \tau \neq \hat{\tau})$ and $H_1 \in (f = \hat{f}, \tau = \hat{\tau})$.

Signal Model

- The Log-likelihood ratio after averaging phase as a nuisance parameter leads to a test statistic of the form.

$$T(Y) = \left| \sum_{k=1}^{M} Y[k] exp(-j2\pi \Delta f k T_{coh}) \right|^2 \qquad (2)$$

- If $\Delta f \rightarrow 0$, then $T(Y) = \left| \sum_{k=1}^{M} Y[k] \right|^2$, coherent integration.
- The false alarm and detection probability,
  - $P_{fa} = P_0\{T(Y) > \gamma\} = \int_{\gamma}^{\infty} f_{T(Y)|H_0}(z|H_0)dz$
  - $P_d = P_1\{T(Y) > \gamma\} = \int_{\gamma}^{\infty} f_{T(Y)|H_1}(z|H_1)dz$
- P_{fa} and P_d depend on the noise variance σ^2.

Noise Uncertainty Model

- Nominal distribution of noise, $w[n] \sim CN(0, \sigma_n^2)$
- Noise variance fluctuation is assumed deterministic but unknown and is bounded as $\sigma^2 \in [\frac{1}{\beta}\sigma_n^2, \beta\sigma_n^2]$, where $\beta \geq 1$.
- Under noise uncertainty model,
  - $P_{fa} = \max_{\sigma^2 \in [\frac{1}{\beta}\sigma_n^2, \beta\sigma_n^2]} P_{fa}(\sigma^2)$
  - $P_d = \min_{\sigma^2 \in [\frac{1}{\beta}\sigma_n^2, \beta\sigma_n^2]} P_d(\sigma^2)$
- Sensitivity limit exists, if,

$$\min_{\sigma^2 \in [\frac{1}{\beta}\sigma_n^2, \beta\sigma_n^2]} E_1\{T(Y)\} \leq \max_{\sigma^2 \in [\frac{1}{\beta}\sigma_n^2, \beta\sigma_n^2]} E_0\{T(Y)\}$$

- The noise uncertainty model captures the residual uncertainty due to imprecise noise variance calibration and/or temperature variations.

Coherent Integration

- The test statistics of coherent integration is expressed as,

$$T_A = \left| \sum_M Y[k] \right|^2, \tag{3}$$

- Coherent integration is non-robust to frequency uncertainty and unknown data-bits.
- Asymptotically robust to noise uncertainty.

$$\min_{\sigma^2 \in [\frac{1}{\beta}\sigma_n^2, \beta\sigma_n^2]} N_{coh}^2 P + 2N_{coh}\sigma^2 \leq \max_{\sigma^2 \in [\frac{1}{\beta}\sigma_n^2, \beta\sigma_n^2]} 2N_{coh}\sigma^2, \tag{4}$$

- The SNR limit for reliable detection is thus

$$SNR_L \leq \frac{2}{N_{coh}} \left(\beta - \frac{1}{\beta} \right), \tag{5}$$

NC-PDI (Quadratic)

- The test statistics of Non-coherent PDI(NC-PDI) is expressed as,

$$T_{B1} = \sum_{M} |Y[k]|^2, \tag{6}$$

- Non-coherent PDI is similar to energy detection.
- Under H_0, $T(Y)$ is distributed as central χ^2 and is distributed as non-central χ^2 under H_1, with $2M$ degrees of freedom.
- The P_{fa} and P_d are expressed as,

$$P_{fa} = \exp\left(-\frac{\gamma}{2N_{coh}\sigma^2}\right) \sum_{k=0}^{M-1} \frac{1}{k!} \left(\frac{\gamma}{2N_{coh}\sigma^2}\right)^k \tag{7}$$

NC-PDI (Quadratic)

$$P_d = Q_M\left(\sqrt{\frac{\lambda}{N_{coh}\sigma^2}}, \sqrt{\frac{\gamma}{N_{coh}\sigma^2}}\right), \tag{8}$$

where λ is the non-centrality parameter and is related to the signal power, γ is the detection threshold.

