Downlink Sum-Rate Maximization for MU & SU TDD Massive MIMO systems: A Majorization-Minimization Approach

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March 27, 2016

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Introduction

- Sum-rate maximization problem for Single-User & Multi-User TDD Massive MIMO systems with finite control overhead.
- 22% overhead of control signaling in current generation cellular systems like LTE & LTE-A. Control overhead much higher for next generation Massive MIMO systems.
- Fixed codebook based beamforming (precoding) in the downlink. Optimization of the achievable sum-rate by assigning non-overlapping beams to different users.
- Only the precoder index and the power allocated to the beams should be sent to the UEs.
- TDD Systems: Reduced Channel feedback overhead, Channel estimation by using channel reciprocity or reverse channel training.

System Model

- Consider a MIMO wireless system with one BS and multiple UEs. Number of transmit and receive antennas are N_t (at the BS) and N_r (at each UE) respectively. Let K be the number of users.
- Scenario: MIMO Broadcast channel. BS transmits signals to all the UEs at the same time instant.
- The transmitted signal $\mathbf{s}$ and received signal $\mathbf{y}_k$ at the k^{th} receiver are given by

$$\mathbf{s} = \sum_{k=1}^{K} \sum_{l=1}^{N} \mathbf{v}_k s_k(l) \tag{1}$$

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{s} + \mathbf{w}_k \tag{2}$$

- $\mathbf{H}_k$ is the $N_r \times N_t$ channel of the k^{th} receiver.
- $\mathbf{w}_k$ is the additive noise at the k^{th} user with distribution $\mathcal{CN}(0, \sigma^2 \mathbf{I}_{N_r})$
- Codebook $\mathbf{C} = [\mathbf{v}_1, ..., \mathbf{v}_N] \in \mathbb{C}^{N_t \times N}$
- The received signal at the k^{th} UE is thus

$$\mathbf{y}_k = \mathbf{H}_k \sum_{j=1}^{K} \mathbf{C}\mathbf{s}_j + \mathbf{w}_k \tag{3}$$

where $\mathbf{s}_j = [s_j(1), ..., s_j(N)]^T$.

- The rate achievable for the k^{th} user is given by

$$R_k = \text{logdet}\left(\mathbf{I}_{N_r} + \mathbf{V}_k^{-1}\mathbf{H}_k\mathbf{C}\boldsymbol{\Phi}_k\mathbf{C}^H\mathbf{H}_k^H\right) \tag{4}$$

where

$$\mathbf{V}_k = \left(\sigma^2\mathbf{I}_{N_r} + \sum_{\substack{j=1\\j\neq k}}^{K}\mathbf{H}_k\mathbf{C}\boldsymbol{\Phi}_j\mathbf{C}^H\mathbf{H}_k^H\right) \tag{5}$$

is the interference plus noise covariance matrix and $\boldsymbol{\Phi}_k = \text{diag}\big([P_k(1), P_k(2), ..., P_k(N)]\big)$ is the signal covariance matrix of the k^{th} user.

- Downlink Sum-Rate of the whole system is given by

$$R_{Tot} = \sum_{k=1}^{K} \text{logdet}\left(\mathbf{I}_{N_r} + \mathbf{V}_k^{-1}\mathbf{H}_k\mathbf{C}\boldsymbol{\Phi}_k\mathbf{C}^H\mathbf{H}_k^H\right) \quad (6)$$

$$= \sum_{k=1}^{K} \text{logdet}\left(\mathbf{I}_{N_r} + \mathbf{V}_k^{-1}\tilde{\mathbf{H}}_k\boldsymbol{\Phi}_k\tilde{\mathbf{H}}_k^H\right) \quad (7)$$

  where $\tilde{\mathbf{H}}_k = \mathbf{H}_k\mathbf{C}$.
- Goal: To maximize the sum-rate R_{Tot} under a total transmit power constraint P_{max}.

Problem Statement

- The problem statement is given by

$$\underset{\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_K}{\text{maximize}} \sum_{k=1}^{K} \text{logdet}\left(\mathbf{I}_{N_r} + \mathbf{V}_k^{-1}\tilde{\mathbf{H}}_k\boldsymbol{\Phi}_k\tilde{\mathbf{H}}_k^H\right) \tag{8}$$

subject to

$$\text{Tr}\left(\sum_{k=1}^{K}\boldsymbol{\Phi}_k\right) \leq P_{max}$$

Majorization-Minimization (or Minorization-Maximization) Principle

- MM algorithm proceeds by solving a simple convex optimization problem in place of a complex non-convex optimization problem.
- Surrogate convex function which bounds the objective function either from above (for minimization) or below (for maximization) is computed.
- A function $g(x|x^{(m)})$ is said to majorize a real-valued function $f(x)$ at $x^{(m)}$ if

$$g(x|x^{(m)}) \geq f(x), \forall x \in \mathbb{C}$$
$$g(x^{(m)}|x^{(m)}) = f(x^{(m)})$$

