Ramanujan Sums in the Context of Signal Processing

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August 24, 2019

Ramanujan Sum

- The Ramanujan sum:

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{j2\pi kn/q} \qquad (1)$$

- $c_{10}(n) = e^{j2\pi n/10} + e^{j6\pi n/10} + e^{j14\pi n/10} + e^{j18\pi n/10}$.
- An arithmetic function $x(n)$: infinite sequence defined for $1 \leq n \leq \infty$, and is usually (but not necessarily) integer valued.
- Examples: the Mobius function $\mu(n)$, Euler's totient function $\Phi(n)$.
- $x(n) = \sum_{q=1}^{\infty} \alpha_q c_q(n), \quad n \geq 1.$
- The Ramanujan Fourier transform expansion (i.e., α_q are the RFT coefficients)

P. P. Vaidyanathan, "Ramanujan Sums in the Context of Signal Processing—Part I: Fundamentals," in IEEE Transactions on Signal Processing, vol. 62, no. 16, pp. 4145-4157, Aug.15, 2014.

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- The q^{th} Ramanujan sum $(q \geq 1)$ is a sequence in n defined as

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{j2\pi kn/q} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} W_q^{-kn},$$

  $W_q = e^{-j2\pi/q}$ is the qth root of unity
- $c_q(0) = \phi(q)$, $c_q(n+q) = c_q(n)$
- the DFT of $c_q(n)$:

$$C_q[k] = \sum_{n=0}^{q-1} c_q(n) W^{nk} = \begin{cases} q & if (k,q) = 1 \\ 0 & otherwise. \end{cases}$$

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- $c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} W_q^{kn} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} W_q^{-kn} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \cos \frac{2\pi kn}{q}$
- The Ramanujan sum has period q in the argument n. Unlike sines and cosines, the quantity $c_q(n)$ is always integer valued, which is often an attractive property.
- The first few Ramanujan sequences, shown for one period $0 \leq n \leq q-1$.
  $c_1(n) = 1,\ c_2(n) = 1, -1,\ c_3(n) = 2, -1, -1,$
  $c_4(n) = 2, 0, -2, 0,\ c_5(n) = 4, -1, -1, -1, -1,$
  $c_6(n) = 2, 1, -1, -2, -1, 1,$
  $c_7(n) = 6, -1, -1, -1, -1, -1, -1,$
  $c_8(n) = 4, 0, 0, 0, -4, 0, 0, 0,\ c_9(n) = 6, 0, 0, -3, 0, 0, -3, 0, 0,$
  $c_{10}(n) = 4, 1, -1, 1, -1, -4, -1, 1, -1, 1.$

Properties:

- α primitive q^{th} root of unity if $\alpha^q = 1$, but $\alpha^n \neq 1$ for any positive integer $n < q$. W^{-kq} a primitive q^{th} root of unity iff $(q, k) = 1$. So the Ramanujan sum $c_q(n)$ can be defined as the sum of n^{th} powers of all the q^{th} primitive roots of unity.
- $\sum_{n=0}^{q-1} c_q(n) = 0, \quad \text{for } q > 1.$
- $\sum_{n=0}^{q-1} c_q^2(n) = q\phi(q)$
- Orthogonality: Any two Ramanujan sums $c_{q_1}(n)$ and $c_{q_2}(n)$ are orthogonal in the sense that

$$\sum_{n=0}^{m-1} c_{q_1}(n) c_{q_2}(n) = 0, \quad q_1 \neq q_2,$$

where $m = \operatorname{lcm}(q_1, q_2)$.

- For q prime: $c_q(n) = \begin{cases} q-1 & \textit{if}\,\mathrm{n}= \text{mul. of} \quad \mathrm{q} \\ -1 & \textit{otherwise}. \end{cases}$
- For $q = p^m$: $c_{p^m}(n) = \begin{cases} 0 & \textit{if}\,\mathrm{p}^{m-1} \not| \, n \\ -p^{m-1} & \textit{if}\,\mathrm{p}^{m-1}|n \text{ but } p^m \not| \, n \\ p^{m-1}(p-1) & \textit{if}\,\mathrm{p}^m|n \end{cases}$
- Multiplicative property: $c_{q_1 q_2}(n) = c_{q_1}(n) c_{q_2}(n)$.
- the Mobius function: $\mu(n) = \begin{cases} 1 & \textit{if}\,\mathrm{n}{=}1 \\ (-1)^K & \textit{if}\,\mathrm{n}{=}\mathrm{p}_1 p_2 \dots p_K \\ 0 & \textit{otherwise}. \end{cases}$
- $c_q(n) = \mu(q), \quad \text{whenever } (q, n) = 1.$

Ramanujan Subspace:

- $\mathbf{B_q} = \begin{bmatrix} c_q(0) & c_q(q-1) & c_q(q-2) & \dots & c_q(1) \\ c_q(1) & c_q(0) & c_q(q-1) & \dots & c_q(2) \\ c_q(2) & c_q(1) & c_q(0) & \dots & c_q(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_q(q-2) & c_q(q-3) & c_q(q-4) & \dots & c_q(q-1) \\ c_q(q-1) & c_q(q-2) & c_q(q-3) & \dots & c_q(0) \end{bmatrix}$
- Ramanujan Space: The column space of $\mathbf{B_q}$ will be called the Ramanujan subspace $\mathbf{S_q}$.
- $c_q(n)$, and all its circularly shifted versions, belong to this space.

