Testing Sparsity over Known and Unknown Bases

Property Testing Algorithms

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Work by

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Introduction

Objectives

- Test the existence of sparse representation
- Design a **property testing algorithm**

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Query input a small number of times

Distinguish between two cases:

- Input satisfies a given property
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- Test the existence of sparse representation
- Design a **property testing algorithm**

Query input a small number of times	**Linear measurements**
Distinguish between two cases: • Input satisfies a given property • Input is "far" from satisfying that property	**Sparse property**

System Model

Linear queries:

$$\tilde{\boldsymbol{Y}} = \boldsymbol{\Phi}\boldsymbol{Y}$$

Test checks for existence of:

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$$

Measurements	$\boldsymbol{Y} \in \mathbb{R}^{m \times p}$
Basis/Dictionary	$\boldsymbol{A} \in \mathbb{R}^{m \times N}$
Sparse representation	$\boldsymbol{X} \in \mathbb{R}^{N \times p}$
Measurement matrix	$\boldsymbol{\Phi} \in \mathbb{R}^{n \times m}$

System Model

$$\tilde{\boldsymbol{Y}} = \boldsymbol{\Phi} \boldsymbol{Y} \qquad \boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$$

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Goal: Given $\tilde{\boldsymbol{Y}}$ check if sparse solution exits

For any fixed $\epsilon, \delta \in (0, 1)$,

$$p_{\text{test}} \begin{cases} > 1 - \delta & \text{if } \boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} \text{ for a } k\text{-sparse solution} \quad \textbf{Completeness} \\ < \delta & \text{if } \|\boldsymbol{Y} - \boldsymbol{A}\boldsymbol{X}\| > \epsilon \ \ \forall k - \text{sparse solutions} \quad \textbf{Soundness} \end{cases}$$

$$p_{\text{test}} \triangleq \mathbb{P}\left\{\text{Tester accepts}\right\}$$

Two Settings

Goal: Check if sparse solution exits

$$p_{\text{test}} \begin{cases} > 1-\delta & \text{if } \boldsymbol{Y} = \boldsymbol{AX} \text{ for a } k\text{−sparse solution} \\ < \delta & \text{if } \|\boldsymbol{Y} - \boldsymbol{AX}\| > \epsilon \;\; \forall k - \text{sparse solutions} \end{cases}$$

Known $\boldsymbol{A}$, $p = 1$

k−sparse solution:

$$\{\boldsymbol{x} \colon \|\boldsymbol{x}\|_0 \leq k\}$$

no structure on $\boldsymbol{A}$

Unknown $\boldsymbol{A}$

k−sparse solution:

$$\{(\boldsymbol{A}, \boldsymbol{X}) \colon \|\boldsymbol{X}_i\|_0 \leq k \;\; \boldsymbol{A} \text{ satisfies RIP}(\epsilon, k)\}$$

Known Design Matrix

Test Design

- **Model:** $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$

Measurements	$\boldsymbol{y} \in \mathbb{R}^{m\times 1}$	
Basis/Dictionary	$\boldsymbol{A} \in \mathbb{R}^{m\times N}$	$\|\boldsymbol{A}_i\| = 1$
Sparse representation	$\boldsymbol{x} \in \mathbb{R}^{N\times 1}$	$\|\boldsymbol{x}\| = 1$

- **Test:** Existence of **k-sparse** solution with **unit norm**

Test Design

- **Model:** $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$

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Sparse representation	$\boldsymbol{x} \in \mathbb{R}^{N\times 1}$	$\lVert \boldsymbol{x} \rVert = 1$

- **Test:** Existence of **k-sparse** solution with **unit norm**

Intuition: Suppose that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$

Since $\boldsymbol{x}$ is $k-$sparse

$$\lVert \boldsymbol{x} \rVert_1 \leq \sqrt{k}\, \lVert \boldsymbol{x} \rVert = \sqrt{k}$$

Thus, $\boldsymbol{y} \in \sqrt{k}\, \mathrm{conv}\left\{\boldsymbol{A} \cup -\boldsymbol{A}\right\} \implies \tilde{\boldsymbol{y}} \in \sqrt{k}\, \mathrm{conv}\left\{\boldsymbol{\Phi}(\boldsymbol{A}_{\pm})\right\}$

Algorithm: Convex-hull membership test

Test: Convex-hull membership

- Set $n = 100\epsilon^{-1} k \log(N/\delta)$, $\boldsymbol{\Phi} \sim \frac{1}{n}\mathcal{N}(0,1)^{n \times m}$
- Observe $\tilde{\boldsymbol{y}} = \boldsymbol{\Phi}\boldsymbol{y}$
- Let $\boldsymbol{A}_{\pm} = \boldsymbol{A} \cup -\boldsymbol{A}$
- Accept iff $\tilde{\boldsymbol{y}} \in \sqrt{k}\,\mathrm{conv}\,\{\boldsymbol{\Phi}(\boldsymbol{A}_{\pm})\}$