- Inverting the marcum-Q function to evaluate the sample complexity M, is not feasible as no closed form solution exists.
- In general, the sample complexity of M to detect weak signal is moderate to high, so the statistics of T(Y) is approximated by Gaussian distribution using CLT.
- The mean and variance of the test statistics under both H_0 and H_1 are well known.

NC-PDI (Quadratic)

- Under H_0,
  - $E\{T_{B1}|H_0\} = 2M(N_{coh}\sigma^2)$
  - $Var\{T_{B1}|H_1\} = 4M(N_{coh}\sigma^2)^2$
- Under H_1,
  - $E\{T_{B1}|H_1\} = M(N_{coh}^2 P) + 2M(N_{coh}\sigma^2)$
  - $Var\{T_{B1}|H_1\} = 4M((N_{coh}\sigma^2)^2 + (N_{coh}^3 P\sigma^2))$
- The expressions for P_{fa} and P_d simplify to,

$$P_{fa} \approx Q\left(\frac{\gamma - 2M(N_{coh}\sigma^2)}{\sqrt{4M}(N_{coh}\sigma^2)}\right), \tag{9}$$

$$P_d \approx Q\left(\frac{\gamma - M(N_{coh}^2 P + 2(N_{coh}\sigma^2))}{\sqrt{4M((N_{coh}\sigma^2)^2 + (N_{coh}^3 P\sigma^2))}}\right) \tag{10}$$

NC-PDI

- In the presence of noise uncertainty, the P_{fa} and P_d are expressed as,

$$\begin{aligned} P_{fa} &= \max_{\sigma^2 \in [\frac{1}{\beta}\sigma_n^2, \beta\sigma_n^2]} Q\left(\frac{\gamma - 2M(N_{coh}\sigma^2)}{\sqrt{4M}(N_{coh}\sigma^2)}\right), \\ &= Q\left(\frac{\gamma - 2M(N_{coh}\beta\sigma_n^2)}{\sqrt{4M}(N_{coh}\beta\sigma_n^2)}\right), \end{aligned} \tag{11}$$

$$\begin{aligned} P_d &= \min_{\sigma^2 \in [\frac{1}{\beta}\sigma_n^2, \beta\sigma_n^2]} Q\left(\frac{\gamma - M(N_{coh}^2 P + 2(N_{coh}\sigma^2))}{\sqrt{4M((N_{coh}\sigma^2)^2 + (N_{coh}^3 P\sigma^2))}}\right) \\ &= Q\left(\frac{\gamma - M(N_{coh}^2 P + 2(N_{coh}\frac{1}{\beta}\sigma_n^2))}{\sqrt{4M((N_{coh}\frac{1}{\beta}\sigma_n^2)^2 + (N_{coh}^3 P\frac{1}{\beta}\sigma_n^2))}}\right) \end{aligned} \tag{12}$$

NC-PDI(Quadratic)

- Simplifying the P_{fa} and P_d expressions, the sample complexity of the NC-PDI technique is expressed as,

$$M = \frac{4\left(Q^{-1}(P_{fa,nu})\beta - Q^{-1}(P_{d,nu})\sqrt{\frac{1}{\beta}\rho_{coh} + \frac{1}{\beta^2}}\right)^2}{\left(\rho_{coh} - 2(\beta - \frac{1}{\beta})\right)^2}, \tag{13}$$

- where, $\rho_{coh} = \frac{N_{coh}P}{\sigma^2}$, SNR at the coherent integration output.
- As $\rho_{coh} \to 2\left(\beta - \frac{1}{\beta}\right)$, $M \to \infty$ i.e., sample complexity becomes unbounded.
- $SNR_{limit} = \frac{2}{N_{coh}}\left(\beta - \frac{1}{\beta}\right)$.
- Noise Uncertainty imposes fundamental sensitivity limits, when the NC-PDI technique is used as the test statistic.

