Performance Analysis of Physical Layer Binary Consensus Protocols

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Outline

- Introduction to Consensus
- Problem Setup
- Physical Layer Binary Consensus Protocols
  - Message Exchange
  - Update Procedure
- Performance Analysis
- Simulation Results

Consensus Problems

- A set of nodes with arbitrary initial data values agree upon a common value
  - Examples: min., max., average, majority value
- Nodes repeatedly exchange msgs & update their values
- Network layer consensus
  - Reliable packet exchanges in the local neighborhood
- Physical layer consensus
  - Data exchanges with all other nodes over noisy wireless links
  - No overhead of control information

Literature Survey: Network Layer Consensus

- Distributed averaging
- [Tsitsiklis 1984] Distributed computing
- [Boyd et al. 2005] Gossip algorithms
- [Benezit et al. 2011] Voting problem as interval consensus

Literature Survey: Physical Layer Consensus

- Distributed detection
- [Oltafi 2006, Chan 2010, Wang 2012] Consensus on test statistic and treat as distributed hypothesis testing
- [Mostofi 2007, 2008, 2010] Exchange hard decisions by broadcasting bits and attain majority consensus
- Our focus
  - Physical layer binary consensus by exchanging hard decisions
  - Two options: (a) Broadcast-based, and (b) Distributed Co-Phasing (DCP)-based consensus

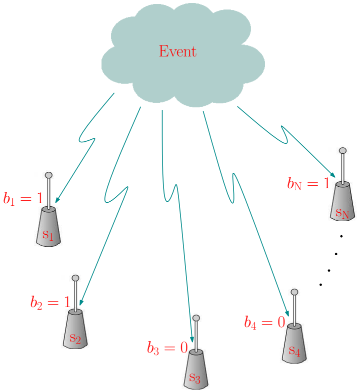
Contributions

- Performance analysis:
  - Probability of correct majority bit detection
  - Average hitting time
  - Average consensus duration
- Analysis captures the effect of channel estimation errors, fading, and noise on consensus performance
- Comparison of broadcast-based and DCP-based consensus protocols

Main Message

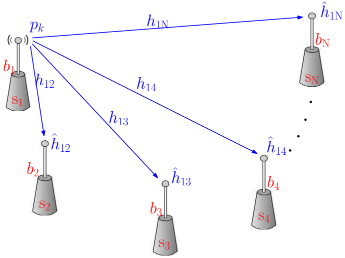
DCP offers advantage over conventional broadcast-based consensus at low to moderate pilot SNRs.

Problem Setup



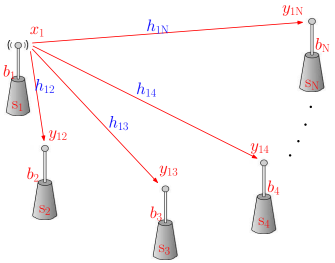
- Nodes $[s_1, s_2, \ldots, s_N]$ have initial values $[b_1, b_2, \ldots, b_N]$
- Goal: To achieve majority consensus

Broadcast-based Data Exchange: Pilot Phase



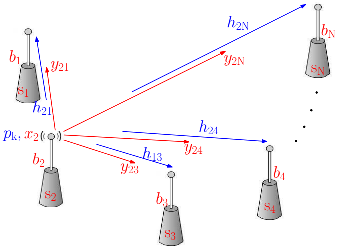
- Node broadcasts a known pilot symbol p_k
- All other nodes estimate the corresponding channel

Broadcast-based Data Exchange: Data Phase



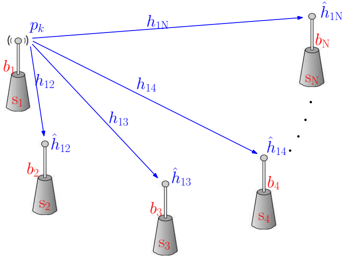
- Node broadcasts a BPSK symbol x_i corresp. to data bit b_i
- At node s_j, $y_{1j} = h_{1j}x_1 + w_j$, where $w_j \sim \mathcal{CN}(0, \sigma^2)$

