# Spectral Graph Theory

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- Various graph properties have linear algebraic interpretations
- Several applications: clustering, clique detection, pagerank, graph property testing

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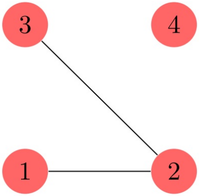
  - Degree matrix $D$: an $n \times n$ diagonal matrix with $D_{ii} = \sum_{j=1}^{n} A_{ij}$
  - Laplacian $L = D - A$

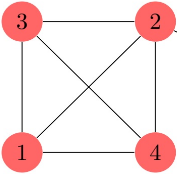
# An example



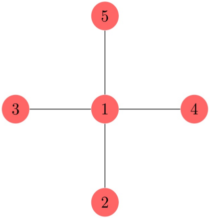

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Some well-known graphs

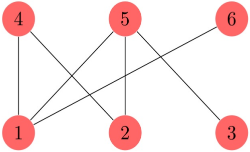
Complete graph




Star graph




Bipartite graph




$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

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  - Connectivity
  - Number of paths, triangles
  - Presence of cliques

# Powers of the adjacency matrix: counting number of edges

- Let $A \in \{0,1\}^n$ be the adjacency matrix of a graph $G = (V, E)$ and $a_i$ be its columns. Then

$$\begin{aligned}(A^2)_{ii} &= (A^\top A)_{ii} \\ &= \|a_i\|_2^2 = d_i\end{aligned}$$

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- This gives

$$\mathrm{Tr}(A^2) = \sum_{i=1}^{n} d_i = 2|E|$$

# Powers of the adjacency matrix: paths between nodes

- Number of paths $N_{ij}$ of length 2 between nodes $i$ and $j$:

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\end{aligned}
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- In general $(A^k)_{ij}$ counts the number of paths of length $k$ between nodes $i$ and $j$

# Graph spectra: Bipartite graphs

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. Its adjacency matrix has the form

$$A = \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix}$$

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$$A = \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix}$$
- The spectrum of a bipartite graph is symmetric around zero

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$$Av = \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Bv_2 \\ B^\top v_1 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

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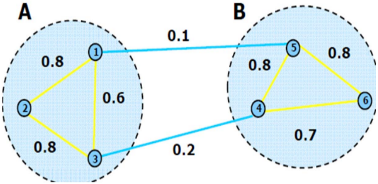
# Graphs with clusters




- Adjacency matrix has a block diagonal structure (roughly)

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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- The all ones vector is an eigenvector with eigenvalue equal to the size of each component
- The next eigenvector reveals component labels

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

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- Has direct application to community detection in large graphs
- Lot of recent work in analyzing spectral algorithms in the stochastic block model setting
- Several other interesting connections: graph coloring and chromatic number, grouping graphs with similar spectra

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  For $n \in \mathbb{N}$ and $p, q \in (0, 1)$, let $\mathcal{G}(n, p, q)$ be the class of random graphs where

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  - each vertex $v$ is assigned a label $\sigma_v \in \{+1, -1\}$ (independently and uniformly at random)
  - each possible edge $(u, v)$ is included with probability $p$ if $\sigma_u = \sigma_v$ and with probability $q$ if $\sigma_u \neq \sigma_v$

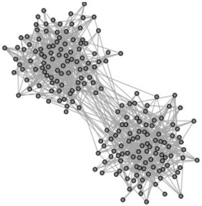

Figure 1: A random graph $G \sim \mathcal{G}(200, \frac{1}{20}, \frac{1}{200})$

- The expected adjacency matrix has a block structure

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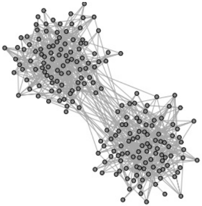

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- For example, with $n = 4$:

$$\mathbb{E}[A] = \begin{bmatrix} p & p & q & q \\ p & p & q & q \\ q & q & p & p \\ q & q & p & p \end{bmatrix}$$

# References


- Luca Trevisan. *Lecture Notes on Expansion, Sparsest Cut, and Spectral Graph Theory.*
- Daniel Spielman. *Spectral Graph Theory, Combinatorial Scientific Computing, Chapman and Hall/CRC Press, 2011.*

Thank you