Deterministic dictionaries for sparse representation: Construction and Applications

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March 19, 2016

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Part I: Dictionaries for sparse representations: Basic concepts

What is Compressed Sensing

- In many applications one would like to recover x from its linear measurements or observed data
- In mathematical terms, the observed data $y \in R^m$ is connected to the signal $x \in R^M$ of interest via

$$y = \Phi x$$

- When the number of measurements m is equal to M, the recovered x is in general $\Phi^{-1}y$
- However, in many applications, it is much more desirable to take fewer measurements provided one can still recover x
- In particular, when $m < M$, the linear system $\Phi x = y$ is under-determined and in general possesses infinitely many solutions

Dictionaries for sparse representations: Basic concepts

- In this case, an interesting question arises: "is it still possible to recover x possessing fewer nonzero components from y through a computationally tractable procedure ?"
- The research area associated to this phenomenon has become popular as **Sparse Representation Theory, Compressed Sensing (CS), Compressive Sampling**
- A vector $x \in \mathbb{R}^M$ is $k-$**sparse** if it has k nonzero coordinates. That is, $\|x\|_0 := |\{i \mid x_i \neq 0\}| = k < M$
- One can recover sparse x from its linear measurements by solving the following optimization problem:

$$P_0 : \min_x \|x\|_0 \text{ subject to } \quad \Phi x = y \tag{1}$$

- This l_0-minimization problem is computationally not tractable[a] in general

[a] Simon Foucart and Holger Rauhut, "A Mathematical Introduction to Compressive Sensing," Birkhauser, Baseln, 2013.

On the solvability of P_0 problem

- There have been attempts to repose or solve P_0 problem via greedy and convex relaxation methods
- D.Donoho et.al.[a] posed an equivalent of this problem as

$$P_1 : \min_x \|x\|_1 \text{ subject to } \quad b = \Phi x \tag{2}$$

- Due to shape of l_1 ball, l_1 minimization promotes sparsity
- An l_1 minimization problem can be recast as a linear programming problem (LPP)
- Fast solvers are available
- The algorithms OMP, STOMP, WMP, MP, ROMP fall under greedy category. Among all, OMP is most popular algorithm

[a] S.S. Chen, D.L. Donoho, and M.A. Saunders, "Atomic Decomposition by Basis Pursuit," SIAM, 2001.

Sufficient conditions for equivalence between P_0 and P_1

- The general question of CS is: "when do both problems (1) and (2) admit same solution ?"

Definition

The mutual-coherence of a given matrix Φ is the largest absolute inner-product between different normalized columns of Φ. Denoting the k-th column in Φ by ϕ_k, the **mutual-coherence** is given by

$$\mu(\Phi) = \max_{1\leq\, i,j\leq\, m,\; i\neq j} \frac{|\ \phi_i^T \phi_j|}{\|\phi_i\|_2 \|\phi_j\|_2}. \tag{3}$$

Dictionaries for sparse representations: Basic concepts

- The coherence parameter provides a kind of measure for checking whether or not P_0 and P_1 are equivalent

Theorem

Let Φ be an $m \times M$ matrix and let $0 \neq x \in \mathbb{R}^M$ be a solution of P_0 satisfying

$$\|x\|_0 < \frac{1}{2}(1 + (\mu(\Phi))^{-1}). \tag{4}$$

Then x is the unique solution [a] of P_0 and P_1.

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[a] D.L. Donoho et. al., "Stable Recovery of Sparse Over complete Representations in the Presence of Noise," IEEE Trans. Inform. Theory, 2006.

- In simulations, the stated equivalence is established through phase diagrams

- Terense Tao and Candes proposed an alternative approach establishing the stated equivalence

Definition

We say that a matrix Φ satisfies **Restricted Isometry Property (RIP)** of order k, if there is a $0 < \delta_k < 1$ such that

$$(1-\delta_k)\|z\|_{l_2} \leq \|\Phi_T z\|_{l_2} \leq (1+\delta_k)\|z\|_{l_2}, \quad z \in \mathcal{R}^k, \qquad (5)$$

holds for all T of cardinality k.

