Discrete Time Linear Systems with Sparse Inputs

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System Model

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$

$$y_k = Cx_k + e_k \tag{2}$$

k is the integer time index, $x_k \in \mathbf{R}^n$ is the state, $e_k \in \mathbf{R}^p$ is the measurement noise, $B \in \mathbf{R}^{n \times m}$ is a wide matrix

- Consider the case of bounded noise $||e_k||_2 \leq \epsilon$
- Assume that at each time at most s input nodes are active (s-sparse input), i.e. $||u_k||_0 \leq s$.

Problem statement

- To recover the initial state of the system x_0 and the inputs u_k given the measurements y_k $k = 1, 2, \ldots, K$
- In the papers they have discussed conditions on the system matrices under which a unique x_0 and a unique sequence of inputs u_k exist and bounds on the l2-norm of the error in the estimated inputs and initial state in the noisy measurement case.

The following is the relation between the measurements, the initial state and the inputs.

$$Y_K = \mathcal{O}_K x_0 + J_K^s U_{K-1}^s$$

Where, Y_K is the vector obtained by stacking the measurements

$$Y_K = [y_0^T,\ y_1^T,\ y_2^T,\ \dots,\ y_K^T]^T$$

U_{K-1}^s is the vector obtained by stacking the inputs.

$$U_{K-1}^s = [(u_0^s)^T,\ (u_1^s)^T,\ (u_2^s)^T,\ \dots,\ (u_{K-1}^s)^T]^T$$

$\mathcal{O}_K$ is the observability matrix.

$$\mathcal{O}_K = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^K \end{bmatrix}$$

$$J_K^s = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ CB_0^s & 0 & \cdots & \cdots & 0 \\ CAB_0^s & CB_1^s & \cdots & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ CA^{K-1}B_0^s & CA^{K-2}B_1^s & \cdots & \cdots & CB_{K-1}^s \end{bmatrix}$$

The set of possible J_K^s is denoted by $\mathcal{J}_K^s$

Optimization Problems of Interest

(P1)

$$\begin{aligned} \min_{(x_k)_{k=0}^{K}(u_k)_{k=0}^{K}} \sum_{k=0}^{K-1} ||u_k||_1 \\ \text{subject to } x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k \end{aligned}$$

(P2)

$$\begin{aligned} \min_{(x_k)_{k=0}^{K}(u_k)_{k=0}^{K}} \sum_{k=0}^{K-1} ||u_k||_1 \\ \text{subject to } x_{k+1} = Ax_k + Bu_k \\ ||y_k - Cx_k||_2 \leq \epsilon \end{aligned}$$

Lemma 1

Suppose that the sequence $(y_k)_{k=0}^{K}$ from noiseless measurements is given, and A, B and C are known. Assume $rank(\mathcal{O}_K) = n$ and the matrix CB satisfies the RIP condition with isometry constant $\delta_{2s} < 1$. Further, assume
$rank([\mathcal{O} J_K^{2s}]) = n + rank(J_K^{2s}) \; \forall J_K^{2s} \in \mathcal{J}_K^{2s}$ Then, there is a unique s-sparse sequence $(u_k)_{k=0}^{K}$ of and a unique sequence of $(x_k)_{k=0}^{K}$ that generate $(y_k)_{k=0}^{K}$.

Lemma 2

Suppose that the sequence $(y_k)_{k=0}^{K}$ is given and generated from sequences $(x_k)_{k=0}^{K}$ and s-sparse $(u_k)_{k=0}^{K}$, where $||e_k||_2 \leq \epsilon$ and A, B, C are known. Then, any solution x_k^* to (P2) obeys

$$||x_k^* - x_k||_2 \leq 2\epsilon \sqrt{\frac{K\sigma_{max}((I - P_{J_K^{2s}})^T(I - P_{J_K^{2s}}))}{\sigma_{min}(\mathcal{O}_K^T\mathcal{O}_K^T)\sigma_{min}((I - P_{J_K^{2s}})^T(I - P_{J_K^{2s}}))}} \quad (3)$$

Since, $y_{k+1} = CAx_k + CBu_k + e_{k+1}$,

$$\begin{aligned} ||CB(u_k^* - u_k)||_2 &= ||e_{k+1}^* + e_{k+1} + CA(x_k - x_k^*)||_2 \\ &\leq ||e_{k+1} + e_{k+1}^*||_2 + ||CA(x_k - x_k^*)||_2 \end{aligned} \quad (4)$$

Proposition

Let Π be the projection onto the orthogonal complement of the column space of the observability matrix $\mathcal{O}_K$. If $\mathcal{O}_K$ is full rank and the projected matrix J_K is incoherent, x_0 and U_K can be uniquely recovered from Y_K as the solution to (P1).