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Run time: poly$(N, k, 1/\epsilon)$

Analysis

1. **Completeness:** $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ for some $k-$sparse vector $\boldsymbol{x} \implies$ tester accepts with probability one
2. **Soundness:** $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\| > \epsilon$ for all sparse vector $\boldsymbol{x} \implies$ test rejects with prob. $\leq \delta$

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Analysis: Preliminaries

Approximate Caratheodory's theorem (ACT)

- Given $\mathcal{C} = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_p\}$ where $\|\boldsymbol{w}_i\| \leq 1$
- For every $\boldsymbol{y} \in \mathrm{conv}(\mathcal{C})$ and $K \in \mathbb{N}$

$$\left\| \frac{1}{K} \sum_{j=1}^{K} \boldsymbol{w}_{i_j} - \boldsymbol{y} \right\| \leq \frac{2}{\sqrt{K}}$$

Analysis: Soundness

$\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\| > \epsilon$ for all sparse vector $\boldsymbol{x} \Longrightarrow$ test rejects with prob. $\leq \delta$

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(1) Let $\boldsymbol{A}_{\tilde{\epsilon}}$ be set of all $\frac{2k}{\tilde{\epsilon}^2}$ uniform convex combinations of $\sqrt{k}\boldsymbol{A}_{\pm}$

Analysis: Soundness

$\|\boldsymbol{y} - \boldsymbol{Ax}\| > \epsilon$ for all sparse vector $\boldsymbol{x} \implies$ test rejects with prob. $\leq \delta$

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(2) $\boldsymbol{A}_{\tilde{\epsilon}}$ is $\tilde{\epsilon}-$cover of $\sqrt{k}\text{conv}(\boldsymbol{A}_{\pm})$ **ACT**

- **ACT:** For every $\boldsymbol{y} \in \text{conv}(\mathcal{C})$ and $K \in \mathbb{N}$,

$$\left\| \frac{1}{K} \sum_{j=1}^{K} \boldsymbol{w}_{i_j} - \boldsymbol{y} \right\| \leq \frac{2}{\sqrt{K}}, \boldsymbol{w}_{i_j} \in \mathcal{C}.$$

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(3) With probability at least $1 - \delta/2$, for $\boldsymbol{z} \in \sqrt{k}\mathrm{conv}(\boldsymbol{A}_{\pm})$

$$(1-\epsilon)\|\boldsymbol{y}-\boldsymbol{z}\| \leq \|\boldsymbol{\Phi}(\boldsymbol{y}-\boldsymbol{z})\| \leq (1+\epsilon)\|\boldsymbol{y}-\boldsymbol{z}\|$$

Query: $\mathcal{O}\left(\epsilon^{-2} k \log(N/\delta)\right)$

Analysis: Soundness

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(3) With probability at least $1 - \delta/2$, for $\boldsymbol{z} \in \sqrt{k}\text{conv}(\boldsymbol{A}_{\pm})$

$$(1-\epsilon)\|\boldsymbol{y}-\boldsymbol{z}\| \leq \|\boldsymbol{\Phi}(\boldsymbol{y}-\boldsymbol{z})\| \leq (1+\epsilon)\|\boldsymbol{y}-\boldsymbol{z}\|$$

(4) $\boldsymbol{\Phi}(\boldsymbol{A}_{\tilde{\epsilon}})$ be $\tilde{\epsilon}-$cover of $\sqrt{k}\text{conv}(\boldsymbol{\Phi}(\boldsymbol{A}_{\pm}))$ **ACT**

- **ACT:** For every $\boldsymbol{y} \in \text{conv}(\mathcal{C})$ and $K \in \mathbb{N}$,

$$\left\|\frac{1}{K}\sum_{j=1}^{K}\boldsymbol{w}_{i_j} - \boldsymbol{y}\right\| \leq \frac{2}{\sqrt{K}}, \boldsymbol{w}_{i_j} \in \mathcal{C}.$$

Analysis: Soundness

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(4) $\boldsymbol{\Phi}(\boldsymbol{A}_{\tilde{\epsilon}})$ be $\tilde{\epsilon}-$cover of $\sqrt{k}\text{conv}(\boldsymbol{\Phi}(\boldsymbol{A}_{\pm}))$ **ACT**

(5) If test accepts $\boldsymbol{y}$ with probability $> \delta$

$$\tilde{\boldsymbol{y}} \in \sqrt{k}\,\text{conv}\{\boldsymbol{\Phi}(\boldsymbol{A}_{\pm})\} \implies \left\|\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{A}}_{\epsilon}\boldsymbol{x}\right\| \leq \epsilon \qquad \textbf{Step 4}$$