The following theorem [a] establishes the equivalence between P_0 and P_1 problems through RIP

[a] E. Candes, "The restricted isometry property and its implications for compressed sensing," Comptes Rendus Mathematique, 2008

Theorem

Suppose an $m \times M$ matrix Φ has the RIP of order $2k$ with constant $\delta_{2k} < \sqrt{2} - 1$, then P_0 and P_1 have same $k-$sparse solution if P_0 has a $k-$sparse solution.

- The widest possible known range of k is of the order $\frac{m}{\log(\frac{M}{m})}$
- The only known constructions yielding matrices that satisfy RIP for this range are based on random matrices[a]

[a] Baraniuk.R, et.al. "A Simple Proof of the Restricted Isometry Property for Random Matrices," Constructive Approximation, 2008

- The following proposition relates the RIP constant δ_k and μ

Proposition

[a] *Suppose that $\Phi_1, \ldots, \Phi_M$ are the unit norm columns of the matrix Φ with coherence μ. Then Φ satisfies RIP of order k with constant $\delta_k = (k-1)\mu$.*

[a] M. Elad, "Sparse and redundant representations; from theory to applications in signal and image processing," Springer, Berlin, 2010.

- Since the role of CS matrix is to provide sparse representations to a given y, it is referred to as **dictionary**
- There are two classes of approaches being developed for generating dictionaries
  - Data-independent dictionaries
  - Data-driven dictionaries

Summary of contributions

The summery of my work is as follows:

- Deterministic construction of general size binary sensing matrices that enjoy RIP compliance
- Analysis of optimal bounds on the column sizes of binary sensing matrices
- Application of our constructions to image reconstruction and retrieval problems
- Analysis and retrieval of large medical databases via a new data-driven method

Part II: Data-independent dictionaries: Theory and applications

Advantage with deterministic constructions

- For random matrices, there is nonzero probability for noncompliance with RIP. This is not case with deterministic matrices

Advantages with deterministic binary constructions

- Binary matrices being sparse and possessing $0, 1$ as elements provide multiplier-less and faster dimensionality reduction operation, which is not possible with their dense counterparts
- These matrices have smaller density than Gaussian matrices. Here, by density, one refers to the ratio of number of nonzero entries to the total number of entries of the matrix

Existing deterministic constructions

- R. Devore[a] has constructed **deterministic binary sensing matrices** of size $p^2 \times p^{r+1}$, where p is a prime power
- Later on, S. Li, F. Gao et. al.[b] have generalized the Devore's work, constructing binary sensing matrices of size $|\mathcal{P}|q \times q^{\mathcal{L}(G)}$, where q is any prime power and $\mathcal{P}$ is the set of all rational points on algebraic curve $\mathcal{X}$ over finite field $\mathbb{F}_q$
- P. Indyk[c] has constructed binary sensing matrices using Hash functions and extractor graphs with sizes $r2^{O(\log\log n)^{O(1)}} \times n$, where $r \ll n$

[a] Ronald A. DeVore, "Deterministic constructions of compressed sensing matrices," Journal of Complexity, 2007.

[b] S. Li et.al., "Deterministic construction of compressed sensing matrices via algebraic curves," IEEE Trans. Inf. Theory, 2012.

[c] P. Indyk, "Explicit constructions for compressed sensing matrices," in Proc. 19th Annu. ACM-SIAM, SODA 2008.

Data-independent dictionaries: Theory and applications

- A. Amini et. al.[a] have constructed binary, bipolar and ternary sensing matrices using OOC codes
- G.Xu et. al.[b] have constructed CS matrices using Fourier matrices
- J. Bourgain et. al.[c] have constructed RIP matrices of order $k \geq m^{\frac{1}{2}+\epsilon}$, for some $\epsilon > 0$ and $M^{1-\epsilon} \leq m \leq M$ using additive combinatorics
- It is remarkable that this construction overcomes the natural barrier $k = O(m^{\frac{1}{2}})$ for those based on coherence
- R. Calderbank et.al. have constructed CS matrices of size $2^m \times 2^{(r+1)m}$ using Delsarte-Geothals codes.

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[a] A. Amini et.al., "Deterministic construction of binary, bipolar and binary compressed sensing matrices," IEEE Trans. Inf. Theory, 2011.