Broadcast-based Data Exchange



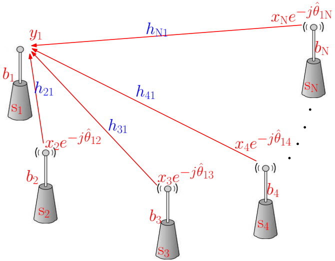
- A bit exchange cycle: nodes broadcast a pilot symbol followed by a data bit, in a round-robin manner
- At the end of a cycle, node s_j will have $\{y_{ij}\}_{i=1,\ldots,N, i\neq j}$

DCP-based Data Exchange: Pilot Phase



- A node broadcasts pilot symbol, other nodes estimate channel phase (same as broadcast-based data exchange)
- Assumption: Channels $h_{ij} \triangleq |h_{ij}|e^{j\theta_{ij}}$ are reciprocal, where $|h_{ij}|$ is Rayleigh distributed and $\theta_{ij} \sim \mathcal{U}[0, 2\pi)$

DCP-based Data Exchange: Data Phase



- The other $N-1$ nodes synchronously transmit their BPSK symbols, pre-rotated with negative of est. channel phase
  - Nodes attempt to coherently combine their signals over the air
- In a cycle, the nodes carry out DCP sessions in a round-robin manner
- At the end of a cycle, node s_j will have an observation y_j

Received Samples at Node s_j

- Broadcast-based Scheme

$$y_{ij} = h_{ij} x_i + w_{ij},$$

$$\text{where } x_i = \pm\sqrt{P},\ i = 1, \ldots, N,\ \ i \neq j$$

$$r_{ij} = Re\{y_{ij} e^{-j\hat{\theta}_{ij}}\} = |h_{ij}| \cos\theta_{ij}^{e}\ x_i + n_{ij}$$

$$\mathbf{r}_j = [r_{1j} \ldots r_{ij} \ldots r_{Nj}]^T = \hat{\mathbf{H}}_j \mathbf{x}_j + \mathbf{n}_j$$

$$\text{where } \hat{\mathbf{H}}_j = diag(|h_{ij}| \cos\theta_{ij}^{e}), \quad \mathbf{x}_j = [x_i], \quad \mathbf{n}_j = [n_{ij}]$$

- DCP-based Scheme

$$y_j = \sum_{i=1}^{N} h_{ij} e^{-j\hat{\theta}_{ij}} x_i + w_j, \text{ where } x_i = \pm\sqrt{\frac{P}{N-1}}$$

$$r_j = Re\{y_j\} = \underline{1}\hat{\mathbf{H}}_j \mathbf{x}_j + n_j$$

Bit Update Procedure: DCP-based scheme

- Since BPSK is employed, the difference of votes $\Delta_j \triangleq \sum_{\substack{i=1 \\ i \neq j}}^{N} x_i$ is a test statistic for detecting majority bit[1]
- Majority bit detection rule

$$f(\hat{\Delta}_j) = \begin{cases} 1 & \hat{\Delta}_j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- DCP-based scheme: A node s_j has one DCP received sample y_j, use $\hat{\Delta}_j = r_j$

[1] For simplicity, the self-bit, i.e., the sensor's own observation, is ignored here.

Bit Update Procedure: Broadcast-based Scheme

- Broadcast-based scheme: A node s_j has $N-1$ received samples, $\mathbf{r}_j = [r_{1j}, \ldots, r_{i-1,j}, r_{i+1,j}, \ldots, r_{Nj}]^T$
- Soft combining: $\hat{\Delta}_j = \sum_{\substack{i=1 \\ i \neq j}}^{N} r_{ij} = \underline{1}^T \mathbf{r}_j$
- LMMSE-based $\hat{\Delta}_j$ estimation
  - $\hat{\Delta}_j = \boldsymbol{\alpha}_j^T \mathbf{r}_j$, where $\mathbf{r}_j = \mathbf{H}_j \mathbf{x_j} + \mathbf{n_j}$
  - Optimization problem: $\boldsymbol{\alpha}_j^* = \arg\min_{\boldsymbol{\alpha}_j} \mathbb{E}[(\hat{\Delta}_j - \Delta_j)^2]$
  - $\boldsymbol{\alpha}_j^* = (\mathbf{H}_j^2 + \Omega_j)^{-1} \mathbf{H}_j \underline{1}$, $\mathbf{H}_j$ is diagonal channel matrix, $\Omega_j$is noise covariance matrix

Prob. of Detecting bit '1'