- Minimization of the surrogate function followed by finding another surrogate function at the new iterate.
- Iterative Algorithm which has monotonic convergence property.
- Monotonic decrease property of the MM algorithm provides numerical stability and solve for the minimizer of $f(x)$ after a certain number of iterations.
- Globally convergent algorithm which will converge to a local optimum point.
- Majorization relation between functions is closed under the formation of sums, nonnegative products, limits, and composition with an increasing function. (This property is exploited for MU case.)
- For more details, refer to the tutorial "A Tutorial on MM algorithms", by D. Hunter and K. Lange.

Proposed Algorithms

- Square-Root-MM (SMM)
- Inverse-MM (IMM)
- SMM & IMM algorithms were designed for Single-User and Multi-User cases respectively.
- SMM was extended to Multi-User case also.
- Both algorithms give same performance but complexity of IMM is lesser than that of SMM.

SMM

Lemma (1)

For matrices $\mathbf{Z}, \mathbf{Y} \succeq 0$*, the non-convex function*

$$f(\mathbf{Z}, \mathbf{Y}) = \text{logdet}\left(\mathbf{Z}^{-1}\mathbf{Y}\right) \tag{9}$$

can be lower bounded by

$$f(\mathbf{Z},\mathbf{Y}) \geq -\Bigg(\text{logdet}\,\mathbf{Z}^{(m)} + \text{Tr}\left(\left(\mathbf{Z}^{(m)}\right)^{-1}\left(\mathbf{Z} - \mathbf{Z}^{(m)}\right)\right) + \text{logdet}\left(\mathbf{Y}^{-1}\right)^{(m)} + \text{Tr}\left(\mathbf{Y}^{(m)}\left(\mathbf{Y}^{-1} - \left(\mathbf{Y}^{(m)}\right)^{-1}\right)\right)\Bigg) \tag{10}$$

with equality attained when $\mathbf{Z} = \mathbf{Z}^{(m)}$ *and* $\mathbf{Y} = \mathbf{Y}^{(m)}$.

- Define the matrix $\mathbf{B}_k = \left(\sigma^2 \mathbf{I}_{N_r} + \sum_{j=1}^{K} \tilde{\mathbf{H}}_k \mathbf{\Phi}_j \tilde{\mathbf{H}}_k^H\right)$
- Majorization Step 1: Applying Lemma 1 to the sum-rate objective function, the optimization problem is reformulated as

$$R_{Tot} \geq \sum_{k=1}^{K} \Bigg\{ -\operatorname{Tr}\left(\left(\mathbf{V}_k^{(m)}\right)^{-1}\left(\sigma^2 \mathbf{I}_{N_r} + \sum_{\substack{j=1 \\ j\neq k}}^{K} \tilde{\mathbf{H}}_k \mathbf{\Phi}_j \tilde{\mathbf{H}}_k^H\right)\right) - \operatorname{Tr}\left(\mathbf{B}_k^{(m)}\left(\sigma^2 \mathbf{I}_{N_r} + \sum_{j=1}^{K} \tilde{\mathbf{H}}_k \mathbf{\Phi}_j \tilde{\mathbf{H}}_k^H\right)^{-1}\right)\Bigg\} \quad (11)$$

- Single User Case: The first term in the above equation is a constant which can be removed from the optimization problem.

$$
\begin{aligned}
\mathbf{\Phi}_K^{(m+1)} &= \underset{\mathbf{\Phi}_K}{\operatorname{argmax}} \Bigg\{ - N_r - \operatorname{Tr}\Bigg(\mathbf{B}_K^{(m)}\big(\sigma^2 \mathbf{I}_{N_r} \\
&\qquad + \tilde{\mathbf{H}}_K \mathbf{\Phi}_j \tilde{\mathbf{H}}_K^H\big)^{-1}\Bigg)\Bigg\} \\
&= \underset{\mathbf{\Phi}_K}{\operatorname{argmin}} \left\{ \operatorname{Tr}\left(\mathbf{B}_K^{(m)} \left(\sigma^2 \mathbf{I}_{N_r} + \tilde{\mathbf{H}}_K \mathbf{\Phi}_K \tilde{\mathbf{H}}_K^H\right)^{-1}\right)\right\}
\end{aligned}
\tag{12}
$$

