The Role of MVUE and CRB in Composite Hypothesis Test

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Outline

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Introduction

- The relation between composite binary hypothesis test and the estimation of the unknown parameters is explored
- If an UMP test exists, it is identical to comparing the MVUE (Minimum variance Unbiased Estimator) for the unknown parameter to a threshold
- The relation between CRB (Cramer Rao Bound) in the estimation theory and the UMP performance bound in detection theory is analyzed
- If an UMP test does not exist, using a good estimator of the unknown parameter as a test statistic can improve the detection performance

CRB

- If the pdf $p(\mathbf{x};\theta)$ satisfies the following regularity condition

$$E\left[\frac{ln(p(\mathbf{x};\theta))}{\partial\theta}\right] = 0, \quad \textit{for all}\theta$$

- Then, the variance of any unbiased estimator $\hat{\theta}$ must satisfy

$$var\hat{\theta} \geq \frac{1}{E\left[\frac{\partial^2 ln(p(\mathbf{x};\theta))}{\partial\theta^2}\right]}$$

- Further an unbiased estimate may be found that attains the bound for all θ iff

$$\frac{\partial ln(p(\mathbf{x};\theta))}{\partial\theta} = I(\theta)(g(\mathbf{x}) - \theta) \quad (1)$$

- $\hat{\theta} = g(\mathbf{x})$ is the MVUE

- The decision rule divides the observation space into two partitions Γ_0 and Γ_1
- It can be shown that the decision rule for a binary hypothesis test can be stated as a comparison of a continuous test statistic with a threshold
- The observation space χ is a normal space and Γ_0 and Γ_1 are connected sets

Theorem

Urysohn lemma*: Let χ be a normal space; Let A and B be disjoint closed subsets of χ. Let $[a, b]$ be a closed interval in the real line. Then, there exists a continuous map $f : \chi \to [a, b]$, such that $f(x) = a$ for $x \in A$ and $f(x) = b$ for $x \in B$*

Theorem

Consider a decision rule $d(\boldsymbol{x}) = 1$ *if* $\boldsymbol{x} \in \Gamma_1$ *and* $d(\boldsymbol{x}) = 0$ *if* $x \in \Gamma_0$ *in which* $x \in \chi$ *and* $\Gamma_1 \cap \Gamma_0 = \emptyset$*,* $\Gamma_1 \cup \Gamma_0 = \chi$*. If* Γ_1 *and* Γ_0 *are connected, then there exists a continuous statistic* $g(\boldsymbol{x})$ *and a threshold* γ *such that*

$$g : \chi \to \mathcal{R} \quad (2)$$

$$d(\boldsymbol{x}) = \left\{ \begin{array}{ll} 1, & g(\boldsymbol{x}) \geq \gamma \\ 0, & g(\boldsymbol{x}) < \gamma \end{array} \right\} \quad (3)$$

- There exists an infinite number of functions $g(\mathbf{x})$ that satisfy (2)

- Consider a one-sided hypothesis testing problem, i.e.,

$$
\begin{aligned}
H_0 &: \mathbf{x} \sim f_x(\mathbf{x};\theta), \theta < \theta_b \\
H_1 &: \mathbf{x} \sim f_x(\mathbf{x};\theta), \theta > \theta_b
\end{aligned}
$$

- where θ_b is known.

Theorem

Consider a one-sided hypothesis test $H_0 : \mathbf{x} \sim f_x(\mathbf{x};\theta), \theta < \theta_0$ *against* $H_1 : \mathbf{x} \sim f_x(\mathbf{x};\theta), \theta > \theta_b$*. Let* $\Omega_0 = \{\theta : \theta < \theta_b\}$*,* $\Omega_1 = \{\theta : \theta > \theta_b\}$*; and* $\Omega = \Omega_0 \cup \Omega_1$*. If the UMP test exists, then it can be stated as comparing the MVUE statistic of* $\theta \in \Omega$ *with a threshold which is set to satisfy the probability of false alarm,*P_{fa}

Outline of the Proof

- The optimal N-P test is obtained from the likelihood ratio

$$LR(\mathbf{x}) = \frac{f_x(\mathbf{x};\theta_1)}{f_x(\mathbf{x};\theta_0)} > \gamma$$

- θ_1 is $\theta \in \Omega_1$ and θ_0 is $\theta \in \Omega_0$
- The log-likelihood ratio is given by

$$lnLR(\mathbf{x}) = ln(f_x(\mathbf{x};\theta_1)) - ln(f_x(\mathbf{x};\theta_0)) > ln(\gamma)$$