Analysis: Soundness

$\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}\| > \epsilon$ for all sparse vector $\boldsymbol{x} \Longrightarrow$ test rejects with prob. $\leq \delta$

(1) Let $\boldsymbol{A}_{\tilde{\epsilon}}$ be set of all $\frac{2k}{\tilde{\epsilon}^2}$ uniform convex combinations of $\sqrt{k}\boldsymbol{A}_{\pm}$

(2) $\boldsymbol{A}_{\tilde{\epsilon}}$ is $\tilde{\epsilon}-$cover of $\sqrt{k}\text{conv}(\boldsymbol{A}_{\pm})$ **ACT**

(3) With probability at least $1-\delta/2$, for $\boldsymbol{z} \in \sqrt{k}\text{conv}(\boldsymbol{A}_{\pm})$

$$(1-\epsilon)\|\boldsymbol{y}-\boldsymbol{z}\| \leq \|\boldsymbol{\Phi}(\boldsymbol{y}-\boldsymbol{z})\| \leq (1+\epsilon)\|\boldsymbol{y}-\boldsymbol{z}\|$$

(4) $\boldsymbol{\Phi}(\boldsymbol{A}_{\tilde{\epsilon}})$ be $\tilde{\epsilon}-$cover of $\sqrt{k}\text{conv}(\boldsymbol{\Phi}(\boldsymbol{A}_{\pm}))$ **ACT**

(5) If test accepts $\boldsymbol{y}$ with probability $> \delta$

$$\tilde{\boldsymbol{y}} \in \sqrt{k}\,\text{conv}\{\boldsymbol{\Phi}(\boldsymbol{A}_{\pm})\} \Longrightarrow \left\|\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{A}}_{\epsilon}\boldsymbol{x}\right\| \leq \epsilon \qquad \textbf{Step 4}$$

$$\Longrightarrow \|\boldsymbol{y} - \boldsymbol{A}_{\epsilon}\boldsymbol{x}\| \leq (1-\epsilon)^{-1}\epsilon \leq 2\tilde{\epsilon} \qquad \textbf{Step 3}$$

Known Design Matrix: Summary

- Fix $\epsilon, \delta \in (0, 1)$
- Fix $\boldsymbol{A}$ such that $\|\boldsymbol{A}_i\| = 1$

Tester: Convex hull membership

1. Query: $\mathcal{O}(\epsilon^{-2} \log(p/\delta))$
2. Properties:
   2.1 Completeness: $p_{test} = 1$ if $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, for some $\boldsymbol{x}_i \in \mathcal{S}_k$
   2.2 Soundness: $p_{test} < \delta$ if $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\| \geq \epsilon$ for all $\boldsymbol{x}_i \in \mathcal{S}_{2k/\epsilon^2}$

$\mathcal{S}_k = \{\boldsymbol{x} : \|\boldsymbol{x}\|_0 \leq k \text{ and } \|\boldsymbol{x}\| = 1\}$

Other Property Tests

Unknown Design Matrix

- **Model:** $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$

Measurements	$\boldsymbol{Y} \in \mathbb{R}^{m \times p}$
Basis/Dictionary	$\boldsymbol{A} \in \mathbb{R}^{m \times N}$
Sparse representation	$\boldsymbol{X} \in \mathbb{R}^{N \times p}$

- **Test:** Existence of decomposition to **RIP-compliant** matrix and **k-sparse** matrix

Unknown Design Matrix

- **Model:** $\boldsymbol{Y} = \boldsymbol{AX}$

Measurements	$\boldsymbol{Y} \in \mathbb{R}^{m \times p}$
Basis/Dictionary	$\boldsymbol{A} \in \mathbb{R}^{m \times N}$
Sparse representation	$\boldsymbol{X} \in \mathbb{R}^{N \times p}$

- **Test:** Existence of decomposition to **RIP-compliant** matrix and **k-sparse** matrix

Intuition: Suppose that $\boldsymbol{Y} = \boldsymbol{AX}$

- RIP-property $\implies 1 - \epsilon \leq \|\boldsymbol{y}_i\| \leq 1 + \epsilon$

Step 1 **Check if average of square of linear measurements lie in the interval**

- Gaussian width of $k-$dimensional subset of $\mathbb{R}^N$ is $\leq 2\sqrt{3k \log(N/k)}$

Step 2 **Check if median of entries of linear queries is $\leq 4\sqrt{3k \log(N/k)}$**

Unknown Design Matrix: Guarantees

- Fix $\epsilon, \delta \in (0,1)$
- Fix $k : (k/m)^{1/8} < \epsilon < 1/100$ and $k \geq 10\log(1/\epsilon)$