- Define matrices

$$\mathbf{F}_K = \text{Cholesky}(\mathbf{B}_K) \tag{13}$$

$$\mathbf{S}_K = \frac{\tilde{\mathbf{H}}_K^H \tilde{\mathbf{H}}_K}{\sigma^2} \tag{14}$$

$$\mathbf{X}_K = \frac{\mathbf{F}_K \tilde{\mathbf{H}}_K \mathbf{\Phi}_K^{\frac{1}{2}}}{\sigma^2} \tag{15}$$

$$\mathbf{Y}_K = \mathbf{I}_N + \mathbf{\Phi}_K^{\frac{1}{2}} \mathbf{S}_K \mathbf{\Phi}_K^{\frac{1}{2}} \tag{16}$$

- Woodbury's Matrix Identity:

$$(\mathbf{A} + \mathbf{UCV})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left(\mathbf{C}^{-1} + \mathbf{VA}^{-1}\mathbf{U}\right)^{-1}\mathbf{VA}^{-1} \tag{17}$$

Lemma (2)

For a diagonal and positive semidefinite square matrix $\mathbf{Q}$ *of any size, the function*

$$f(\mathbf{Q}) = \mathrm{Tr}\left(\mathbf{A}\left(\mathbf{B} + \mathbf{C}\mathbf{Q}\mathbf{C}^H\right)^{-1}\mathbf{A}^H\right) \tag{18}$$

can be upper bounded by

$$\begin{aligned} f(\mathbf{Q}) \leq \mathrm{Tr}(\mathbf{K}^{(m)}) - \mathrm{Tr}\Bigg(&\Bigg(\left(\mathbf{Y}^{-1}\mathbf{X}^H\right)^{(m)}\mathbf{A}\mathbf{B}^{-1}\mathbf{C} \\ &+ \mathbf{C}^H\left(\mathbf{B}^{-1}\right)^H\mathbf{A}^H\left(\mathbf{X}\mathbf{Y}^{-1}\right)^{(m)}\Bigg)\mathbf{Q}^{\frac{1}{2}} \\ &- \left(\mathbf{Y}^{-1}\mathbf{X}^H\mathbf{X}\mathbf{Y}^{-1}\right)^{(m)}\mathbf{Q}^{\frac{1}{2}}\mathbf{C}^H\mathbf{B}^{-1}\mathbf{C}\mathbf{Q}^{\frac{1}{2}}\Bigg) \end{aligned} \tag{19}$$

where $\mathbf{X} = \mathbf{A}\mathbf{B}^{-1}\mathbf{C}\mathbf{Q}^{\frac{1}{2}}$, $\mathbf{Y} = \mathbf{I} + \mathbf{Q}^{\frac{1}{2}}\mathbf{C}^H\mathbf{B}^{-1}\mathbf{C}\mathbf{Q}^{\frac{1}{2}}$ *and*

$$\begin{aligned} \mathbf{K} =& \mathbf{A}\mathbf{B}^{-1}\mathbf{A}^H + \mathbf{Y}^{-1}\mathbf{X}^H\mathbf{X} - \mathbf{Y}^{-1}\mathbf{X}^H\mathbf{X}\mathbf{Y}^{-1}\mathbf{Y} \\ &+ \mathbf{Y}^{-1}\mathbf{X}^H\mathbf{X}\mathbf{Y}^{-1} + \mathbf{X}\mathbf{Y}^{-1}\mathbf{X}^H \end{aligned} \tag{20}$$

- Majorization Step 2: Applying (17) and Lemma 2 to the optimization problem (12),

$$\mathbf{\Phi}_K^{(m+1)} = \underset{\mathbf{\Phi}_K}{\operatorname{argmin}} \left\{ \operatorname{Tr} \left(\mathbf{W}_{1,K}^{(m)} \mathbf{\Phi}_K^{\frac{1}{2}} + \mathbf{W}_{2,K}^{(m)} \mathbf{\Phi}_K^{\frac{1}{2}} \mathbf{S}_K \mathbf{\Phi}_K^{\frac{1}{2}} \right) \right\} \quad (21)$$

where

$$\mathbf{W}_{1,K} = -\left\{ \frac{\mathbf{Y}_K^{-1} \mathbf{X}_K^H \mathbf{F}_K \tilde{\mathbf{H}}_K + \tilde{\mathbf{H}}_K^H \mathbf{F}_K^H \mathbf{X}_K \mathbf{Y}_K^{-1}}{\sigma^2} \right\} \quad (22)$$

$$\mathbf{W}_{2,K} = \mathbf{Y}_K^{-1} \mathbf{X}_K^H \mathbf{X}_K \mathbf{Y}_K^{-1}$$

Lemma (3)

