Zero-crossings Based Nonparametric Goodness-of-Fit Tests for Spectrum Sensing in Cognitive Radios under Noise Uncertainty

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Goodness-of-Fit Tests (GoFT)

- In the classical Neyman-Pearson hypothesis testing, both hypotheses are known
- In GoFT, distribution of the test statistic is known under H_0 and assumed to be unknown under H_1
- Examples : Tests for independence, tests for deviation from a specific distribution under H_0

GoFT for Spectrum Sensing (SS) in CR

The choice of a GoFT depends on:

- Statistics of noise
- Knowledge of noise variance
- Number of observations
- Signal characteristics of the Primary

Existing GoFTs in SS context

- Wang et al. [2009] - a GoFT based on the Anderson-Darling Statistic (ADD)
- Shen et al. [2011] - a modified GoFT based on ADD and called it the Blind Detector (BD)
- Denkovski et al. [2012] - a higher order statistic based GoFT
- Rostami et al. [2012] - an ordered statistic based GoFT

This work...

- We propose a new GoFT based on a modification of an existing test using zero-crossings (Kedem et al. 1982)
- Advantages of our detector:
  - Can be applied under Gaussian and Laplacian noise distributions
  - Robust to noise uncertainty
  - Closed form for the optimal threshold

System Model

- A single CR node carrying out SS with M observations.
- Under Gaussian noise, The hypothesis testing problem is

$$\begin{aligned} H_0 &: Y_i \sim \mathcal{N}(0, \sigma_n^2) \\ H_1 &: Y_i \nsim \mathcal{N}(0, \sigma_n^2) \end{aligned}$$

- Under Laplacian noise, The hypothesis testing problem is

$$\begin{aligned} H_0 &: Y_i \sim \mathcal{L}(\sigma_n^2) \\ H_1 &: Y_i \nsim \mathcal{L}(\sigma_n^2) \end{aligned}$$

Energy Detector (ED)

- When σ_n^2 is known, the ED has the following critical region

$$\left\{ Y_i, i \in \mathcal{M} : \sum_{i=1}^{M} Y_i^2 > \tau_{\text{ED}} \right\}, \tag{1}$$

- When $n_i \sim \mathcal{N}(0, \sigma_n^2)$,

$$\tau_{\text{ED}} = \gamma^{-1}\left(1-\alpha, \frac{M-1}{2}, \frac{2\sigma_n^2}{M}\right), \tag{2}$$

where $\gamma^{-1}(x, A, B)$ is the normalized inverse gamma CDF evaluated at x, with parameters A and B.

Anderson Darling Detector (ADD) - 1/2

- The Anderson-Darling statistic is defined as

$$A_c^2 \triangleq -\frac{\sum_{i=1}^{M}(2i-1)(\ln Z_i + \ln(1-Z_{M+1-i}))}{M} - M \qquad (3)$$

  with $Z_i = F_0(Y_i)$, where $F_0(\cdot)$ is the distribution under $\mathcal{H}_0$. Also, $Y_1 \leq Y_2 \leq \cdots \leq Y_M$.
- The ADD has the following critical region

$$\left\{ Y_i, i \in \mathcal{M} : A_c^2 \geq \tau_{\text{ADD}} \right\}, \qquad (4)$$

Anderson Darling Detector (ADD) - 2/2

- For any $p_f = \alpha$ and for moderate values of M, τ_{ADD} satisfies

$$1 - \frac{\sqrt{(2\pi)}}{\tau_{\text{ADD}}} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma(0.5+\ell)}{\Gamma(0.5)\ell!} \exp\left(-\frac{\pi^2(4\ell+1)^2}{8\tau_{\text{ADD}}}\right) \times (4\ell+1)\int_0^{\infty} \exp\left(\frac{\tau_{\text{ADD}}}{8(w^2+1)} - \frac{\pi^2 w^2 (4\ell+1)^2}{8\tau_{\text{ADD}}}\right) dw = \alpha. \quad (5)$$

Blind Detector (BD) - 1/2

- When the noise process is i.i.d. Gaussian, the construction of the BD is such that the test statistic is independent of σ_n^2.
- M observations are divided into n windows of m observations each and the test statistic is constructed as follows. Define

$$X_l \triangleq \sum_{u=0}^{m-1} \frac{Y_{ml-u}}{m}, \quad S_l^2 \triangleq \sum_{u=0}^{m-1} \frac{(Y_{ml-u} - X_l)^2}{m-1}, \tag{6}$$

$$\text{and } B_l \triangleq \frac{X_l}{S_l/\sqrt{m}}, l = 1, \cdots, m. \tag{7}$$