Proof

Since $\mathcal{O}_K$ is full rank, it follows from (10) that we can solve for the initial condition x_0 as a function of the unknown input sequence U_K as

$$x_0 = (\mathcal{O}_K^T \mathcal{O}_K)^{-1} \mathcal{O}_K^T (Y_K - J_K U_K).$$

Substituting this expression for x_0 back into the optimization problem in (P1) we obtain

$$\min_{U_K} ||U_K||_1 \text{ s.t. } Y_\Pi = J_\Pi U_K$$

where $Y_\Pi = \Pi Y_K, J_\Pi = \Pi J_K$, and Π is the projection matrix onto the orthogonal complement of the column space of the observability matrix $\mathcal{O}_K$, which is given by:

$$\Pi = I - \mathcal{O}_K (\mathcal{O}_K^T \mathcal{O}_K)^{-1} \mathcal{O}_K^T,$$

Controllability under sparse constraints

Complete state controllability (or simply controllability) describes the ability of an external input (the vector of control variables) to move the internal state of a system from any initial state to any other final state in a finite time interval.

Necessary and sufficient condition for general systems

$$rank(\begin{bmatrix} B & AB & A^2B & . & . & A^{n-1}B \end{bmatrix}) = n \tag{5}$$

PBH test

$$rank(\begin{bmatrix} \lambda I - A & B \end{bmatrix}) = n \ \forall \lambda \tag{6}$$

Necessary and sufficient condition for systems with sparse inputs

$$\begin{gathered} \exists S_0, S_1, ..., S_K \text{ s.t. } |S_0| \leq s, ..., |S_K| \leq s, \\ rank(\begin{bmatrix} B_{S_0} & AB_{S_1} & A^2B_{S_2} & . & . & A^K B_{S_K} \end{bmatrix}) = n \end{gathered} \tag{7}$$

PBH test

- $rank(\begin{bmatrix} \lambda I - A & B \end{bmatrix}) = n \ \forall \lambda$
- $\exists S$ s.t. $|S| \leq s$ and $rank(\begin{bmatrix} A & B_S \end{bmatrix}) = n$

Stabilizability under sparse constraints

The pair (A, B) is stabilizable if for any $x(0) = x_0$, there exists a sequence u_k

$$\lim_{t \to \infty} x(t) = 0 \tag{8}$$

Necessary and sufficient condition for general systems

$$\text{rank}[\lambda I - A \quad B] = n \; \forall |\lambda| \geq 1 \tag{9}$$

Necessary and sufficient condition for systems with sparse inputs
Same as the necessary and sufficient conditions for general systems.

Proof

If A is similar to another matrix Σ i.e $A = V^{-1}\Sigma V$ then,

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ Vx(k+1) &= \Sigma Vx(k) + VBu(k) \\ \tilde{x}(k+1) &= \Sigma\tilde{x}(k) + \tilde{B}u(k) \end{aligned} \tag{10}$$

Lemma Any matrix A with l distinct eigen values (algebraic multiplicity of l) is similar to a matrix of the form

$$\begin{pmatrix} T_1 & 0 & . & . & 0 \\ 0 & T_2 & . & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & T_l \end{pmatrix} \tag{11}$$

where T_i is of the form

$$\begin{pmatrix} \lambda_i & \times & . & . & \times \\ 0 & \lambda_i & . & . & \times \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & \lambda_i \end{pmatrix} \tag{12}$$

Let us denote the states that correspond to $|\lambda_i| >= 1$ as $x_1(k)$ and the states that correspond to eigen values less than 1 as $x_0(k)$.

$$x_1(k+1) = \Sigma_1 x(k) + B_1' u(k) \tag{13}$$

$$x_0(k+1) = \Sigma_0 x(k) + B_0' u(k) \tag{14}$$

where, Σ_1, B_1' and Σ_0, B_0' have only the rows corresponding to $|\lambda_i| \geq 1$ and < 1 respectively. $x_0(k)$ eventually dies down to 0 because the eigen values are less than 1 in magnitude. But $x_1(k)$ needs to be driven to 0 using sparse inputs. This is is ensured if the 2 conditions for sparse controllability are satisfied

$$a) \begin{bmatrix} \lambda I - \Sigma_1 & B_1' \end{bmatrix} \text{ has full row rank} \tag{15}$$

$$b) \begin{bmatrix} \Sigma_1 & (B_1')_S \end{bmatrix} \text{ has full row rank} \tag{16}$$

Problem Statement for Future Work

- Given x_0 and x_f find sparse $u_0, u_1, ...u_k$ s.t. the system at state x_0 at time 0, transitions to state x_f at some time K.
  - Design Algorithm
  - Extension: Bounded l2 norm on u_i
  - Guarentees (based on RIP) when A,B are random.
- x_0 is uknown, just have a final state x_f. Based on y, want to develop an online algorithm to design sparse u_k s.t. we reach x_f at some finite time. Goal: Reach x_f and stay there.
  - Recipe: Apply 0 input till the system becomes observable then use 1.
  - Charecterize the set of states x_f st $(I - A)x_f = Bu$ for sparse u.

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