NC-PDI (Non-Quadratic)

- The test statistics of Non-coherent PDI(non-quadratic) is expressed as,

$$T_{B2} = \sum_{M} |Y[k]|, \tag{14}$$

- The performance is similar to that of quadratic case (non-robust to noise uncertainty).
- Analytical characterization for P_{fa} and P_d is not known in closed form.
- The mean and variance of T_{B2} under both the hypotheses are expressed as:

$$E\{T_{B2}|H_0\} = M\sqrt{\frac{\pi N_{coh}\sigma^2}{2}}, \tag{15}$$

$$Var\{T_{B2}|H_0\} = M\left(\frac{4-\pi}{2}\right)N_{coh}\sigma^2, \tag{16}$$

NC-PDI (Non-Quadratic)

$$E\{T_{B2}|H_1\} = M\sqrt{\frac{\pi N_{coh}\sigma^2}{2}}e^{-\frac{\rho_s}{4}}\left[\left(1+\frac{\rho_s}{2}\right)I_0\left(\frac{\rho_s}{4}\right)\frac{\rho_s}{2}I_1\left(\frac{\rho_s}{4}\right)\right],$$

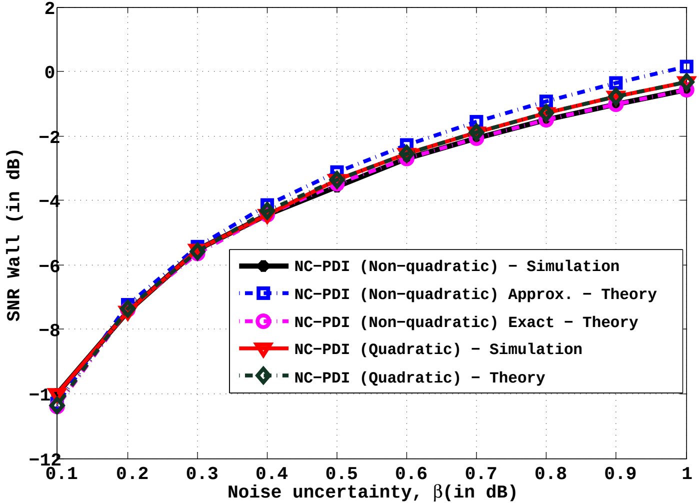
$$Var\{T_{B2}|H_1\} = M\left(2N_{coh}\sigma^2\left(1+\frac{\rho_s}{2}\right)-\left(E\left\{T_{B2}|H_1\right\}\right)^2\right),$$

- Using polynomial approximation and assuming low SNR,

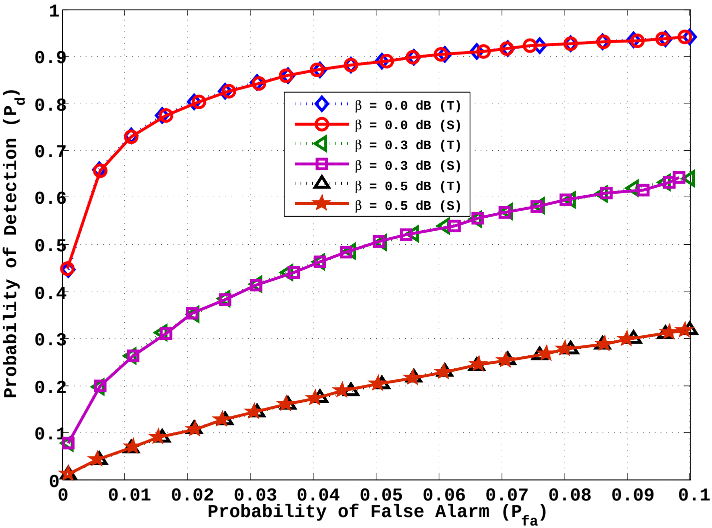
$$\frac{1}{\sqrt{\beta}}\left(\frac{\beta\rho_{coh}}{4}-\frac{(\beta\rho_{coh})^2}{96}+\frac{(\beta\rho_{coh})^3}{768}\right) \le \left(\sqrt{\beta}-\frac{1}{\sqrt{\beta}}\right). \quad (17)$$

- Neglecting higher order terms in the above expression, the SNR limit can be evaluated as $\rho_{coh} \rightarrow SNR_L \triangleq 4\left(1-\frac{1}{\beta}\right)$.