For $\mathbf{Q}$, *a diagonal and positive semidefinite square matrix of any size and matrices* $\mathbf{A}$, $\mathbf{B} \succeq 0$, *the function*

$$f(\mathbf{Q}) = \operatorname{Tr}(\mathbf{AQBQ}) \tag{23}$$

can be upper bounded by

$$\begin{aligned} f(\mathbf{Q}) \leq \operatorname{Tr}\Big(&\mathbf{A}\mathbf{Q}^{(m)}\mathbf{B}\mathbf{Q}^{(m)} - \big((\mathbf{B}-\lambda\mathbf{I})\mathbf{Q}^{(m)}\mathbf{A} \\ &+ \mathbf{A}\mathbf{Q}^{(m)}(\mathbf{B}-\lambda\mathbf{I})\big)\mathbf{Q}^{(m)}\Big) + \operatorname{Tr}\Big(\big((\mathbf{B}-\lambda\mathbf{I})\mathbf{Q}^{(m)}\mathbf{A} \\ &+ \mathbf{A}\mathbf{Q}^{(m)}(\mathbf{B}-\lambda\mathbf{I})\big)\mathbf{Q}\Big) + \lambda \operatorname{Tr}\left(\mathbf{A}\mathbf{Q}^2\right) \end{aligned} \tag{24}$$

where λ *is the largest eigenvalue of the matrix* $\mathbf{B}$.

- Majorization Step 3: Applying Lemma 3 to the optimization problem,

$$\mathbf{\Phi}_K^{(m+1)} = \underset{\mathbf{\Phi}_K}{\text{argmin}} \left\{ \text{Tr}\left(\mathbf{W}_{A,K}^{(m)} \mathbf{\Phi}_K^{\frac{1}{2}} + \mathbf{W}_{B,K}^{(m)} \mathbf{\Phi}_K \right) \right\} \tag{25}$$

where

$$\mathbf{W}_{A,K} = \mathbf{W}_{1,K} + \left(\mathbf{S}_K - \lambda_{max}\mathbf{I}_N\right) \mathbf{\Phi}_K^{\frac{1}{2}} \mathbf{W}_{2,K} + \mathbf{W}_{2,K} \mathbf{\Phi}_K^{\frac{1}{2}} \left(\mathbf{S}_K - \lambda_{max}\mathbf{I}_N\right) \tag{26}$$

$$\mathbf{W}_{B,K} = \lambda_{max} \mathbf{W}_{2,K} \tag{27}$$

and λ_{max} is the largest eigenvalue of the matrix $\mathbf{S}_K$.

- The Lagrangian is given below:

$$\sum_{i=1}^{N}\left([\mathbf{W}_{A,K}]_{(i,i)}^{(m)}P_K(i)^{\frac{1}{2}}+[\mathbf{W}_{B,K}^{(m)}]_{(i,i)}P_K(i)\right) + \eta\left(\sum_{i=1}^{N}P_K(i)-P_{max}\right) \tag{28}$$

The analytical solution for (28) is given below.

$$P_K(i)=\left(\frac{\left[\mathbf{W}_{A,K}^{(m)}\right]_{(i,i)}}{2\left(\left[\mathbf{W}_{B,K}^{(m)}\right]_{(i,i)}+\eta\right)}\right)^2 \tag{29}$$

IMM

- Majorization Step 1 is same as that of SMM.

$$R_{Tot} \geq \sum_{k=1}^{K} \Bigg\{ -\operatorname{Tr}\left(\left(\mathbf{V}_k^{(m)}\right)^{-1}\left(\sigma^2 \mathbf{I}_{N_r} + \sum_{\substack{j=1\\ j\neq k}}^{K} \tilde{\mathbf{H}}_k \mathbf{\Phi}_j \tilde{\mathbf{H}}_k^H\right)\right) - \operatorname{Tr}\left(\mathbf{B}_k^{(m)}\left(\sigma^2 \mathbf{I}_{N_r} + \sum_{j=1}^{K} \tilde{\mathbf{H}}_k \mathbf{\Phi}_j \tilde{\mathbf{H}}_k^H\right)^{-1}\right)\Bigg\} \quad (30)$$

