Group Discussion
Dual frames as Analysis operator

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Organization

- Introduction to analysis model.
- Basics of frame theory.
- Role of dual frames in sparse representation.
- Existing results on learning analysis operator.
- Dual frames as analysis operator.

Introduction of synthesis model

- Given a measurement vector $y \in \mathbb{R}^m$ and its sensing matrix $\Psi_{m \times d}$, the problem of recovering $x \in \mathbb{R}^d$ can be formulated the following way as finding the sparsest solution of linear equations $y = \Psi x$:

$$P_0 : \min_x \|x\|_0 \text{ subject to } \quad \Psi x = y.$$

- The convex relaxation of P_0 problem can be posed as

$$P_1 : \min_x \|x\|_1 \text{ subject to } \quad \Psi x = y.$$

Introduction of analysis model

- Alongside synthesis, there is an analysis counterpart model.
- Let $\Omega \in \mathbb{R}^{p \times d}$ be a signal transformation or an analysis operator. Its rows are the vectors $(\omega_j)_{j=1}^{p}$ that will be applied to the signals.
- Applying Ω to y, we obtain the (analysis) representation Ωy of y. To capture various aspects of the information in y, we typically have $p \geq d$.

Definition

The co-sparsity ℓ of a signal $y \in \mathbb{R}^d$ with respect to $\Omega \in \mathbb{R}^{p \times d}$ (or simply the co-sparsity of y) is defined to be: $\ell := p - \|\Omega y\|_0$.

- The index set of the zero entries of Ωy is called the co-support of y.
- y is co-sparse when the co-sparsity of y is large, that is, ℓ being close to p.

Synthesis vs Analysis

- In the synthesis model, it is the columns $\Psi_j, j \in T$, associated with the index set T of nonzero coefficients that define the signal subspace. Removing columns from Ψ not in T leaves this subspace unchanged.
- In contrast, it is the rows ω_j associated with the index set T' such that $\langle \omega_j, y \rangle = 0, j \in T'$ that define the analysis subspace. In this case removing rows from Ω for which $\langle \omega_j, y \rangle \neq 0$ leaves the subspace unchanged.

Basics of Frame Theory

Definition

A family of vectors $(\psi_i)_{i=1}^n$ in $\mathbb{R}^m$ is called a frame for $\mathbb{R}^m$, if there exists constants $0 < A \leq B < \infty$ such that

$$A \|x\|^2 \leq \sum_{i=1}^{n} |\langle x, \psi_i \rangle|^2 \leq B \|x\|^2, \forall x \in \mathbb{R}^m$$

- If $A = B$, then $(\psi_i)_{i=1}^n$ is an $A-$tight frame.
- If $A = B = 1$, then it is a Parseval frame.
- If there exits a constant c such that $\|\psi_i\| = c$ for all $i = 1, 2, \ldots, n$ then $(\psi_i)_{i=1}^n$ is an equal norm frame.
- If $c = 1$ then it is a unit-norm frame.

J. Cahill , P.G. Casazza , G. Kutyniok, "Operators and frames," Journal of Operator Theory,2013

Definition

Let $\Psi = (\psi_i)_{i=1}^n$ be a frame in $\mathbb{R}^m$.

- The associated analysis operator $T : \mathbb{R}^m \to \mathbb{R}^n$ is defined by $Tx = \Psi^* x = (\langle x, \psi_i \rangle)_{i=1}^n, x \in \mathbb{R}^m$.
- The associated synthesis operator is defined to be the adjoint operator T^*. A short computation shows that $T^* : \mathbb{R}^n \to \mathbb{R}^m$ is given by $T^* c = \Psi c = \sum_{i=1}^n c_i \psi_i, c = (c_i)_{i=1}^n \in \mathbb{R}^n$.
- The associated frame operator is the map $S : \mathbb{R}^m \to \mathbb{R}^m$ defined to be
$Sx = T^* Tx = \Psi\Psi^* x = \sum_{i=1}^n \langle x, \psi_i \rangle \psi_i, x \in \mathbb{R}^m$.