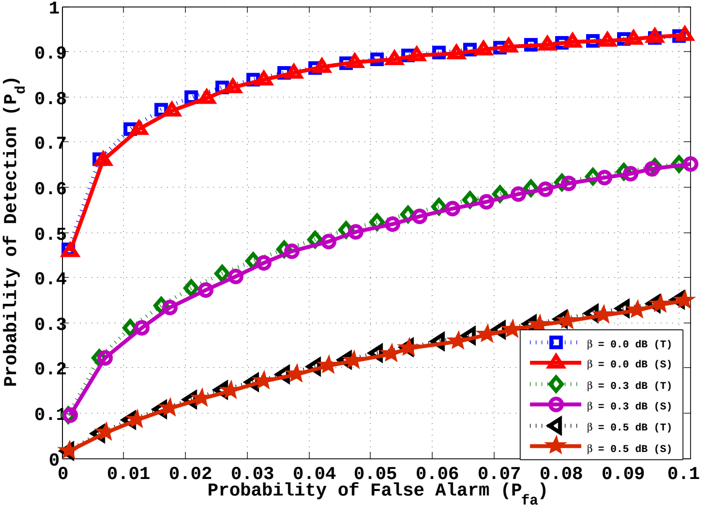
Simulation Results (NC-PDI SNR WALL)



Simulation Results (NC-PDI - Quadratic)



Simulation Results (NC-PDI - Non-Quadratic)



D-PDI

- The test statistics of Differential PDI (D-PDI) is expressed as,

$$T_C = Re\left\{\sum_{m=1}^{M} Y[m](Y[m-1])^{\star}\right\} \tag{18}$$

- Let $Y[m] = I[M] + jQ[m]$, then the test statistics becomes,

$$T_C = \sum_{m=1}^{M} I[m]I[m-1] + Q[m]Q[m-1] \tag{19}$$

- The test statistic is not robust to navigation data-bits and as well as large frequency offset. An alternative is to use D-PDI (Abs) i.e.,

$$T = \left|\sum_{m=1}^{M} Y[m](Y[m-1])^{\star}\right| \tag{20}$$

- However, the above detector is analytically intractable.

D-PDI

- The P_{fa} expression is given by:

$$P_{fa} = \begin{cases} \sum_{k=1,\lambda_k>0}^{M} C_k \exp\left(-\frac{\gamma}{2\lambda_k N_{coh}\sigma^2}\right), & \gamma > 0, \\ 1 - \sum_{k=1,\lambda_k<0}^{M} C_k \left(\exp\left(-\frac{\gamma}{2\lambda_k N_{coh}\sigma^2}\right)\right), & \gamma < 0. \end{cases} \quad (21)$$

- The P_d expression is evaluated numerically as:

$$P_d = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \frac{\sin(\theta(u))}{uc(u)} du, \quad (22)$$

where, $\theta(u)$ and $c(u)$ are derived from the characteristic function as

$$\theta(u) = \sum_{k=1}^{M}\left[\tan^{-1}(2\lambda_k N_{coh}\sigma^2 u) + \frac{\mu_{y_k}^2 \lambda_k N_{coh}\sigma^2 u}{(1+4\lambda_k^2 N_{coh}^2\sigma^4 u^2)}\right] - \gamma u,$$

$$c(u) = \prod_{k=1}^{M}[(1+4\lambda_k^2 N_{coh}^2\sigma^4 u^2)^{\frac{1}{2}}]\exp\left(\frac{1}{2}\sum_{k=1}^{M}\frac{(\mu_{y_k} 2\lambda_k N_{coh}\sigma^2 u)^2}{(1+4\lambda_k^2 N_{coh}^2\sigma^4 u^2)}\right).$$

D-PDI

- To evaluate the sample complexity, the mean and variance are given by:

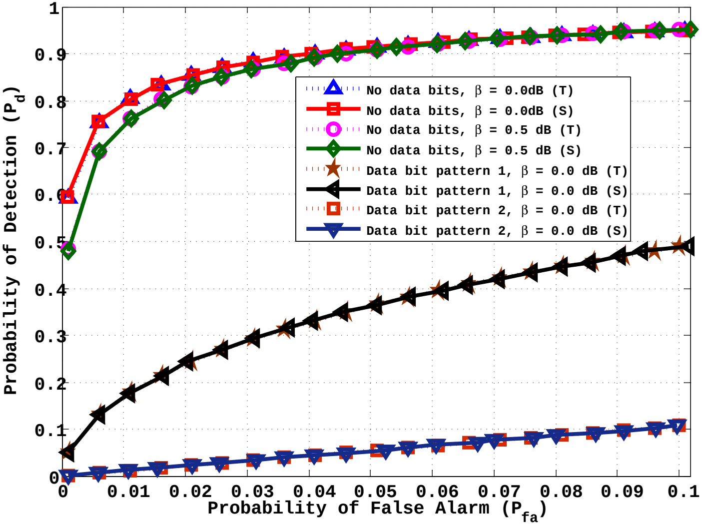
$$\begin{aligned}
E\{T_C|H_0\} &= 0; \\
Var\{T_C|H_0\} &= 2(M-1)N_{coh}^2\sigma^4, \\
E\{T_C|H_1\} &= (M-1)N_{coh}^2 P; \\
Var\{T_C|H_1\} &= 2(M-1)N_{coh}^2\sigma^4 + 2(2M-3)N_{coh}^3 P\sigma^2.
\end{aligned}$$

- The sample complexity in the presence of noise uncertainty:

$$M = 1 + \frac{2\left(Q^{-1}(P_{fa,nu})\beta - Q^{-1}(P_{d,nu})\sqrt{\frac{2\rho_{coh}}{\beta} + \frac{1}{\beta^2}}\right)^2}{\rho_{coh}^2}. \tag{23}$$

- The D-PDI technique is robust to uncertainty in the noise variance. The same analysis can be extended to test-statistic with different spans.

Simulation Results (D-PDI)



Modified-PDI (Data)

- The test statistic is expressed as,

$$T_{D1} = \sum_{k=1}^{M} \left\{ \left| \sum_{n=1}^{N} x_k[n] \right|^2 - \sum_{n=1}^{N} |x_k[n]|^2 \right\}, \tag{24}$$

- where $x_k[n]$ is the output of the T_{coh} milliseconds of coherent integration in the k^{th} PDI block
- T_{coh} typically a fraction of T_b, the data bit duration;
- N is the number of coherent integration outputs within a data bit duration.
- The coherent integration duration is limited and hence it is robust to frequency uncertainty.
- In addition, the coherent integration spans within a data-bit and hence the technique is also robust to noise uncertainty.

Modified-PDI (Data)

- The P_{fa} expression is evaluated numerically as in D-PDI,

$$\begin{aligned}\theta_{P_{fa}}(u) &= \sum_{k=1}^{2}\left[MN_k \tan^{-1}(2\lambda_k N_{coh}\sigma^2 u)\right] - \gamma u,\\ c_{P_{fa}}(u) &= \prod_{k=1}^{2}(1+4\lambda_k^2 N_{coh}^2\sigma^4 u^2)^{\frac{MN_k}{2}}.\end{aligned} \tag{25}$$

- Similarly for P_d,

$$\theta_{P_d}(u) = \sum_{k=1}^{2}\left[MN_k \tan^{-1}(2\lambda_k N_{coh}\sigma^2 u) + K_1\right] - \gamma u,$$

- where $K_1 = MN_k\mu_{y_k}^2\lambda_k N_{coh}\sigma^2 u(1+4\lambda_k^2 N_{coh}^2\sigma^4 u^2)^{-1}]$

$$c_{P_d}(u) = \prod_{k=1}^{2}\left[(1+4\lambda_k^2 N_{coh}^2\sigma^4 u^2)^{\frac{MN_k}{2}}\right]\exp\left(\frac{1}{2}\sum_{k=1}^{2}\frac{(MN_k\mu_{y_k}2\lambda_k N_{coh}\sigma^2 u)^2}{(1+4\lambda_k^2 N_{coh}^2\sigma^4 u^2)}\right) \tag{26}$$

Modified-PDI (Data)