- We form the extended channel matrix, signal covariance matrix as follows

$$\mathbf{\Phi} = \text{diag}\left(\mathbf{\Phi}_1, ..., \mathbf{\Phi}_K\right) \in \mathbb{R}^{KN \times KN} \tag{31}$$

$$\tilde{\mathbf{\Phi}} = \text{diag}\left(\mathbf{\Phi}_1, ..., \mathbf{\Phi}_K, \sigma^2 \mathbf{I}_{N_r}\right) \mathbb{R}^{(KN+N_r) \times (KN+N_r)} \tag{32}$$

$$\mathbf{\Psi}_k = \left[\tilde{\mathbf{H}}_k, ..., \tilde{\mathbf{H}}_k, \mathbf{I}_{N_r}\right] \in \mathbb{C}^{N_r \times (KN+N_r)}, k = 1, ..., K \tag{33}$$

$$\mathbf{\Xi}_k = \mathbf{\Psi}_k \tilde{\mathbf{\Phi}} \mathbf{\Psi}_k^H \in \mathbb{C}^{N_r \times N_r}, k = 1, ..., K \tag{34}$$

- We consider the two terms of (30) separately and combine to get the final optimization problem.

$$\sum_{k=1}^{K} \mathrm{Tr}\left(\left(\mathbf{V}_k^{(m)}\right)^{-1} \sum_{\substack{j=1 \\ j\neq k}}^{K} \tilde{\mathbf{H}}_k \mathbf{\Phi}_j \tilde{\mathbf{H}}_k^H\right) \tag{35}$$

$$= \sum_{k=1}^{K} \mathrm{Tr}\left(\tilde{\mathbf{H}}_k^H \left(\mathbf{V}_k^{(m)}\right)^{-1} \tilde{\mathbf{H}}_k \sum_{\substack{j=1 \\ j\neq k}}^{K} \mathbf{\Phi}_j\right)$$

$$= \sum_{k=1}^{K} \mathrm{Tr}\left(\mathbf{Q}_k^{(m)} \tilde{\mathbf{\Phi}}\right) = \mathrm{Tr}\left(\mathbf{Q}^{(m)} \tilde{\mathbf{\Phi}}\right) \tag{36}$$

where

$$\mathbf{R}_k = \tilde{\mathbf{H}}_k^H \left(\mathbf{V}_k^{(m)}\right)^{-1} \tilde{\mathbf{H}}_k \tag{37}$$

$$\mathbf{Q}_k^{(m)} = \mathrm{diag}\left(\mathbf{R}_k, ..., \mathbf{0}_N, ..., \mathbf{R}_k, \mathbf{0}_{N_r}\right) \tag{38}$$

$$\mathbf{Q}^{(m)} = \sum_{k=1}^{K} \mathbf{Q}_k^{(m)} \tag{39}$$

Result from Matrix Analysis:

For a matrix $\mathbf{A} \succeq 0$ and $\mathbf{R} = \mathbf{T}\mathbf{S}\mathbf{T}^H$, we can upper bound the function $f(\mathbf{R}) = \text{Tr}\left(\mathbf{A}\mathbf{R}^{-1}\right)$ as

$$\text{Tr}\left(\mathbf{A}\mathbf{R}^{-1}\right) \leq \text{Tr}\left(\mathbf{A}\left(\mathbf{R}^{(m)}\right)^{-1}\mathbf{T}\mathbf{S}^{(m)}\mathbf{S}^{-1}\mathbf{S}^{(m)}\mathbf{T}^H\left(\mathbf{R}^{(m)}\right)^{-1}\right) \tag{40}$$

with equality achieved at $\mathbf{S} = \mathbf{S}^{(m)}$.

- The second term in (30)

$$\sum_{k=1}^{K} \text{Tr}\left(\mathbf{B}_k^{(m)}\left(\sigma^2\mathbf{I}_{N_r} + \sum_{j=1}^{K}\tilde{\mathbf{H}}_k\mathbf{\Phi}_j\tilde{\mathbf{H}}_k^H\right)^{-1}\right)$$
$$= \sum_{k=1}^{K} \text{Tr}\left(\mathbf{B}_k^{(m)}\mathbf{\Xi}_k^{-1}\right) \qquad (41)$$

Applying (40) in (41),

$$\begin{aligned}
&\sum_{k=1}^{K} \mathrm{Tr}\left(\mathbf{B}_k^{(m)} \boldsymbol{\Xi}_k^{-1}\right) \\
&\leq \sum_{k=1}^{K} \mathrm{Tr}\left(\mathbf{B}_k^{(m)} \left(\boldsymbol{\Xi}_k^{(m)}\right)^{-1} \boldsymbol{\Psi}_k \tilde{\boldsymbol{\Phi}}^{(m)} \tilde{\boldsymbol{\Phi}}^{-1} \tilde{\boldsymbol{\Phi}}^{(m)} \boldsymbol{\Psi}_k^H \left(\boldsymbol{\Xi}_k^{(m)}\right)^{-1}\right) \\
&= \sum_{k=1}^{K} \mathrm{Tr}\left(\tilde{\boldsymbol{\Phi}}^{(m)} \boldsymbol{\Psi}_k^H \left(\boldsymbol{\Xi}_k^{(m)}\right)^{-1} \boldsymbol{\Psi}_k \tilde{\boldsymbol{\Phi}}^{(m)} \tilde{\boldsymbol{\Phi}}^{-1}\right) \\
&= \mathrm{Tr}\left(\mathbf{Z}^{(m)} \tilde{\boldsymbol{\Phi}}^{-1}\right) \qquad (42)
\end{aligned}$$