Role of dual frames in sparse representation

- A dual frame Φ of Ψ provides a representation of a signal satisfying $\Psi x = y$, namely, $x_{\Phi_y} = \Phi^* y$.
- Is it possible to find a dual frame Φ such that $\Phi^* y$ is sparse?

$$D_0 : \arg\min_{\Phi \in S_D} \|\Phi^* y\|_0$$

- $S_D = \{\Phi : \Psi\Phi^* = I\}$ is closed and convex.
- l_1 relaxation of D_0,

$$D_1 : \arg\min_{\Phi \in S_D} \|\Phi^* y\|_1$$

Relation between P_0 and D_0

- Let us define, $S_{D_y} = \{\Phi^* y : \Phi \in S_D\}$ and $S_{P_y} = \{x : \Psi x = y\}$.
- Claim: $S_{D_y} = S_{P_y}$
- It is easily verifiable that $S_{D_y} \subseteq S_{P_y}$.
- Now it is enough to show that $S_{P_y} \subseteq S_{D_y}$.
- Let $x \in S_{P_y}$, then to show $x \in S_{D_y}$, that is $x = \Phi_x^* y$ for some $\Phi_x \in S_D$.
- Let Ψ^+ is canonical dual frame and $x' = (\Psi^+)^* y$. Now x' is a non-trivial solution of $\Psi x = y$.
- Then we have $x = x' + z$ for some $z \in \text{Null}(\Psi)$.

Relation between P_0 and D_0

- As $y \neq 0$, there exist i such that the $i-$th entry y_i of y is non-zero.
- Form a matrix $(Z_y^x)^*_{n\times m}$ such that the $i-$th column is $\frac{1}{y_i}z$ and all other columns are zero columns(note that selection of Z_y^x is not unique).
- $x = x' + z = (\Psi^+ + Z_y^x)^* y = (\Phi_Z)^* y$, where $\Phi_Z = \Psi^+ + Z_y^x$.
- $\Psi\Phi_Z^* = I$ shows that $\Phi_Z \in S_D$.
- Therefore if $\hat{\Phi}$ is the solution of D_0 (respectively D_1), then $\hat{\Phi}^* y$ is the solution of P_0 (respectively P_1).

Continued ...

- $\Phi_1, \Phi_2 \in S_D$ belongs to same equivalence class with respect to y if $\Phi_1^* y = \Phi_2^* y$.

Theorem

For a given system of equation $\Psi x = y$, let $\hat{x}$ be an unique solution for P_0 (respectively P_1). Then there exist an unique solution $\hat{\Phi}$ (up to equivalence) of D_0(respectively D_1) such that $\|\hat{x}\|_0 = \|\hat{\Phi}^* y\|_0$(respectively $\|\hat{x}\|_1 = \|\hat{\Phi}^* y\|_1$).

- Above theorem states that the properties like Restricted Isometry Property (RIP) and Null Space Property (NSP), which establish equivalence between P_0 and P_1, also establish equivalence between D_0 and D_1.
- Therefore, for a given y, it is possible to find a candidate Φ from the set of dual frames of a good synthesis operator Ψ, so that Φ^T acts as an analysis operator on y, that is, the co-sparsity of y with respect to Φ^T becomes large, that is, $\|\Phi^T y\|_0$ is small.

Learning analysis operator

- Let us consider a linear system $\Psi X = Y$.
- We can pose the following optimization problem to learn an analysis operator:

$$A_1 : \arg\min_{\Omega \in \zeta} \|\Omega y\|_1$$

- To exclude trivial solutions, Ω is to restricted to an admissible set ζ.
- ζ is taken as the set of unit norm tight frames.

M. Yaghoobi, S. Nam, R. Gribonval, and M. E. Davies, "ANALYSIS OPERATOR LEARNING FOR OVERCOMPLETE COSPARSE REPRESENTATIONS," 19th European Signal Processing Conference (EUSIPCO 2011)

Sparse Bayesian Learning

- If we use SBL method on $\Psi X = Y$, then we can find a sparse X^*.
- We can find a dual frame Ψ_d corresponding to X^* via solving the following optimization:

$$F_0 : \min_{\Psi_d} f(\Psi_d) \text{ subject to } \quad \Psi\Psi_d^T = I, \quad \Psi_d^T Y = X^*,$$

  where f is convex function of Ψ_d.
- Let Ψ_{d^*} be a dual frame such that $\Psi_{d^*}^T Y = X^*$.
- Let $V = \text{span}\{Y\}$ and $\dim(V) < m$. Select $(P - d)$ vectors from $V^\perp$.
- Form an analysis operator $\Omega_{P\times m}$ by concatenation of $(\Psi_{d^*})_{i=1}^d$ and previously selected $(P - d)$ vectors in any arbitrary order.

Dual frame as analysis operator

- Let $S \in \mathbb{R}^{m \times P}$ be a good synthesis operator.
- Take $\zeta = \{S_d^T \in \mathbb{R}^{P \times m} : SS_d^T = I\}$, that is, S_d is a dual frame of S.
- Assume $SX = Y$ has a sparse solution, then there exist dual frame S_d of S such that $S_d^T Y$ is sparse.
- Intuition: Consider a linear system $SZ = Y$. As S is a good synthesis operator then we can expect to recover a sparse Z^*. According to the previous discussion then we can have a dual frame S_{d^*} of S such that $S_{d^*}^T Y = Z^*$.

References

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Thank You