[b] G. Xu et. al., Compressed sensing matrices from Fourier matrices, IEEE Transactions on Information Theory, 2015.

[c] J. Bourgain et. al., "Explicit constructions of RIP matrices and related problems," Duke Math. J., 2011.

Motivation

- All existing binary and ternary constructions are given for specific sets of row sizes
- Other constructions provide dense CS matrices

Objective

- To construct **general size and sparse binary sensing matrices** which are useful for fast processing

- The sparse CS matrix may contribute to fast processing with low computational complexity in Compressed Sensing[a]

[a] A. Gilbert et. al., "Sparse recovery using sparse matrices," Proceedings of IEEE, 2010.

A. Construction of binary sensing matrices through multi-variable polynomials

- In the present work, we extend the work in [4] and construct binary sensing matrices based on **multi-variable homogeneous polynomials**
- The advantage of using such polynomials is that a different class of CS matrices can be constructed
- The size of the matrix obtained in [4] is $p^2 \times p^{(r+1)}$ and of our matrix is $p^3 \times p^{(r+1)} - 1$

Theorem

The matrix $\Phi_0 = \frac{1}{p}\Phi$ *satisfies the RIP with* $\delta_k = \frac{k-1(r(p-1)+1)}{p^2}$ *for any* $k < \frac{p^2}{r(p-1)+1} + 1$.

- The field that is considered in the above construction is $\mathbb{Z}_p$
- If we consider any finite field $\mathbb{F}_q, q = p^i, i \in \mathbb{Z}^+$ in place of $\mathbb{Z}_p$, then sizes of Devore's and our matrices become $p^{2i} \times p^{i(r+1)}, p^{3i} \times p^{i(r+1)} - 1$ respectively
- Further, we also extend our construction to deal with circulant matrices

Theorem

The circulant matrix $\Phi_1 = \frac{1}{p}\Phi_0$ *has the RIP with* $\delta_k = 2^4(k-1)\frac{r}{p}$ *whenever* $k - 1 < \frac{p}{2^4 r}$

- These constructions can further be generalized through n variable polynomials

B. On the role of extremal set theory and set systems in CS

- Now we attempt to relate the notions of **extremal set theory**[a] **and set systems to CS**
- In particular, in the first result using the ideas from extremal set theory, we bound the column size of binary sensing matrix
- The maximum possible column size of a binary sensing matrix is at most $\frac{\binom{m}{r}}{\binom{k}{r}}$, where m is the row size, k is the number of ones each column contains and $r-1$ is the cardinality of overlap between any two columns
- We also prove the existence of binary sensing matrices having optimal column size asymptotically for given r and k

[a] Vojtech Rodl, "On a packing and covering problem," European journal of combinatorics, 5, 69-78, 1985.

- In the second result, using the ideas from set systems we generate sparse binary sensing matrices from existing binary sensing matrices

Theorem

Suppose $f(x_1, x_2, \ldots, x_m) = x_1 + x_2 + \ldots + x_m + \sum_{i<j} x_i x_j$ is a symmetric polynomial. Let Φ be a binary sensing matrix of size $m \times M$ such that $\frac{m(m+1)}{2} < M$ with the coherence being at most $\frac{r}{k}$. Here k represents the number of nonzero elements that each column of Φ has. Then there exists a binary sensing matrix Ψ of size $\frac{m(m+1)}{2} \times M$ whose coherence is at most $\frac{r+\binom{r}{2}}{k+\binom{k}{2}}$.

- The following theorem concludes the RIP compliance of $\Psi_0 = \frac{1}{\sqrt{k+\binom{k}{2}}}\Psi$

Theorem

The matrix $\Psi_0 = \frac{1}{\sqrt{k+\binom{k}{2}}}\Psi$ *has the RIP with* $\delta = (k-1)\left(\frac{r+\binom{r}{2}}{k+\binom{k}{2}}\right)$ *whenever* $k-1 < \frac{k+\binom{k}{2}}{r+\binom{r}{2}}$.

Remark: The density of the new matrix Ψ is $\frac{k+\binom{k}{2}}{m+\binom{m}{2}}$, which is smaller than $\frac{k}{m}$, the density of Φ.