where

$$\mathbf{Z}^{(m)} = \sum_{k=1}^{K} \tilde{\boldsymbol{\Phi}}^{(m)} \boldsymbol{\Psi}_k^H \left(\boldsymbol{\Xi}_k^{(m)}\right)^{-1} \boldsymbol{\Psi}_k \tilde{\boldsymbol{\Phi}}^{(m)} \qquad (43)$$

The Lagrangian is given by

$$\sum_{k=1}^{K}\sum_{i=1}^{N}\left(\left[\mathbf{Q}^{(m)}\right]_{((k-1)N+i,(k-1)N+i)} P_k(i) + \left[\mathbf{Z}^{(m)}_{((k-1)N+i,(k-1)N+i)}\right]\frac{1}{P_k(i)}\right) + \eta\left(\sum_{k=1}^{K}\sum_{i=1}^{N} P_k(i) - P_{max}\right)$$

The solution for the optimal power allocation is

$$P_k(i) = \left(\frac{\left[\mathbf{Z}^{(m)}\right]_{((k-1)N+i,(k-1)N+i)}}{\left[\mathbf{Q}^{(m)}\right]_{((k-1)N+i,(k-1)N+i)} + \eta_{opt}}\right)^{\frac{1}{2}} \tag{44}$$

$\forall i = 1, ..., N$ and $k = 1, ..., K$. Since the objective function is strictly decreasing, the Lagrangian multiplier can be computed by line search.

SMM for Multi-User Case

- The lower bound for the sum-rate is taken from (30)

$$R_{Tot} \geq \sum_{k=1}^{K} \Bigg\{ -\operatorname{Tr}\left(\left(\mathbf{V}_k^{(m)}\right)^{-1}\left(\sigma^2 \mathbf{I}_{N_r} + \sum_{\substack{j=1 \\ j\neq k}}^{K} \tilde{\mathbf{H}}_k \mathbf{\Phi}_j \tilde{\mathbf{H}}_k^H\right)\right) - \operatorname{Tr}\left(\mathbf{B}_k^{(m)}\left(\sigma^2 \mathbf{I}_{N_r} + \sum_{j=1}^{K} \tilde{\mathbf{H}}_k \mathbf{\Phi}_j \tilde{\mathbf{H}}_k^H\right)^{-1}\right)\Bigg\} \quad (45)$$

- The first term in (45) is handled in the same way as IMM.

- We define an extended channel matrix and covariance matrix as follows

$$\tilde{\mathbf{\Psi}}_k = \left[\tilde{\mathbf{H}}_k, \ldots, \tilde{\mathbf{H}}_k\right] \in \mathbb{C}^{N_r \times KN}, k = 1, \ldots, K \tag{46}$$

$$\mathbf{\Phi} = \operatorname{diag}\left(\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_K\right) \in \mathbb{R}^{KN \times KN} \tag{47}$$

- Second term in (45) is thus

$$\sum_{k=1}^{K} \operatorname{Tr}\left(\mathbf{B}_k^{(m)}\left(\sigma^2 \mathbf{I}_{N_r} + \sum_{j=1}^{K} \tilde{\mathbf{H}}_k \mathbf{\Phi}_j \tilde{\mathbf{H}}_k^H\right)^{-1}\right)$$
$$= \sum_{k=1}^{K} \operatorname{Tr}\left(\mathbf{B}_k^{(m)}\left(\sigma^2 \mathbf{I}_{N_r} + \tilde{\mathbf{\Psi}}_k \mathbf{\Phi} \tilde{\mathbf{\Psi}}_k^H\right)^{-1}\right) \tag{48}$$

- (48) can be handled in the same way as Single User case.