- LMMSE-based scheme
$$\Pr\{\hat{\Delta}_j \geq 0\} = Q\left(\frac{-\sqrt{2}\boldsymbol{\alpha}_j\hat{\mathbf{H}}_j\mathbf{x}_j}{\sqrt{\boldsymbol{\alpha}_j^T\boldsymbol{\alpha}_j}}\right)$$
- Soft combining
$$\Pr\{\hat{\Delta}_j \geq 0\} = Q\left(\frac{-\sqrt{2}\hat{\mathbf{H}}_j\mathbf{x}_j}{\sqrt{(N-1)\sigma^2}}\right) = Q\left(\frac{-\sqrt{2}\hat{\mathbf{H}}_j\mathbf{b}_j}{\sqrt{(N-1)\sigma^2}}\right)$$
- DCP-based scheme
$$\Pr\{\hat{\Delta}_j \geq 0\} = Q\left(\frac{-\sqrt{2}\hat{\mathbf{H}}_j\mathbf{x}_j}{\sqrt{\sigma^2}}\right) = Q\left(\frac{-\sqrt{2}\hat{\mathbf{H}}_j\mathbf{b}_j}{\sqrt{(N-1)\sigma^2}}\right)$$

Multiple Cycles of Bit Exchanges and Update

- Bit exchanges happen over noisy fading channels
- Multiple cycles are required to achieve consensus
- Define network state $[b_1(t) \ b_2(t) \ \ldots \ b_N(t)]$ collection of decision bits at the N nodes
- After every update cycle, network will be in one of the $M = 2^N$ states
  - The all-zero and all-one states are consensus states
- Current network state depends on previous network state, current channel states, and current receiver noise: Markovian evolution

Network State Evolution as a Markov chain

- State distribution vector: $\pi(t) = \mathbf{P}(t)\pi(t-1)$; $\mathbf{P}(t)$ is the one-step transition probability matrix (tpm)
- Leads to: $\pi(t) = \mathbf{P}(t)\mathbf{P}(t-1)\ldots\mathbf{P}(1)\pi(0)$, i.e., a time *inhomogeneous* Markov chain
- In such scenarios, the average tpm is considered, $\bar{\pi}(t) = (\bar{\mathbf{P}})^t\pi(0)$
- The average tpm is *irreducible*. Thus, the stationary state distribution vector $\bar{\pi}_\infty$ will have equal entries
  - Memoryless consensus: the final consensus state is independent of the initial state of the system
  - This is bad news!

Good News: Transient Period of the Markov Chain

- During the initial transient period: the network reaches accurate consensus with high probability
- Largest eigen value of the tpm is 1
- Second largest eigen value of the tpm: the closer it is to 1, the longer the transient period
- Need a way to decide when to stop the consensus procedure
- Average hitting time and average consensus duration

Average Hitting Time

- Average hitting time: average number of cycles required to reach consensus state for the first time
- $f_{ij}^{(n)}$ prob. of starting from state i and hitting state j in n cycles
- $f_{ij}^{(n)} = \sum_{\substack{k=1 \\ k \neq j}}^{N} p_{ik} f_{kj}^{(n-1)}$
- $[f_{ij}^{(n)}]_{i=1,\ldots,N} = \mathbf{Q}[f_{kj}^{(n-1)}]_{k=1,\ldots,N}$
- $\mathbf{Q}$ = matrix formed by removing j^{th} column of $\mathbf{P}$

Average Hitting Time Contd.

- $[f_{ij}^{(n)}]_{i=1,\ldots,N} = \mathbf{Q}^{n-2}[f_{kj}^{(1)}]_{k=1,\ldots,N}$
- $[f_{ij}^{(n)}]_{i=1,\ldots,N} = \mathbf{Q}^{n-1}[p_{kj}]_{k=1,\ldots,N}$
- $[p_{ij}]_{i=1,\ldots,N} = j^{th}$ column of $\mathbf{P}$
- Average hitting time

$$\tau_h = \sum_{n=1}^{\infty} n f_{ij}^{(n)}$$

Average Consensus Duration

- $\bar{P}_c =$ average probability of remaining in consensus after the next cycle, once the network is already in consensus
- Prob. of staying in consensus for n consecutive cycles

$$(\bar{P}_c)^n(1-\bar{P}_c)$$

- Average consensus duration: average number of cycles for which the network stays in consensus state

$$\tau_c = \sum_{n=1}^{\infty} n\left(\bar{P}_c\right)^n(1-\bar{P}_c) = \frac{\bar{P}_c}{1-\bar{P}_c}$$

Avg. Prob. of Incorrect Majority Bit Detection

- Received sample at node s_j

$$y_j = h_p(+1) + h_n(-1) + n_j$$
$$y_j = (h_p - h_n)(+1) + n_j$$

- h_p and h_n (sum of Rayleigh RVs) $\approx$ Nakagami RVs.

