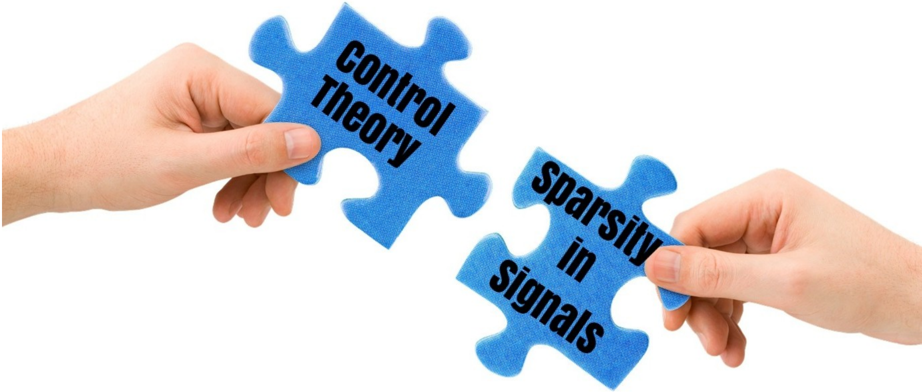
# Controllability of A Linear Dynamical System with Sparsity Constraints

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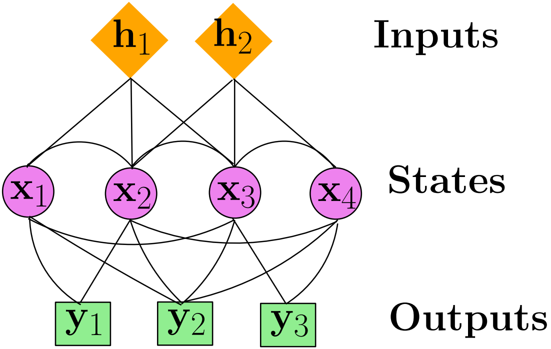
# Controllability and Observabiilty

- **State-space representation**: Mathematical model of a physical system as a set of input, output and state variables




- **Controllability:** indicates if the behaviour of a system can be controlled by acting on its inputs
- **Observability:** indicates if the internal behavior of a system can be detected at its outputs

# Outline

- Basics:
  - Controllability definition
  - Controllability tests
- **Controllability under sparsity constraints on inputs**
- **Characterization of length of input sequence**

# Linear State Space Model

- System Model:

$$\boldsymbol{x}_{k+1} = \boldsymbol{D}\boldsymbol{x}_k + \boldsymbol{H}\boldsymbol{h}_k$$

  - $\boldsymbol{x}_k \in \mathbb{R}^N$: state variables at time instant $k$
  - $\boldsymbol{h}_k \in \mathbb{R}^L$: input at time instant $k$
  - $\boldsymbol{D} \in \mathbb{R}^{N \times N}$: nonzero system transfer matrix
  - $\boldsymbol{H} \in \mathbb{R}^{N \times L}$: input matrix

## Controllable System

For any initial state $\boldsymbol{x}_0$ and any final state $\boldsymbol{x}_f$, there exists an input sequence $\{\boldsymbol{h}_k\}_{k=1}^{K}$ that transfers $\boldsymbol{x}_0$ to $\boldsymbol{x}_f$ in a finite time $K$

- Controllability only relates inputs and states
- Property of the pair $(\boldsymbol{D}, \boldsymbol{H})$