C. Construction of sparse CS matrices from existing ones

1. **Input:** Two matrices Ψ, Ψ' of sizes $nk'' \times M$, $n'k' \times M'$
2. Suppose $|\text{supp}(\Psi_i) \cap \text{supp}(\Psi_j)| \leq r$ and $|\text{supp}(\Psi'_i) \cap \text{supp}(\Psi'_j)| \leq r'$, $\forall\ i \neq j$
3. Assume $n' > n, k \leq \min\{k', k''\}$ and $r \leq r' \leq k \leq n$
4. Set $S_i = (((\text{supp}(\Psi_i) - 1)(\mod n))^{\mathbf{T}} + \mathbf{1}^{\mathbf{T}})$, $|S_i| = k''$, $\forall\ i = 1, 2, \ldots, M$
5. Similarly, S'_j with $|S'_j| = k'$ are defined for $j = 1, 2, \ldots, M'$
6. Set $S''_{i,j} = S'_{j,k} - \mathbf{1} + n' S_{i,k}$, $S'_{j,k}$ and $S_{i,k}$ are first k entries of S'_j and S_i
7. $|S''_{i,j}| = k$ and Let $S''_{i,j} = (S''_{i,j,1}, S''_{i,j,2}, \ldots, S''_{i,j,k})$ $1 \leq S''_{i,j,l} \leq nn', \forall l = 1, 2, ..., k$
8. From each $S''_{i,j}$, create a vector $v_{i,j}$ of length $nn'k$ and $v_{i,j} = 1$ at $(l-1).nn' + S''_{i,j,l}$ for $l = 1, 2, \ldots, k$ and zero elsewheres
9. **Output:** Φ', a matrix of size $nn'k \times MM'$, whose columns are $v_{i,j}$.

D. Construction via Euler Squares and applications

- Using Euler squares, we construct general size binary CS matrices

Definition

An Euler Square of order n, degree k and index n, k is a square array of $n^2, k-$ads, $(a_{ij1}, a_{ij2}, \ldots, a_{ijk})$, where $a_{ijr} = 0, 1, 2, \ldots, n-1;\; r = 1, 2, \ldots, k;$ with $i, j = 1, 2, \ldots, n$ and $n > k$; $a_{ipr} \neq a_{iqr}$ and $a_{pjr} \neq a_{qjr}$ for $p \neq q$ and $(a_{ijr}+1)(a_{ijs}+1) \neq (a_{pqr}+1)(a_{pqs}+1)$ for $i \neq p$ and $j \neq q$.

Harris F. MacNeish [a] has constructed Euler Squares for the following cases:

[a] H. F. MacNeish, "Euler squares," Ann. Math., 1922.

- Index $p, p-1$, where p is a prime number, more generally Index $p^r, p^r - 1$, for a prime p
- Index n, k, where $n = 2^r p_1^{r_1} p_2^{r_2} \ldots, p_l^{r_l}$ for distinct odd primes $p_1, p_2, \ldots, p_l$ and $k = \min\{2^r, p_1^{r_1}, p_2^{r_2}, \ldots, p_l^{r_l}\} - 1$
- Using the Euler square of index n, k, we define the elements of a binary sensing matrix Φ as: For $1 \leq i \leq nk, 1 \leq j \leq n^2$,

$$\phi_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } (a_j)_{\lfloor \frac{i-1}{n} \rfloor + 1} \equiv i - 1 (\text{mod } n) \\ 0 & \text{otherwise,} \end{array} \right\}, \quad (6)$$

  where (a_j) is the j^{th} $k-$ad, $(a_j)_l$ is l^{th} element in j^{th} $k-$ad and $\lfloor x \rfloor$ is the largest integer not greater than x
- The following lemma finds a bound on mutual coherence of Φ

Lemma

The coherence of Φ is atmost equal to $\frac{1}{k}$.

Using the above Lemma and Proposition (5), we conclude the RIP compliance of Φ.

Theorem

The matrix $\Phi_0 = \frac{1}{\sqrt{k}}\Phi$ satisfies RIP with $\delta_{k'} = \frac{k'-1}{k}$ for any $k' < k+1$.

