Nested Periodic Matrices and Dictionaries: New Signal Representations for Period Estimation

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Periodic Signal

A discrete time signal $x(n)$ is said to be periodic with period P if P is the smallest positive integer for which

$$x(n+P) = x(n) \qquad \forall n \in \mathbb{Z}$$

Objective

Given a finite length noisy version of a periodic signal $x(n)$, objective is to estimate its unknown period. In addition, search for any 'hidden periodicities' in the signal. For instance, if we add two periodic signals with periods P_1 and P_2, the resulting signal is periodic too, but its period can be any divisor of $lcm(P_1, P_2)$.

The Space ν_P

Defining the Space ν_P

Consider a signal that repeats every P samples, that is, $x(n) = x(n+P)$ for all $n \in \mathbb{Z}$. Let ν_P denote the set of all such signals. This is therefore the set of all signals with period P or a proper divisor of P. It is clear that if $x(n) \in \nu_P$ and $y(n) \in \nu_P$, then $\alpha x(n) + \beta y(n) \in \nu_P$ for any α, β. So ν_P is a linear space.

Basis for the space ν_P

Consider any basis for $\mathbb{C}^P$. If we periodically extend every vector in this basis, we will obtain a basis for ν_P.

Nested Periodic Matrices

Let d_i, $1 \leq i \leq K$ denote the divisors of P in increasing order, so that $d_1 = 1$ and $d_K = P$. Let $\phi(d_i)$ denote the Euler totient function. Now consider a matrix of the form

$$\mathbf{A} = [\mathbf{C}_{d_1} \mathbf{C}_{d_2} \ldots \mathbf{C}_{d_K}]$$

with the following properties:

1. Each C_{d_i} is a $P \times \phi(d_i)$ matrix so that the total number of columns in $\mathbf{A}$ is $\sum_{d_i|P} \phi(d_i) = P$. Thus $\mathbf{A}$ is a P×P matrix.
2. Each column of C_{d_i} is a length P segment of a sequence with period d_i.
3. $\mathbf{A}$ has full rank P.

Continue ...

Since **A** is full rank, its Columns form a basis for $\mathbb{C}^P$, so that by periodically extending the columns of **A** we can obtain a basis for ν_P. Such a matrix will be referred to as a Nested Periodic Matrix (NPM), and the basis for ν_P obtained by periodically extending the columns of **A** will be referred to as a Nested Periodic Basis (NPB).

Properties of NPM

- For any integer $q|P$, there are precisely q columns of the $P \times P$ matrix **A** with period q or a divisor of q. Furthermore, any $P \times 1$ vector y with period q can be spanned by these q columns.
- Let z be a column of **A** with period $d|P$. Then it cannot be expressed as a finite sum

  $$z = \sum_i z_i$$

  where z_i have periods $r_i|P$ with $r_i < d$.

Continue ...

- Let $\mathbf{A}$ be a $P \times P$ nested periodic matrix and consider the linear combination $v = \sum_{i=1}^{L} \beta_i a_{k_i}$ of a subset of L distinct columns. Let $n_i =$ period of a_{k_i}, and assume all $\beta_i \neq 0$ without loss of generality. Then the period of v is precisely $lcm\{n_i\}$ (i.e., it cannot be a proper divisor of this lcm).
- Given any nested periodic matrix $\mathbf{A}$, any $P \times 1$ vector y with period $q|P$ can be expressed as $y = \mathbf{Ac}$ where all those components of c are zero, that do not pair up with the q columns a_{k_j} of $\mathbf{A}$ that have periods equal to a divisor of q.