# Controllability

## Controllable System: $(\boldsymbol{D}, \boldsymbol{H})$

For any initial state $\boldsymbol{x}_0$ and any final state $\boldsymbol{x}_f$, there exists an input sequence $\{\boldsymbol{h}_k\}_{k=1}^{K}$ that transfers $\boldsymbol{x}_0$ to $\boldsymbol{x}_f$ in a finite time $K$

$$\boldsymbol{x}_{k+1} = \boldsymbol{D}\boldsymbol{x}_k + \boldsymbol{H}\boldsymbol{h}_k$$

$$\Downarrow$$

$$\boldsymbol{x}_f - \boldsymbol{D}^K\boldsymbol{x}_0 = \underbrace{\begin{bmatrix} \boldsymbol{D}^{K-1}\boldsymbol{H} & \boldsymbol{D}^{K-2}\boldsymbol{H} & \dots & \boldsymbol{H} \end{bmatrix}}_{\text{Controllability matrix}} \underbrace{\begin{bmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \vdots \\ \boldsymbol{h}_K \end{bmatrix}}_{\text{input sequence}} = \tilde{\boldsymbol{H}}_{(K)}\boldsymbol{h}_{(K)}$$

## Controllability Problem

When does the above system always has a solution $\boldsymbol{h}_{(K)}$, for some finite $K$?

# Classical Controllability Tests

## Two equivalent Conditions

(i) **Kalman test:** The controllability matrix $\tilde{\boldsymbol{H}}_{(N)}$ has full row rank:

$$\tilde{\boldsymbol{H}}_{(N)} = \begin{bmatrix} \boldsymbol{D}^{N-1}\boldsymbol{H} & \boldsymbol{D}^{N-2}\boldsymbol{H} & \dots & \boldsymbol{H} \end{bmatrix} \in \mathbb{R}^{N \times NL}$$

(ii) **PBH test[a]:** The rank of $\begin{bmatrix} \lambda \boldsymbol{I} - \boldsymbol{D} & \boldsymbol{H} \end{bmatrix} \in \mathbb{R}^{N \times N+L}$ is $N$, for all $\lambda \in \mathbb{R}$

[a] Popov-Belevitch-Hautus

$L$ : length of input vector

$N$ : length of state vector

$\boldsymbol{D}$ : system transfer matrix

$\boldsymbol{H}$ : input matrix

- Tests are independent of $K$ (the length of input sequence)

# Linear Dynamical System With Sparsity Constraints

- **Goal:** Develop a similar theory for sparse inputs
- Controllability with the input constraint: $\|\boldsymbol{h}_k\|_0 \leq s \leq L, \forall k$
- Equivalent linear system:

$$\boldsymbol{x}_K - \boldsymbol{D}^K \boldsymbol{x}_0 = \tilde{\boldsymbol{H}}_{(K)} \begin{bmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \vdots \\ \boldsymbol{h}_K \end{bmatrix} = \tilde{\boldsymbol{H}}_{(K)} \boldsymbol{h}_{(K)}$$

piece-wise sparse vector $\nearrow$

## Sparse-Controllability Problem

When does the above system always has a piece-wise sparse solution $\boldsymbol{h}_{(K)}$, for some finite $K$?

# Kalman Test Equivalent

## Controllability

$\tilde{\boldsymbol{H}}_{(N)}$ has full row rank:

$$\begin{bmatrix} \boldsymbol{D}^{N-1}\boldsymbol{H} & \boldsymbol{D}^{N-2}\boldsymbol{H} & \dots & \boldsymbol{H} \end{bmatrix} \in \mathbb{R}^{N \times NL}$$

## s-sparse-Controllability

There exists a submatrix of $\tilde{\boldsymbol{H}}_{(K)}$ with full row rank of the following form:

$$\begin{bmatrix} \boldsymbol{D}^{N-1}\boldsymbol{H}_{\mathcal{S}_1} & \boldsymbol{D}^{N-2}\boldsymbol{H}_{\mathcal{S}_2} & \dots & \boldsymbol{H}_{\mathcal{S}_N} \end{bmatrix} \in \mathbb{R}^{N \times Ks},$$

such that $\mathcal{S}_i \subseteq \{1, 2, \dots, L\}$ and $|\mathcal{S}_i| = s$

- $s$-sparse-controllable $\implies$ $\tilde{s}$-sparse-controllable, for all $s \leq \tilde{s} \leq L$
- The column space of the $\boldsymbol{H}_{\mathcal{S}_K}$ should span the left null space of $\boldsymbol{D}$