- Euler Square of index n, k gives matrix of size $nk \times n^2$
- In this matrix, each column contains k number of ones and coherence is atmost $\frac{1}{k}$
- The maximum possible column size is thus $\frac{\binom{nk}{2}}{\binom{k}{2}} = O(n^2) = O((m\mu)^2)$

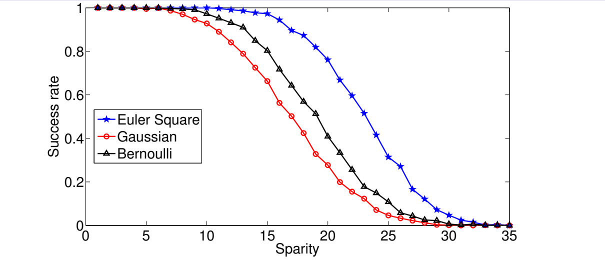
- The density is $\frac{1}{n}$, which is very small for large n

The **following theorem summarizes the main result:**

Theorem

Suppose m is any positive integer different from p, p^2 for a prime p. Then there exists a CS matrix whose row size is m and the column size is $(m\mu)^2$, where μ is a coherence parameter.

Comparison of the numerical performance:



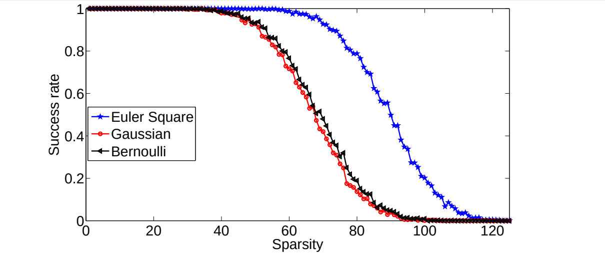


Figure : Comparison of the reconstruction performances of Euler Square based, Bernoulli random and the Gaussian random matrices when the matrices are of size (a) 55×121 (top plot) and (b) 230×529 (bottom plot). These plots indicate that the Euler Square based matrix shows superior performance for some sparsity levels, while for other levels all matrices result in the same performance. The x and y axes in both plots refer respectively to the sparsity level and the success rate (in % terms).

- The present construction is simple in the sense that it does not involve function evaluations like in [4] and gives matrices with small density
- To generate an Euler square matrix of size $p, p-1$, it is only required to store two cyclic permutations of length p and $p-1$ respectively
- For index p^i, p^{i-1}, it is sufficient to store at most $\frac{p^i}{2}$ permutations
- To the best of our knowledge, the constructions possessing sparsity $k' = \sqrt{m}$ (that is, coherence $\mu = \frac{1}{\sqrt{m}}$) exist for non-binary matrices with row size m being prime or prime power
- We construct the binary matrices that provide guarantees for signals of sparsity up to $k' = \lfloor \sqrt{m} \rfloor$ for different class of row sizes such as $p^i(p^i - j)$, where p is prime $i \geq 1$, and $j = 1, 2$

Data independent dictionaries: Theory and Applications

- For an arbitrary binary matrix, if the inner-product between any two columns is at most 1, every column contains fixed ($\sqrt{m}$) number of ones (that is coherence is at most $\frac{1}{\sqrt{m}}$) and row size is m, then the maximum possible column size $M = O(m)$ as mentioned earlier
- Using Euler squares, we have constructed matrices that provide guarantees for signals of sparsity up to $k' = \lfloor \sqrt{m} \rfloor$ with $M = O(m)$
- Through an extension of our construction methodology, it is possible to generate ternary matrices, for which $k' = \sqrt{m}$ holds (that is, coherence is in the order of $\sqrt{\frac{M-m}{m(M-1)}}$) with column size $M = O(m^{\frac{3}{2}})$

