Application of Ramanujan graph in matrix completion problem

Pradip Sasmal

IISc Bangalore

25/05/2019

Ramanujan graph

Let $\mathcal{G}$ be a connected $d-$regular graph with n vertices, and let $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ be the eigenvalues of the adjacency matrix G of $\mathcal{G}$. Because $\mathcal{G}$ is connected and $d-$regular, its eigenvalues satisfy $d = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq -d$. Whenever there exists λ_i with $|\lambda_i| < d$, define $\lambda(G) = \max_{|\lambda_i|<d} |\lambda_i|$.

- $\mathbf{1}$ is the eigen vector corresponding to eigen value d.
- $\lambda_1(G) \geq 2\sqrt{d-1} - \epsilon$

Definition

A $d-$regular graph $\mathcal{G}$ is a Ramanujan graph if $\lambda(G) \leq 2\sqrt{d-1}$, that is the second largest eigen value $\lambda_1(G) \leq 2\sqrt{d-1}$.

Definition

A $(d_1, d_2)-$biregular graph $\mathcal{G}$ is a Ramanujan graph if $\lambda(G) \leq \sqrt{d_1 - 1} + \sqrt{d_2 - 1}$, that is the second largest singular value $\lambda_1(G) \leq \sqrt{d_1 - 1} + \sqrt{d_2 - 1}$.

Problem Statement

- Let $M \in \mathbb{R}^{n_1 \times n_2}$ be a rank$-r$ matrix and let $n_1 \geq n_2$. Define $n = \max\{n_1, n_2\} = n_1$. Let $M = U\Sigma V^T$ be the SVD of M and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ be the singular values of M.
- We observe a small number of entries of M indexed by a set $\Omega \in [n_1] \times [n_2]$. That is, we observe $M_{ij}, \forall (i,j) \in \Omega$. Define the sampling operator $P_\Omega : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{n_1 \times n_2}$ as:

$$P_\Omega(M) = \begin{cases} M_{ij}, & \text{if } (i,j) \in \Omega \\ 0, & \text{if } (i,j) \notin \Omega \end{cases} \quad (1)$$

Objective

The goal in universal matrix completion is to design a set Ω and a recovery algorithm, s.t., all rank$-r$ matrices M can be recovered using only $P_\Omega(M)$.

Connection to bipartite graph

- Define a bipartite graph associated with the sampling operator P_Ω.
- let $\mathcal{G} = (V, E)$ be a bipartite graph where $V = \{1, 2, \ldots, n_1\} \cup \{1, 2, \ldots, n_2\}$ and $(i, j) \in E$ iff $(i, j) \in \Omega$.
- Let $G \in \mathbb{R}^{n_1 \times n_2}$ be the biadjacency matrix of the bipartite graph $\mathcal{G}$ with $G_{ij} = 1$ iff $(i, j) \in \Omega$.
- Note that, $P_\Omega(M) = M.G$, where . denotes the Hadamard product.

Assumptions

Assumptions on G/Ω

- (G 1) Top singular vectors of G are all ones vector.
- (G 2) $\sigma_1(G) = d$ and $\sigma_2(G) \leq C\sqrt{d}$.

Incoherence Assumptions

- (A 1) $\|U^i\|_2^2 \leq \frac{\mu_0 r}{n_1}, \forall\, i$ and $\|V^j\|_2^2 \leq \frac{\mu_0 r}{n_2}, \forall\, i$
- (A 2) $\| \sum_{k \in S} \frac{n_1}{d} U^k {U^k}^T - I\|_2 \leq \delta_d,\ \forall\, S \subset [n_1], |S| = d$ and $\| \sum_{k \in S} \frac{n_2}{d'} V^k {V^k}^T - I\|_2 \leq \delta_d,\ \forall\, S \subset [n_2], |S| = d, d' = \frac{dn_2}{n_1}$.

Main Results

Matrix approximation

[1]

Theorem

Let $\mathcal{G}$ be a $d-$regular bipartite graph satisfying (G 1) and (G 2). Let M be a rank$-r$ matrix that satisfies assumption (A 1). Then,

$$\|\frac{n}{d}P_{\Omega}(M) - M\|_2 \leq \frac{C\mu_0 r}{\sqrt{d}}\|M\|_2,$$

That is,

$$\|\frac{n}{d}P_k(P_{\Omega}(M)) - M\|_2 \leq \frac{C\mu_0 r}{\sqrt{d}}\|M\|_2.$$

for any $k \geq r$, where $P_k(A)$ is the best rank$-k$ approximation of A and can be obtained using top$-k$ singular vectors of A.

---

[1] Srinadh Bhojanapalli and Prateek Jain, "Universal Matrix Completion," Proceedings of the 31 st International Conference on Machine Learning, Beijing, China, 2014.