Examples of Nested Periodic Matrices

Natural Basis Matrices

For each divisor d of P, we only take the first $\phi(d)$ columns of the $d \times d$ identity matrix and periodically extend them to length P. By collecting such columns for all the divisors of P, we obtain a matrix satisfying the conditions of a nested periodic matrix. For example, consider the following 8×8 example, where sets of columns corresponding to different periods have been separated by spaces.

$$\mathbf{A} = \begin{bmatrix} 1 & \quad 1 & \quad 1 & 0 & \quad 1 & 0 & 0 & 0 \\ 1 & \quad 0 & \quad 0 & 1 & \quad 0 & 1 & 0 & 0 \\ 1 & \quad 1 & \quad 0 & 0 & \quad 0 & 0 & 1 & 0 \\ 1 & \quad 0 & \quad 0 & 0 & \quad 0 & 0 & 0 & 1 \\ 1 & \quad 1 & \quad 1 & 0 & \quad 0 & 0 & 0 & 0 \\ 1 & \quad 0 & \quad 0 & 1 & \quad 0 & 0 & 0 & 0 \\ 1 & \quad 1 & \quad 0 & 0 & \quad 0 & 0 & 0 & 0 \\ 1 & \quad 0 & \quad 0 & 0 & \quad 0 & 0 & 0 & 0 \end{bmatrix}$$

The Ramanujan Periodicity Transform Matrices

- Ramanujan Sum: $c_q(n) = \sum_{\substack{k=1 \\ \gcd(k,q)=1}}^{q} e^{j2\pi kn/q}$
- its $q-$point DFT is such that $C_q[k] = q$ when $(k, q) = 1$ and is zero otherwise.
- let q be a divisor of P and define the $P \times 1$ vector $\mathbf{c}_q = [c_q(0) c_q(1) \ldots \ldots c_q(P-1)]^T$, so that there are P/q repetitions of the fundamental period of $c_q(n)$. Now define the $P \times \phi(q)$ matrix $\mathbf{C}_q = \begin{bmatrix} \mathbf{c}_q & \mathbf{c}_q^{(1)} & \mathbf{c}_q^{(2)} & \ldots & \mathbf{c}_q^{(\phi(q)-1)} \end{bmatrix}$, where $c_q^{(i)}$ is the circularly downshifted version of c_q.

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 0 & 4 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 4 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 4 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 4 \\ 1 & 1 & 2 & 0 & -4 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & -4 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & -4 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & -4 \end{bmatrix}$$

DFT Matrices

The P columns of the $P \times P$ DFT matrix can be partitioned into K classes, one for each divisor $d_i|P$. In class i there are $\phi(d_i)$ columns (which can be gathered into a $P \times \Phi(d_i)$ matrix C_{d_i}) and they have the form $W_{d_i}^{k_i n}$ where $(k_i, d_i) = 1$, and all these have period exactly d_i. None of the columns of this DFT matrix can have a period other than a divisor $d_i|P$.

For example, for the (column permuted) 8×8 DFT matrix shown below,

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j & e^{\frac{-j\pi}{4}} & e^{\frac{-j3\pi}{4}} & e^{\frac{j3\pi}{4}} & e^{\frac{j\pi}{4}} \\ 1 & 1 & -1 & -1 & -j & j & -j & j \\ 1 & -1 & j & -j & e^{\frac{-3j\pi}{4}} & e^{\frac{-j\pi}{4}} & e^{\frac{j\pi}{4}} & e^{\frac{j3\pi}{4}} \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -j & j & e^{\frac{3j\pi}{4}} & e^{\frac{j\pi}{4}} & e^{\frac{-j\pi}{4}} & e^{\frac{-j3\pi}{4}} \\ 1 & 1 & -1 & -1 & j & -j & j & -j \\ 1 & -1 & j & -j & e^{\frac{j\pi}{4}} & e^{\frac{j3\pi}{4}} & e^{\frac{-j3\pi}{4}} & e^{\frac{-j\pi}{4}} \end{bmatrix}$$

Hadamard Matrices

Sylvester method

$$\mathbf{H}_0 = 1, \qquad \mathbf{H}_n = \begin{bmatrix} \mathbf{H}_{n-1} & \mathbf{H}_{n-1} \\ \mathbf{H}_{n-1} & -\mathbf{H}_{n-1} \end{bmatrix} \qquad \forall n \in \mathbb{N}$$

For $n = 0, 1, 2, 3 \ldots$, the $2n \times 2n$ Hadamard matrix constructed using the Sylvester method is a nested periodic matrix.