- Derived pdf for difference of two Nakagami RVs

- Average prob. of incorrect majority bit detection[2]

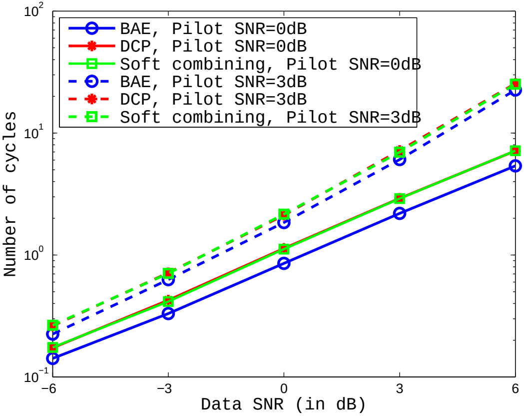
$$\kappa \int_1^{\infty} \frac{{}_2F_1\left(1, m_1 + m_2 + \frac{1}{2}; m_1 + m_2 - \frac{k+l-2}{2}; \frac{\frac{x^2}{2\sigma^2} + \frac{m_1 m_2}{m}}{\frac{x^2}{2\sigma^2} + \frac{m_1}{\Omega_1}}\right)}{\left(\frac{x^2}{2\sigma^2} + \frac{m_1}{\Omega_1}\right)^{m_1+m_2+\frac{1}{2}}} \mathrm{d}x$$

[2] κ is a func. of Nakagmai parameters

Simulation Setup

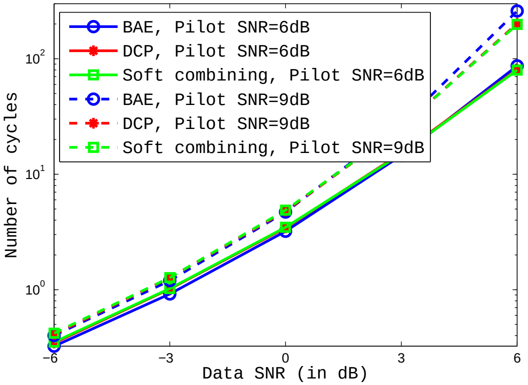
- Number of nodes, $N = 8$
- Channel coefficients, $h_{ij} \sim \mathcal{CN}(0, 1)$
- Receiver noise, $n_j \sim \mathcal{CN}(0, 1)$
- Averaged over 20000 instantiations

Average Consensus Duration Vs. Data SNR



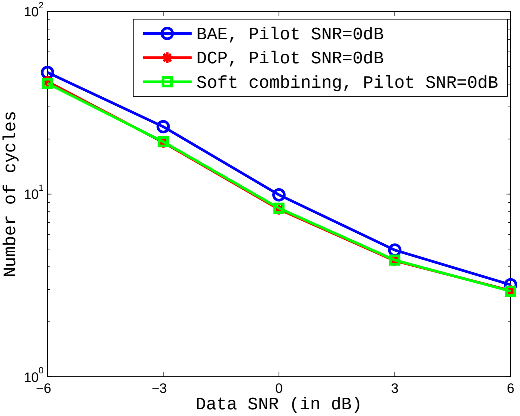
- At low to intermediate pilot SNRs, DCP-based scheme performs better

Average Consensus Duration Vs. Data SNR



- At high pilot SNRs, broadcast-based LMMSE scheme has better performance

Average Hitting Time Vs. Data SNR



- At low to intermediate pilot SNRs, DCP-based scheme has better average hitting time performance

Summary

- Compared broadcast-based and DCP-based consensus protocols in terms of average hitting time and average consensus duration
- Analyzed the average prob. of incorrect majority bit detection performance for DCP-based scheme
- At low to intermediate pilot SNRs, DCP-based consensus outperforms the broadcast-based LMMSE scheme
- At high pilot SNRs, DCP-based scheme is comparable to broadcast-based LMMSE scheme (which uses full CSI)

Thank You