Nuclear norm minimization

Convex relaxation

$$\begin{aligned} \min_{X} \quad & \text{rank}(X) \\ \text{subject to} \quad & P_{\Omega}(X) = P_{\Omega}(M), \end{aligned} \tag{2}$$

converted to

$$\begin{aligned} \min_{X} \quad & \|X\|_* \\ \text{subject to} \quad & P_{\Omega}(X) = P_{\Omega}(M), \end{aligned} \tag{3}$$

where $\|X\|_*$ denote the nuclear norm of X.

Results on Nuclear norm minimization

Existing result

Nuclear norm minimization technique is a popular technique for the low-rank matrix completion problem and has been shown to provably recover the true matrix, assuming that Ω is sampled uniformly at random and $|\Omega| \geq cnr \log n$.[2]

Universal recovery result

Theorem

Let M be an $n_1 n_2$ matrix of rank r satisfying assumptions (A 1) and (A 2) with $\delta_d \leq \frac{1}{6}$, and Ω is generated from a $d-$regular graph $\mathcal{G}$ that satisfies the assumptions (G 1) and (G 2). Also, let $d \geq 36C^2\mu_0^2 r^2$, i.e., $|\Omega| = nd \geq 36C^2\mu_0^2 r^2 \max\{n_1, n_2\}$. Then M is the unique optimum of (3).

---

[2] E. J. Candes and T. Tao, "The Power of Convex Relaxation: Near-Optimal Matrix Completion," in IEEE Transactions on Information Theory, vol. 56, no. 5, pp. 2053-2080, May 2010.

Random $d-$regular graph

- The second singular value of a random $d-$regular graph is $\leq 2\sqrt{d-1}+\epsilon$, for every *epsilon* > 0, with high probability [3]. Hence, a random $d-$regular graph, with high probability, obeys (G 1) and (G 2).

Theorem

Let M be an $n_1 n_2$ matrix of rank r satisfying assumptions (A 1) and (A 2) with $\delta_d \leq \frac{1}{6}$, and Ω is generated from a random $d-$regular graph $\mathcal{G}$, M is the unique optimal solution of (3) when $d \geq 36 * 4\mu_0^2 r^2$, with high probabality.

[3] Joel. A Friedman, "proof of alon's second eigenvalue conjecture," In Proceedings of the thirty-fifth annual ACM symposium on Theory of computing, pp. 720–724. ACM, 2003.

Alternative of Nuclear norm minimization

Definition

$\gamma_2(M) = \min_{UV^*=M} \|U\|_{\ell_2 \to \ell_\infty^{n_1}} \|V\|_{\ell_2 \to \ell_\infty^{n_2}}$, the minimum is taken over all possible factorizations of $M = UV^*$, and the norm $\|X\|_{\ell_2 \to \ell_\infty^n} = \max_i \sqrt{\sum_j X_{ij}^2}$ returns the largest ℓ_2 norm of a row. Equivalently,

$$\gamma_2(M) = \min_{UV^*=M} \max_{i,j} \|u_i\|_2 \|v_j\|_2.$$

- $\gamma_2(M) \leq \sqrt{\text{rank}(M)} \|M\|_\infty$
- $\gamma_2(M) \leq \|M\|_*$

Optimization problem

$$\begin{aligned} \min_{X} \quad & \gamma_2(X) \\ \text{subject to} \quad & P_\Omega(X) = P_\Omega(M), \end{aligned} \tag{4}$$

Theorem

[4] *Solving (4),* $\|\frac{1}{n^2}P_\Omega(M) - M\|_F^2 \leq c\gamma_2(M)^2\frac{\eta}{d}$, *where* Ω *is* $d-$*regular graph and* η *is the second largest singular value.*

[4] Eyal Heiman, Gideon Schechtman, and Adi Shraibman. Deterministic algorithms for matrix completion. Random Structures Algorithms, 45(2):306–317, September 2014.

Continue ..

Theorem

[5] *Solving (4),* $\|\frac{1}{n_1 n_2} P_\Omega(M) - M\|_F^2 \leq c\gamma_2(M)^2 \frac{\eta}{\sqrt{d_1 d_2}}$*, where* Ω *is* $(d_1, d_2)-$*biregular graph and* η *is the second largest singular value.*

Noisy matrix completion

$$\begin{aligned} \min_X \quad & \gamma_2(X) \\ \text{subject to} \quad & \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} ((P_\Omega(X))_{ij} - M_{ij}) \leq \delta^2, \end{aligned} \tag{5}$$

[5] Gerandy Brito, Ioana Dumitriu and Kameron Decker Harris, " Spectral gap in random bipartite biregular graphs and applications"' arXiv:1804.07808

Theorem

Suppose $Z_{ij} = M_{ij} + \epsilon_{ij}$ *with*

$$\frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} \epsilon_{ij}^2 \leq \delta^2.$$

Then, solving (5), $\|\frac{1}{n_1 n_2} P_\Omega(M) - M\|_F^2 \leq c\gamma_2(M)^2 \frac{\eta}{\sqrt{d_1 d_2}} + 4\delta^2$, *where* Ω *is* $(d_1, d_2)-$*biregular graph and* η *is the second largest singular value.*

THANK YOU!