- Final Optimal power allocations:

$$P(i) = \left(\frac{\left[\mathbf{W}_A^{(m)} \right]_{(i,i)}}{2 \left(\left[\mathbf{W}_B^{(m)} \right]_{(i,i)} + \left[\mathbf{Q}^{(m)} \right]_{(i,i)} + \eta_{opt} \right)} \right)^2, i = 1, ..., KN \tag{49}$$

where

$$\mathbf{W}_A = \sum_{k=1}^{K} \mathbf{W}_{A,k}$$

$$\mathbf{W}_B = \sum_{k=1}^{K} \mathbf{W}_{B,k}$$

SMM Algorithm

Input: $\mathbf{H}_K$, $\mathbf{C}$, P_{max}, σ
Output: $P_K(1), ..., P_K(N)$

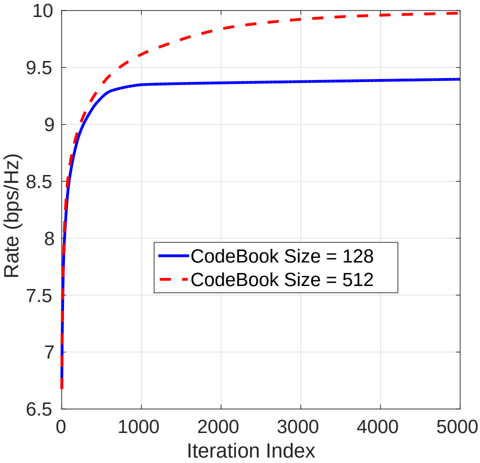
Initialize $P_K(1), ..., P_K(N)$ with random positive values which satisfies the maximum power constraint
Compute $\mathbf{S}_K$, $\mathbf{B}_K$ using (14)
$\tilde{\mathbf{H}}_K = \mathbf{H}_K\mathbf{C}$
λ_{max} = maximum of eigen values of $\mathbf{S}_K$
$\mathbf{F}_K = Cholesky(\mathbf{B}_K)$
repeat
    $\mathbf{\Phi}_K = \text{diag}\big(P_K(1), ..., P_K(N)\big)$
    Compute $\mathbf{X}_K$, $\mathbf{Y}_K$ using (15), (16) respectively
    Compute $\mathbf{W}_{1,K}, \mathbf{W}_{2,K}$ using (22) respectively
    Compute $\mathbf{W}_{A,K}, \mathbf{W}_{B,K}$ using (26), (27) respectively
    Calculate Lagrange multiplier η using line search to satisfy maximum power constraint P_{max}
    for $i = 1$ to N **do**
        Compute $P_K(i)$ using (29)
    end for
until convergence criterion is met

IMM

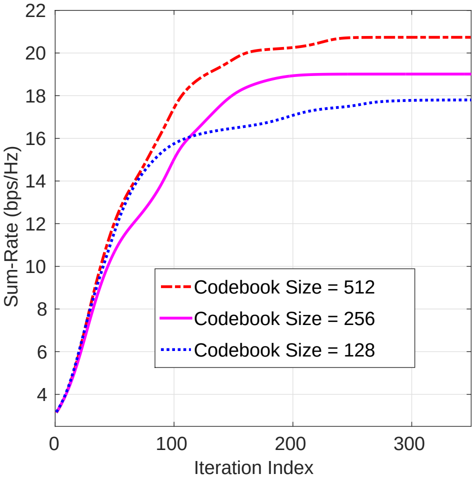
Input: $\mathbf{H}_1, \dots, \mathbf{H}_K, \mathbf{C}, K, P_{max}, \sigma$
Output: $P_1(1), \dots, P_1(N), \dots, P_K(1), \dots, P_K(N)$

Initialize $P_1(1), \dots, P_1(N), \dots, P_K(1), \dots, P_K(N)$ with random positive values which satisfies the maximum power constraint
for $k = 1$ to K **do**
  $\tilde{\mathbf{H}}_k = \mathbf{H}_k \mathbf{C}$
end for
Compute $\mathbf{\Psi}_1, \dots, \mathbf{\Psi}_K$ using (33)
repeat
  for $k = 1$ to K **do**
    $\mathbf{\Phi}_k = \text{diag}\big(P_k(1), \dots, P_k(N)\big)$
    Compute $\mathbf{V}_k$ using (5)
  end for
  Compute $\mathbf{\Phi}, \tilde{\mathbf{\Phi}}$ using (31), (32) respectively.
  for $k = 1$ to K **do**
    Compute $\mathbf{\Xi}_k, \mathbf{R}_k, \mathbf{Q}_k$ using (34), (37), (38) respectively.
  end for
  Compute $\mathbf{Q}, \mathbf{Z}$ using (39), (43) respectively.
  Calculate Lagrange multiplier η using line search to satisfy maximum power constraint P_{max}
  for $k = 1$ to K **do**
    for $i = 1$ to N **do**
      Compute $P_K(i)$ using (44)
    end for
  end for
until convergence criterion is met