- The mean and variance of the test statistic are expressed as,

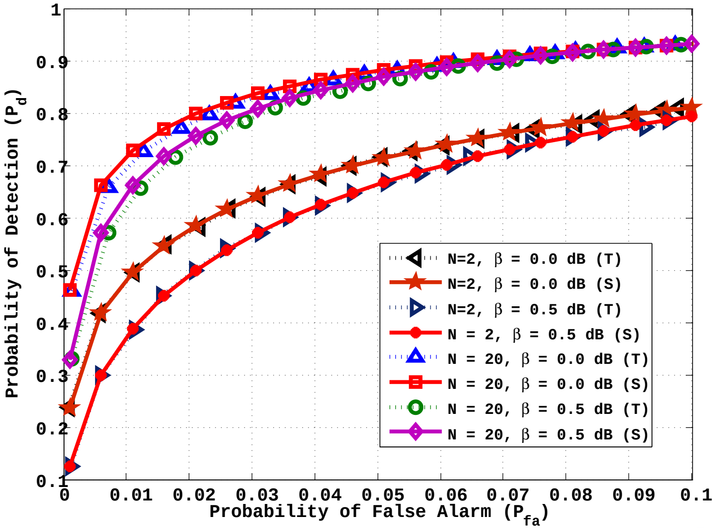
$$\begin{aligned} E\{T_{D1}|H_0\} &= 0, \\ Var\{T_{D1}|H_0\} &= 4MN(N-1)N_{coh}^2\sigma^4 \\ .E\{T_{D1}|H_1\} &= M(N-1)N_{coh}^2 P\alpha^2, \\ Var\{T_{D1}|H_1\} &= 4M(N-1)N_{coh}\sigma^2\left(NN_{coh}\sigma^2 + (N-1)N_{coh}^2 P\alpha^2\right). \end{aligned}$$

- The sample complexity in the presence of uncertainty in the noise variance is given by

$$M = \frac{4\left(Q^{-1}(P_{fa,nu})\sqrt{\frac{N}{(N-1)}\beta^2} - Q^{-1}(P_{d,nu})\sqrt{\frac{N}{(N-1)}\frac{1}{\beta^2} + \frac{\rho_{coh}}{\beta}}\right)^2}{\rho_{coh}^2}, \tag{27}$$

- The modified technique is robust to uncertainty in the noise variance.

Simulation Results (Modifed-PDI - Data)



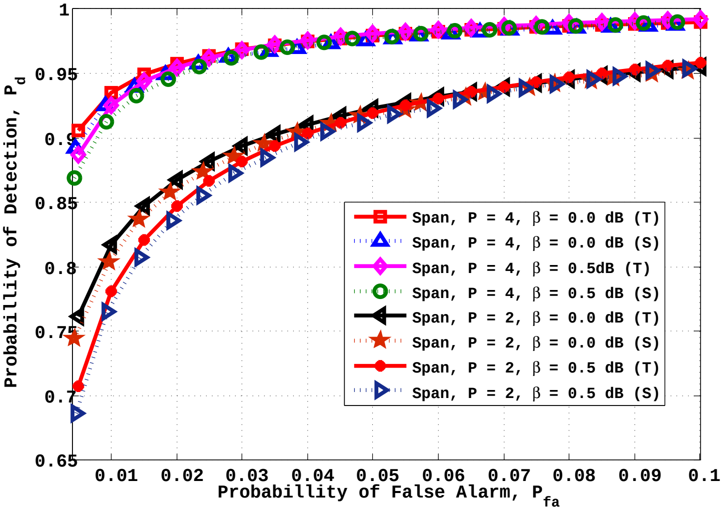
Modified-PDI (Pilot)

- The test statistic is expressed as

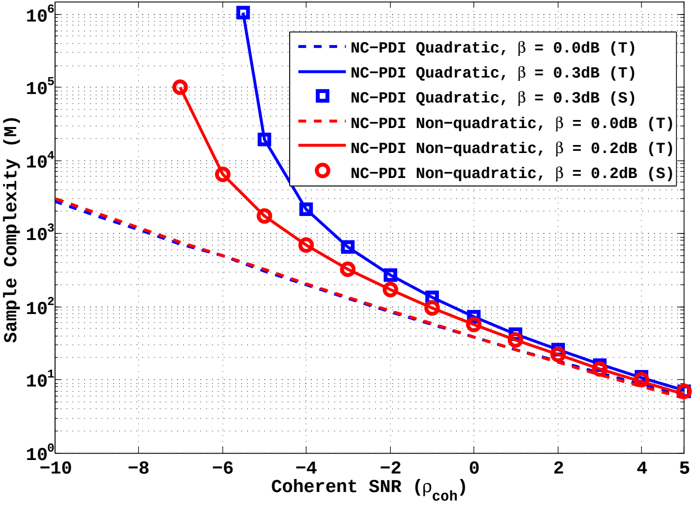
$$T_{D2} = \sum_{p=1}^{P} Re\left\{ \sum_{k=p+1}^{M} x[k]x^*[k-p]\mathrm{sinc}(2pf_{max}T_{coh}) \right\}, \qquad (28)$$