# PBH Test Equivalent

## Controllability

The rank of $\begin{bmatrix} \lambda \boldsymbol{I} - \boldsymbol{D} & \boldsymbol{H} \end{bmatrix} \in \mathbb{R}^{N \times N+L}$ is $N$, for all $\lambda \in \mathbb{R}$

## s-sparse-controllability

(a) The rank of $\begin{bmatrix} \lambda \boldsymbol{I} - \boldsymbol{D} & \boldsymbol{H} \end{bmatrix}$ is $N$, for all $\lambda \in \mathbb{R}$

(b) There exists an index set $\mathcal{S}$ with $s$ entries such that the rank of $\begin{bmatrix} \boldsymbol{D} & \boldsymbol{H}_{\mathcal{S}} \end{bmatrix}$ is $N$

- Extra condition: There exist an index set $S$ such that the $\boldsymbol{H}_S$ span the left null space of $\boldsymbol{D}$

$$min\left\{\text{Rank}\left\{\boldsymbol{H}\right\}, s\right\} \geq N - \text{Rank}\left\{\boldsymbol{D}\right\}$$

- If $\boldsymbol{D}$ is invertible, controllability $\iff$ sparse-controllability, $\forall s$

# Comparison of Two Tests

| Test type | Kalman-type | PBH-type |
|---|---|---|
| Full row rank test | $\begin{bmatrix} \boldsymbol{D}^{K-1}\boldsymbol{H}_{\mathcal{S}_1} & \ldots & \boldsymbol{H}_{\mathcal{S}_K} \end{bmatrix}$ for some submatrix of $\tilde{\boldsymbol{H}}_{(K)}$ and $K \leq N$ | $\begin{bmatrix} \lambda \boldsymbol{I} - \boldsymbol{D} & \boldsymbol{H} \end{bmatrix}, \forall \lambda$ <br> $\begin{bmatrix} \boldsymbol{D} & \boldsymbol{H}_{\mathcal{S}} \end{bmatrix}$, for some $\mathcal{S}$ |
| Insight | Length of input sequence | Uncontrollable and sparse-uncontrollable parts |
| Rank computations | At most $\binom{L}{s}^N$ | At most $N + \binom{L}{s}$ |
| Numerical stability | Powers of $\boldsymbol{D}$ | No issues |

# Length of Input Sequence

The min. no. of inputs $K^*$ required to steer any given state to any other state:

## Unconstrained System

$$\frac{N}{R_{\boldsymbol{H}}} \leq K^* \leq \min\left\{q, N - R_{\boldsymbol{H}} + 1\right\} \leq N$$

## Sparse System

$$\frac{N}{\min\left\{R_{\boldsymbol{H}}, s\right\}} \leq K^* \leq \min\left\{q\left\lceil\frac{R_{\boldsymbol{H}}}{s}\right\rceil, N + 1 - R^*_{\boldsymbol{H},s}\right\} \leq N$$

$$R^*_{\boldsymbol{H},s} = \max_{\substack{\mathcal{S}\in\{1,2,\ldots,L\}\\ |\mathcal{S}|=s}} \text{Rank}\left\{\boldsymbol{H}_{\mathcal{S}}\right\} \geq \max\left\{1, N - R_{\boldsymbol{D}}\right\}$$

Bounds are invariant under multiplication of ***D*** or ***H*** by an invertible matrix

# Summary

- **System:** Linear dynamical systems with sparsity constraints on the input
- **Results:**
  - Necessary and sufficient conditions for controllability of
    - Algebraic conditions based on rank computations of suitable matrices
    - Classical results can be seen as special cases
  - Characterize the length of the shortest input which ensures the controllability of the system
- **Future work:** Extension to the case when the magnitude of the sparse vectors are bounded

Thank You!