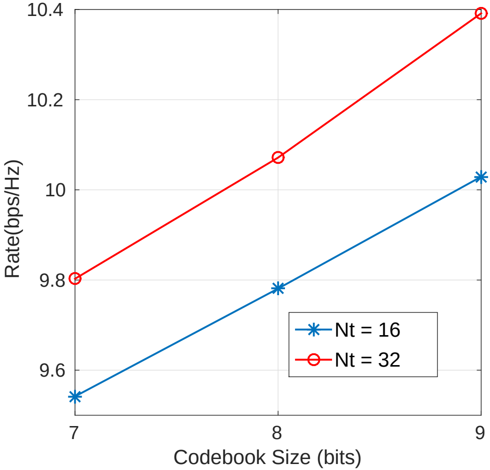
SMM Convergence



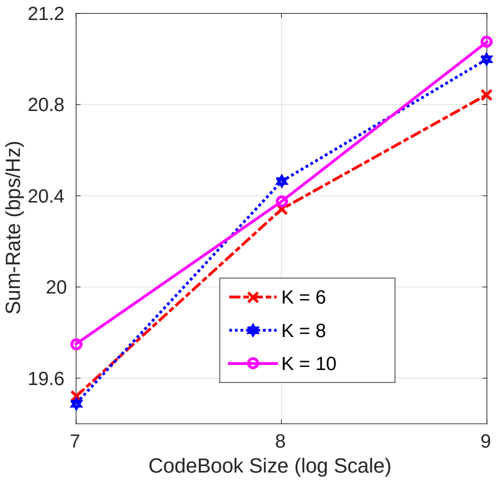
IMM Convergence



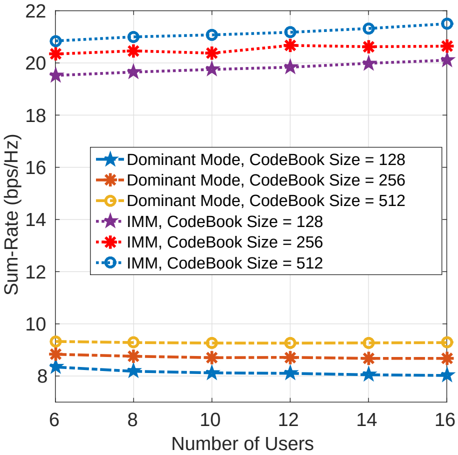
Performance of SMM Algorithm vs Codebook Size



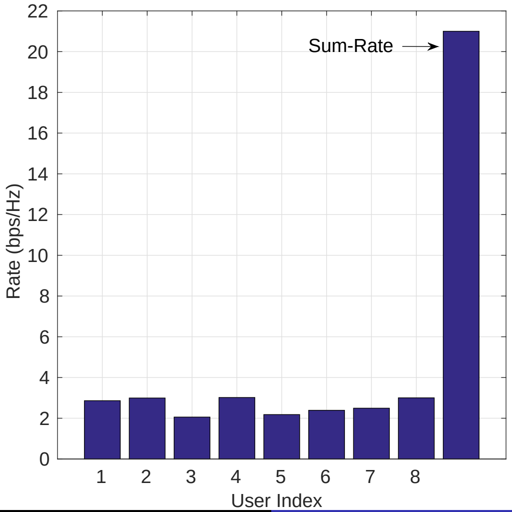
Performance of IMM Algorithm vs Codebook Size



Performance of IMM vs Number of Users



Individual Users Rate



Computational Complexity of SMM

Table: Flop Count Analysis for SMM

Matrix	Size	Flop Counts
$\mathbf{S}_K$	NxN	$N^2(2N_r - 1)$
$\mathbf{X}_K$	N_rxN	$2NN_r(N + N_r - 1)$
$\mathbf{Y}_K$	NxN	$2N^2(2N - 1)$
$\mathbf{W}_{1,K}$	NxN	$N(2NN_r - 2N - r^2 - 1)$
$\mathbf{W}_{2,K}$	NxN	$N^2(4N_r - 1) - NN_r$
$\mathbf{W}_{A,K}$	NxN	$2N(2N - 1)$
$\mathbf{W}_{B,K}$	NxN	N

Computational Complexity of IMM

Table: Flop Count Analysis for IMM

Matrix	Size	Flop Counts
$\boldsymbol{\Xi}_k$	N_rxN_r	$N_r(KN+2N_r)$ $(2(KN+N_r)-1)$
$\mathbf{Z}$	$(KN+N_r)$x $(KN+N_r)$	$K\big(2N_r(KN+2N_r)$ $-1+(K-1)N\big)$
$\mathbf{Q}$	$(KN+N_r)$x $(KN+N_r)$	$KN(K-1+$ $(2N_r-1)(N+N_r))$