- Rank and dimension: There are $\phi(q)$ nonzero eigenvalues, and $\mathbf{B_q}$ has rank $\phi(q)$. So, $\mathbf{S_q}$ has dimension $\phi(q)$.
- Positive semidefiniteness: $\mathbf{B_q}$ is Hermitian with nonnegative eigenvalues $\in \{0, q\}$, it is positive semidefinite.
- Factorization: The circulant $\mathbf{B_q}$ can be factorized as

$$\mathbf{B}_q = \underbrace{\mathbf{V}}_{q\times\phi(q)} \underbrace{\mathbf{V}^\dagger}_{\phi(q)\times q},$$

  where V a submatrix of the DFT matrix W obtained by retaining the "coprime columns", i.e., columns numbered k_i such that $(k_i, q) = 1$.
- Any consecutive $\phi(q)$ columns of $\mathbf{B_q}$ are linearly independent.

Finite duration (FIR) signals:

- Arithmetic function expansion:

$$x(n) = \sum_{q=1}^{\infty} \alpha_q c_q(n), \quad n \geq 1,$$

  where

$$\alpha_q = \frac{1}{\phi(q)} \left(\lim_{M \to \infty} \frac{1}{M} \sum_{n=1}^{M} x(n) c_q(n) \right).$$

- Arithmetic functions (for which Ramanujan-sum expansions were originally used) are infinite duration sequences, and the coefficients have to be evaluated through the limiting process. But in the FIR case, with $x(n)$ equal to zero for all n except possibly in $1 \leq n \leq N$.

The First Ramanujan FIR Representation:

- $$\lim_{M\to\infty}\sum_{n=1}^{M} x(n)c_q(n)/M = \lim_{M\to\infty}\sum_{n=1}^{N} x(n)c_q(n)/M \to 0,$$

  which shows that $\alpha_q \to 0$ for each q. Thus, the conventional approach does not lead to a correct expansion.
- Consider the expansion:

  $$x(n) = \sum_{q=1}^{N} a_q c_q(n), \quad 0 \leq n \leq N-1,$$

  where the first N sequences $c_q(n)$ are all used.

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- $$\underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\mathbf{x}} = \mathbf{A}_N \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}}_{\mathbf{a}},$$

  where $\mathbf{A}_N = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \ldots \mathbf{c}_N]$ and
  $\mathbf{c}_m = [c_m(0) \quad c_m(1) \quad c_m(2) \quad \ldots \quad c_m(N-1)]^T$
- Theorem: The matrix $\mathbf{A_N}$ has full rank N and $\det(\mathbf{A_N}) = (-\mathbf{1})^{\mathbf{N-1}}\mathbf{N}!$
- $\mathbf{A_N}$ is not an orthogonal matrix.

A Second Ramanujan Representation for FIR Signals

Replacing $c_q(n)$ With the Subspace $S_q(n)$

- Equivalent representation of $x(n)$:

$$\mathbf{x} = [\mathbf{c}_{q_1} \quad \mathbf{c}_{q_2} \quad \dots \quad \mathbf{c}_{q_K}]\,\mathbf{d},$$

  where q_i are the K divisors of N.
- An arbitrary FIR sequence of length N cannot be represented as in conventional way if α_q is in conventional form. Only those FIR sequences which are in this column space can be represented.
- Each c_{q_i} represents just one vector in the Ramanujan space S_{q_i}, which has dimension $\phi(q_i)$. Replace c_{q_i} with the matrix

$$\mathbf{G}_{q_i} = \begin{bmatrix} \mathbf{c}_{q_i} & \mathbf{c}_{q_i}^{(1)} & \dots & \mathbf{c}_{q_i}^{(\phi(q_i)-1)} \end{bmatrix}_{N\times\phi(q_i)},$$

  where $\mathbf{c}_{\mathbf{q_i}}^{(\mathbf{k})}$ represents circular downshifting by k.

Properties of $\mathbf{F}_N$

- $\mathbf{F}_N = [\mathbf{G}_{q_1} \quad \mathbf{G}_{q_2} \quad \ldots \quad \mathbf{G}_{q_K}]_{N\times N}$, $(\sum_{q_i|N} \phi(q_i) = N.)$
- Theorem (Orthogonality): For $i \neq k$, the columns of the submatrices $\mathbf{G_{q_i}}$ and $\mathbf{G_{q_k}}$ in the matrix $\mathbf{F}_N$ span orthogonal subspaces of $\mathbb{C}^N$.
- Theorem: Any length N sequence can be represented as a linear combination of the form

$$x(n) = \sum_{q_i|N} \underbrace{\sum_{l=0}^{\phi(q_i)-1} \beta_{il} c_{q_i}(n-l)}_{x_{q_i}(n)},$$

where q_i are divisors of N and $c_{q_i}(n)$ is the q_i^{th} Ramanujan sum.

- Even though the columns of $\mathbf{G}_{q_i}$ are orthogonal to those of $\mathbf{G}_{q_k}$ for $i \neq k$, the $\phi(q_i)$ columns of each $\mathbf{G}_{q_i}$ are in general not orthogonal, so $\mathbf{F_N}$ itself is in general not an orthogonal matrix.
- Theorem: $\mathbf{F}_N$ is an orthogonal matrix iff $N = 2^m$.

P. P. Vaidyanathan, "Ramanujan Sums in the Context of Signal Processing—Part II: FIR Representations and Applications," in IEEE Transactions on Signal Processing, vol. 62, no. 16, pp. 4158-4172, Aug.15, 2014.