- where f_{max} is the upper bound on the frequency offset which depends on the frequency uncertainty in the coarse synchronization stage.
- L is the number of D-PDI spans included in the test statistic.
- The analytical characterization follows on the similar lines as in the previous section.
- The sample complexity computation is recursive and has to be evaluated numerically.
- Incorporating additional spans in the test-statistic improves detection performance.

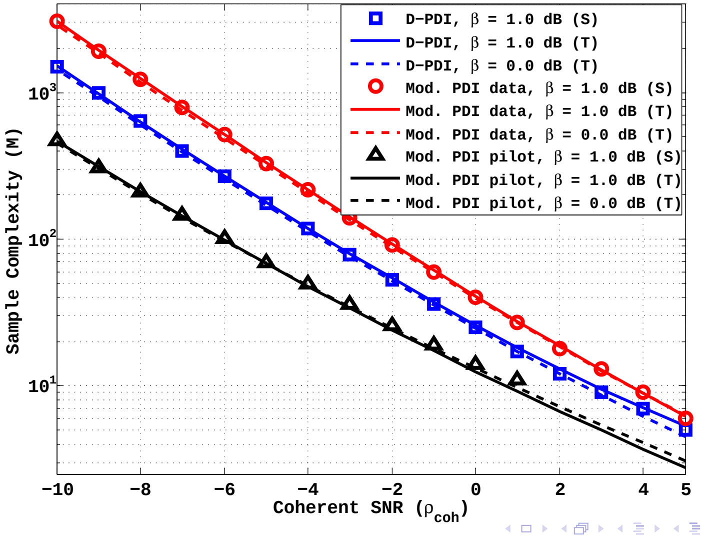
Simulation Results (Modifed-PDI - Pilot)



Simulation Results (Sample Complexity - 1)



Simulation Results (Sample Complexity - 2)



Conclusion

- The performance of the different PDI techniques were analyzed under noise uncertainty.
- It was shown that fundamental sensitivity exists when NC-PDI (both quadratic and non-quadratic) is used as the test statistics.
- The differential PDI (D-PDI) is robust to uncertainty in the noise variance, but it's performance degrades in the presence of data-bits
- A modified PDI technique (Data) was proposed and is shown to be robust to noise, frequency and data-bit uncertainties.
- The modified PDI technique for pilot channel is proposed and the performance improvement obtained by using additional spans is highlighted.

References

- J. Chandrasekhar and C. R. Murthy, "GNSS signal detection under noise uncertainty," *in Proceedings of IEEE International Conference on Communications (ICC´10)*, May 2010.
- J. Chandrasekhar and C. R. Murthy, “Robust GNSS signal detection in presence of navigation data bits,” *Accepted in Proc. IEEE Int. Conf. on Acoustics, Speech and Sig. Proc. (ICASSP)*, May 2011.
- J. Chandrasekhar and C. R. Murthy, “Performance Analysis of Post Detection Integration Techniques in the Presence of Model Uncertainties,” *Submitted to IEEE Trans. on Signal Processing*.
- R. Tandra and A. Sahai, "SNR walls for signal detection," *IEEE J. of Sel. Topics in Sig. Proc.*, vol. 2, no. 1, pp. 4–17, Feb. 2008.
- S.M. Kay, *Fundamentals of Statistical Signal Processing: Detection Theory*. Englewood Cliffs: Prentice-Hall, vol. 2, 1998
- M.H. Zarrabizadeh, E.S. Sousa, "A Differentially Coherent PN Code Acquisition Receiver for CDMA Systems," *IEEE Trans. on Communications*, vol.45, no. 11, pp. 1456–1465, Nov. 1997.