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Random Periodic Matrices

For each divisor d of P, we generate a $d \times 1$ vector with random entries, and repeat it to length P to get the first column of C_d. The other $\phi(d) - 1$ columns can be obtained by cyclically shifting this first column. The $P \times P$ matrix so obtained was observed to be full rank with a high probability.

Example

$$\mathbf{A} = \begin{bmatrix} 6 & 7 & 0 & 2 & 4 & 6 & 9 & 9 \\ 6 & 4 & 2 & 0 & 5 & 4 & 6 & 9 \\ 6 & 7 & 1 & 2 & 4 & 5 & 4 & 6 \\ 6 & 4 & 2 & 1 & 8 & 4 & 5 & 4 \\ 6 & 7 & 0 & 2 & 5 & 8 & 4 & 5 \\ 6 & 4 & 2 & 0 & 9 & 5 & 8 & 4 \\ 6 & 7 & 1 & 2 & 9 & 9 & 5 & 8 \\ 6 & 4 & 2 & 1 & 6 & 9 & 9 & 5 \end{bmatrix}.$$

Nested Periodic Dictionary

Suppose that the input data is $x(n)$, $n = 0, 1, \ldots, N-1$. Consider a particular family of nested periodic matrices – for example, the Ramanujan construction. For each d going from 1 to P_{max}, where P_{max} is the largest expected period, construct the $d \times d$ nested periodic matrix and take only its $\phi(d)$ columns that have period d. Extend these columns periodically to length N, truncating the last period if necessary. We form a dictionary $\mathbf{A}$ by collecting such columns for each d going from 1 to P_{max}. We will call such a dictionary as a Nested Periodic Dictionary.

Example

For $N = 5$ and $P_{max} = 5$ is shown below using the Ramanujan construction. The first column corresponds to period 1, the second to period 2, the third and fourth to period 3, the fifth and sixth to period 4 and the remaining to period 5.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & -1 & 2 & 0 & 4 & -1 & -1 & -1 \\ 1 & -1 & -1 & 2 & 0 & 2 & -1 & 4 & -1 & -1 \\ 1 & 1 & -1 & -1 & -2 & 0 & -1 & -1 & 4 & -1 \\ 1 & -1 & 2 & -1 & 0 & -2 & -1 & -1 & -1 & 4 \\ 1 & 1 & -1 & 2 & 2 & 0 & -1 & -1 & -1 & -1 \end{bmatrix}$$

The sum of Euler totient function from 1 to P_{max} is $O(\frac{3p_{max}^2}{\pi^2})$

Continue ...

If the given signal $x(n)$ is periodic with period less than P_{max}, then it has to be a linear combination of the columns of the dictionary. This is because, if it was periodic with period P, then the columns with periods being divisors of P must be able to span it. So the following system of linear equations must have a solution y, where $x = [x(0)x(1)\dots x(N-1)]^T$

$$x = \mathbf{A}y$$

But since the dictionary is fat, it is likely that $x = \mathbf{A}y$ has multiple solutions. The solution that we are interested in is the one that involves subspaces corresponding to period P and its factors.

Formulating a Convex Program

ℓ_2-norm minimization

$$\min \ \|\mathbf{D}y\|_2 \quad s.t. \quad x = \mathbf{A}y,$$

where D is a diagonal matrix whose i^{th} diagonal entry is $f(P_i)$, where P_i is the period of the i^{th} column of A and $f(\cdot)$ is some increasing function.

ℓ_1-norm minimization

$$\min \ \|\mathbf{D}y\|_1 \quad s.t. \quad x = \mathbf{A}y$$

Choice of $f(\cdot)$

$f(P) = P^2$ or $f(P) = \phi(P)$.

Reference

S. V. Tenneti and P. P. Vaidyanathan, “Nested Periodic Matrices and Dictionaries: New Signal Representations for Period Estimation,” in IEEE Transactions on Signal Processing, vol. 63, no. 14, pp. 3736-3750, July15, 2015.