Image reconstruction via Euler Squares

- It is observed that the CS matrices constructed from the Euler Squares of indices p, s for a prime p and $s = \lfloor \frac{p}{2} \rfloor$ to $p-1$, give relatively superior performance than their Gaussian counterparts
- We demonstrate the efficacy of Euler Square based matrix using image reconstruction from lower dimensional patches, where the patches are generated via the sensing matrices
- The reconstructions shown in Figure below correspond to different down-sampling factors, viz 2.6 and 1.6
- Here by down-sampling factor, we mean the ratio of original patch size to reduced patch size which is same as $\frac{M}{m}$ (where, $m \times M$ is the size of the matrix used for projecting data to the lower dimensional space)

Image reconstruction via Euler Squares

Figure : Orginal image

(b) (c) (d) (e)

Figure : For the original image, the images in (b) and (d) are those reconstructed via the Euler Square based matrices with down-sampling factors 2.6 and 1.6 respectively. The images in (c) and (e) are those obtained via the corresponding Gaussian matrices. This figure states that Euler Square based CS matrices provide competitive reconstruction performance.

The associated reconstruction errors in terms of SNR are:

Down-sampling factor ($\frac{M}{m}$)	Euler recovery SNR error	Gaussian recovery SNR error (Average error)
4	13.36	13.44
2.6	16.44	15.03
2	19.63	18.14
1.6	20.61	19.74

Table : A comparative error analysis of reconstruction by Euler based and Gaussian matrices for different down-sampling factors ($\frac{M}{m}$) 4, 2.6, 2, 1.6. **The average error over 1000 iterations is reported for Gaussian matrices.**

- Signal-to-noise ratio (SNR) of x is computed using

$$SNR(x) = 10.\log_{10}\left(\frac{\|x\|_2}{\|x-\tilde{x}\|_2}\right) dB.$$

Application to CBIR via Euler square matrices

- In the recent past, Gaussian matrices have been used to project data into lower dimension for classification[a]
- The problem[b] of searching for similar images in a large image repository based on content is called Content Based Image Retrieval (CBIR)
- It is demonstrated[c] that the proposed binary sensing matrices project data into lower dimensional spaces in such a way that the reduced vectors are useful for the purpose of CBIR

[a] C. M. Fira et.al., "Ecg compressed sensing based on classification in compressed space and specified dictionaries," in Proc. EUSIPCO, 2011.

[b] Y.C. Chen et.al, In-plane rotation and scale invariant clustering using dictionaries, Image Processing on IEEE Transactions, 2013.

[c] Simulation results are not included here

- Additionally, the proposed dimensionality reduction technique through binary sensing matrices allows for reconstruction, which is very important in fields like tele-medicine
- The other dimensionality reduction techniques and even the data-driven Dictionary based methods in general do not provide this advantage

Part III: Conclusions and Future Work

Conclusions and Future Work

- So far the objectives behind my work have centered around constructing binary and ternary sensing matrices
- I am now interested in constructing more general matrices, through non-coherence arguments and Majorization and minimization methods

Definition

The smallest number of columns that are linearly dependent is called as a spark of the matrix.

- If $spark(\Phi) > 2k$ then every $k-$sparse signal x can exactly recovered by l_0- minimization
- It is necessary and sufficient condition for l_0- minimization

- Donoho et.al., have given the lower bound for spark through coherence. i.e, $Spark(\Phi) \geq 1 + \frac{1}{\mu(\Phi)}$
- Later Shu-Tao Xia et.al., have improved this bound for binary sensing matrices in the following way:

Theorem

Let Φ be a binary matrix with minimum column weight $k > 0$ and maximum inner-product of any two different columns of Φ is $r > 0$. Then $spark(\Phi) \geq \frac{2k}{r}$.

- Also, they have given a binary construction which further maximizes the above mentioned bound
- J. Bourgain et.al., have proved the following lemma

Lemma

Let $k \geq 2^{10}$ and s be a positive integer. Assume that the coherence parameter of the matrix Φ is $\mu \leq \frac{1}{k}$. Also, assume that for some $\delta \geq 0$ and any disjoint $J_1, J_2 \subset \{1, 2, \ldots, M\}$ with $|J_1| \leq k$, $|J_2| \leq k$ we have $|\langle \sum_{j \in J_1} u_j, \sum_{j \in J_2} u_j \rangle| \leq \delta k$. Then Φ satisfies the RIP of order $2sk$ with constant $44s\sqrt{\delta} \log k$.

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Thank you