Tester

1. Query: $\mathcal{O}(\epsilon^{-2}\log(p/\delta))$
2. Properties:
   2.1 Completeness: $p_{test} \geq 1-\delta$ if $\boldsymbol{Y} = \boldsymbol{AX}$, for some $\boldsymbol{A} \in \mathcal{A}_{\epsilon,k}$ $\boldsymbol{X}_i \in \mathcal{S}_k$
   2.2 Soundness: $p_{test} < \delta$ if $\boldsymbol{Y} \neq \boldsymbol{A}(\boldsymbol{X}+\boldsymbol{Z})+\boldsymbol{W}$ for any

$$\begin{cases} \boldsymbol{A} : \boldsymbol{A} \in \mathcal{A}_{\epsilon,k} & \boldsymbol{Z} : \|\boldsymbol{Z}_i\| \leq \epsilon^2 \\ \boldsymbol{X} : \boldsymbol{X}_i \in \mathcal{S} & \boldsymbol{W} : \|\boldsymbol{W}_i\| \leq \mathcal{O}(\epsilon^{1/8}) \end{cases}$$

$\mathcal{A}_{\epsilon,k} : \{\boldsymbol{A} : \|\boldsymbol{A}_i\| = 1 \text{ and } (1-\epsilon) \leq \|\boldsymbol{Ax}\| \leq (1+\epsilon), \forall \boldsymbol{x} \in \mathcal{S}_k\}$
$\mathcal{S}_k = \{\boldsymbol{x} : \|\boldsymbol{x}\|_0 \leq k \text{ and } \|\boldsymbol{x}\| = 1\}$

Dimensionality Test: Design

- **Input:** $\boldsymbol{Y} \in \mathbb{R}^{m \times p}$
- **Test:** Check if rank exceeds k

Dimensionality Test: Design

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- **Test:** Check if rank exceeds k

Intuition: Suppose that $\boldsymbol{Y} = \boldsymbol{AX}$

- Gaussian width of $k-$dimensional subset of $\mathbb{R}^N$ is $\leq 2\sqrt{3k\log(N/k)}$

Algorithm **Check if median of entries of linear queries is $\leq 4\sqrt{3k\log(N/k)}$**

Dimensionality Test: Guarantee

- Fix $\epsilon, \delta \in (0, 1)$
- Fix $k : k \geq 10\epsilon^2 \log(m)$

Tester

1. Query: $\mathcal{O}(\log \delta^{-1})$
2. Properties:
   2.1 Completeness: $p_{test} \geq 1 - \delta$ if $\text{Rank}\{\boldsymbol{Y}\} \leq k$
   2.2 Soundness: $p_{test} < \delta$ if $\text{Rank}\{\boldsymbol{Y} + \boldsymbol{Z}\} \geq \frac{20k}{\epsilon^2}\ \forall \boldsymbol{Z} : \|\boldsymbol{Z}\|_\infty \leq \epsilon$

Reference

- “Testing Sparsity over Known and Unknown Bases.”
    - Siddharth Barman,Arnab Bhattacharyya, Suprovat Ghoshal
    - *arXiv preprint arXiv:1608.01275.* 2016 Aug 3.

Conclusion

Summary

Sparsity/Rank: k $\epsilon, \delta \in (0,1)$

	Known $\boldsymbol{A}$	Unknown $\boldsymbol{A}$	Dimensionality
Query:	$\mathcal{O}(\epsilon^{-1} k \log(N/\delta))$	$\mathcal{O}(\epsilon^{-2} \log(p/\delta))$	$\mathcal{O}(\log(1/\delta))$

$$\tilde{\boldsymbol{Y}} = \boldsymbol{\Phi} \boldsymbol{Y} \qquad \boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$$

Measurements	$\boldsymbol{Y} \in \mathbb{R}^{m \times p}$
Basis/Dictionary	$\boldsymbol{A} \in \mathbb{R}^{m \times N}$
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Measurement matrix	$\boldsymbol{\Phi} \in \mathbb{R}^{n \times m}$

Questions?

Preliminaries

Estimating Gaussian Width using linear queries

- For any $u > 4$, $\epsilon \in (0, 1/2)$ and $\delta > 0$
- Given set $\mathcal{S} \subseteq \{\boldsymbol{v} \in \mathbb{R}^m, \|\boldsymbol{v}\| \in [1 \pm \epsilon]\}$
- Median $\hat{w}$ of $\mathcal{O}(\log(1/\delta\,|\mathcal{S}|)$ linear queries satisfy with probability $1 - \delta$

$$w(\mathcal{S}) - u \leq \hat{w} \leq w(\mathcal{S}) + u$$