- From the CRB theorem:

$$\frac{\partial ln(f_x(\mathbf{x};\theta))}{\partial\theta} = I(\theta)[g(\mathbf{x}) - \theta] \tag{4}$$

Outline of the Proof

- $g(\mathbf{x})$ is the MVUE statistic; $I(\theta)$ is the Fisher information matrix

$$ln(f_x(\mathbf{x};\theta)) = \int I(\theta)[g(\mathbf{x}) - \theta]d\theta + C(\mathbf{x}) \quad (5)$$

- $C(\mathbf{x})$ is a function of only **x**

$$\begin{aligned} ln(LR(\mathbf{x})) &= \int I(\theta_1)[g(\mathbf{x}) - \theta_1]d\theta_1 \\ &= \int I(\theta_0)[g(\mathbf{x}) - \theta_0]d\theta_0 \end{aligned}$$

- Log-likelihood is increasing in

$$\left[\int I(\theta_1)d\theta_1 - \int I(\theta_0)d\theta_0\right] g(\mathbf{x}) \quad (6)$$

Outline of the proof

- The Fisher information is positive, i.e., $I(\theta) > 0$, $I(\theta)$ is an increasing function of θ
- From the one-sided hypothesis test, $\theta_1 > \theta_0$, therefore

$$\left[\int I(\theta_1)d\theta_1 - \int I(\theta_0)d\theta_0\right] > 0 \tag{7}$$

- Hence $\left[\int I(\theta_1)d\theta_1 - \int I(\theta_0)d\theta_0\right] g(\mathbf{x})$ is increasing in $g(\mathbf{x})$
- The UMP statistic rejects H_0 if $g(\mathbf{x}) > \gamma$, where γ satisfies the P_{fa} constraint

Vector Parameter

- $\Theta = [\theta_1, \theta_2, \ldots, \theta_N]^T$
- From CRB theorem, for MVUE:

$$ln(f_x(\mathbf{x}, \Theta)) = \int (I(\Theta)[g(\mathbf{x}) - \Theta])^H d\Theta \tag{8}$$

- The log-likelihood ratio is increasing in

$$g(\mathbf{x})^H \left[\int I(\Theta_1)^H d\Theta_1 - \int I(\Theta_0)^H d\Theta_0 \right] \tag{9}$$

- UMP for the vector parameter is given by the linear combination of the elements of MVUE

Vector Parameter

- The co-efficients in the linear combination do not simplify into constants
- In general, the co-efficients may be function of unknown parameters
- By properly choosing the co-efficients of the linear combination of the MVUE, UMP performance bound can be achieved
- In summary, if a good estimator for the function $g(\mathbf{x})$ is chosen (reaching CRB), then UMP performance can be achieved

Example-I

- Consider the following hypothesis testing problem

$$\begin{aligned} H_0 &: \mathbf{x} = \mathbf{n} \\ H_1 &: \mathbf{x} = \theta\mathbf{s} + \mathbf{n} \end{aligned}$$

- **s** is the known signal with N samples and $\theta > 0$
- Under H_0, $\mathbf{x} \sim N(0,\ C)$ and under H_1, $\mathbf{x} \sim N(\theta\mathbf{s},\ C)$
- In the WGN case, $C = \sigma^2 I$.
- From the likelihood ratio, the test statistic for detection

$$T(\mathbf{x}) = \mathbf{x}^T C^{-1}\mathbf{s} > \gamma \tag{10}$$

Example-I

- γ is the detection threshold. This is an UMP test
- Generalized matched filter
- The MVUE for θ is

$$\theta_{MVUE} = \frac{\mathbf{x}^T C^{-1} \mathbf{s}}{\mathbf{s}^T C^{-1} \mathbf{s}}$$

- The denominator is known positive constant; C is positive definite
- The UMP detection statistic is nothing but MVUE compared to a threshold

Example-II

- Consider the hypothesis testing problem

$$\begin{aligned} H_0 &: \quad x_i = n_i \\ H_1 &: \quad x_i = x_0 + n_i \end{aligned}$$

- The pdf of n_i is Cauchy distributed and $x_0 > 0$
- No UMP exists and MVUE of center parameter also does not exist
- An unbiased estimation which tends to meet CRB as the number of observation increases

$$I_k(\mathbf{x}) = \frac{y_0}{\pi} \int_{-\infty}^{\infty} x_0^k \prod_{i=1}^{N} \frac{1}{y_0^2 + (x_i - x_0)^2]} dx_0$$

- $0 \leq k \leq 2N - 1$. Unbiased estimator of x_0 is $\hat{x_0} = \frac{I_1(\mathbf{x})}{I_0(\mathbf{x})}$