Then, the BD has the following critical region

$$\{Y_i, i \in \mathcal{M} : B_l \geq \tau_{\text{BD}}\}. \tag{8}$$

Blind Detector (BD) - 2/2

- It is known that when $n_i \sim \mathcal{N}(0, \sigma_n^2)$, the statistic B_l is student-t distributed with parameter $m-1$. Therefore, for a given p_f level α, τ_{BD} satisfies

$$\frac{1}{2} - \frac{\tau_{\mathrm{BD}}\,\Gamma\left(\frac{m}{2}\right)\,{}_2\mathcal{F}_1\left(\frac{1}{2}, \frac{m}{2}; \frac{3}{2}; -\frac{\tau_{\mathrm{BD}}^2}{m-1}\right)}{\sqrt{\pi(m-1)}\,\Gamma\left(\frac{m-1}{2}\right)} = \alpha, \qquad (9)$$

with ${}_2\mathcal{F}_1(\cdot;\cdot;\cdot)$ representing the Kummer's hypergeometric function.

Disadvantages of ED, ADD and BD

- Both ED and ADD requires the knowledge of σ_n^2
- Additionally, calculation of τ_{ADD} needs evaluation of an integral over an infinite series
- The analysis of BD fails in non-gaussian noise. Extending the same idea to other distributions is not easy.
- Both ADD and BD are limited to small sample sizes.

Basics of Zero-crossings (ZC) (1/2)

- Let ∇^k denote the k^{th} order difference operator.

$$\begin{aligned} \nabla Y_i &\triangleq Y_i - Y_{i-1} && (10) \\ \nabla^2 Y_i &= \nabla(\nabla Y_i) = Y_i - 2Y_{i-1} + Y_{i-2} && (11) \\ &\vdots \\ \nabla^k Y_i &= \sum_{j=0}^{k} \binom{k}{j} (-1)^j Y_{i-j}, \quad i \in \mathcal{M} && (12) \end{aligned}$$

The k^{th} order ZC of $\{Y_i, i \in \mathcal{M}\}$ is defined as the number of ZCs in $\nabla^{k-1} Y_i$. Also called the *Higher Order Crossings* (HOC), they are denoted by $D_{k,M}$.

Basics of Zero-crossings (ZC) (2/2)

- Let $\Delta_{j,M}$, and $\mu_{j,M}$ be defined as

$$\Delta_{j,M} \triangleq \begin{cases} D_{1,M}, & j = 1, \\ D_{j,M} - D_{j-1,M}, & j = 2, \cdots, k-1 \\ (M-1) - D_{k-1,M}, & j = M, \end{cases} \tag{13}$$

$$\text{and } \mu_{j,M} \triangleq \mathbb{E}\Delta_{j,M}, j = 1, \cdots, k, \tag{14}$$

  where $\mathbb{E}(\cdot)$ denotes the expectation operator. Observe that $\sum_{j=1}^{k} \Delta_{j,M} = M - 1$.

- Under Gaussian noise, it can be shown that

$$\mathbb{E}D_{k,M} = (M-1)\left\{\frac{1}{2} + \frac{1}{\pi}\sin^{-1}\left(\frac{k-1}{k}\right)\right\}, \tag{15}$$

The Ψ^2 statistic and the ΨSD

- The goodness-of-fit measure Ψ^2_M upto a given order k is defined as

$$\Psi^2_M \triangleq \sum_{j=1}^{k} \frac{(\Delta_{j,M} - \mu_{j,M})^2}{\mu_{j,M}}. \tag{16}$$

- Under Gaussian noise and for moderately large M, $\Psi^2_M \sim \chi^2_3(11)$, very closely
- The ΨSD has a critical region

$$\left\{ Y_i, i \in \mathcal{M} : \Psi^2_M > \tau_{\Psi\mathrm{SD}} \right\}, \tag{17}$$

where for a given p_f level α, $\tau_{\Psi\mathrm{SD}}$ satisfies

$$Q_{\frac{3}{2}}(\sqrt{11}, \sqrt{\tau_{\Psi\mathrm{SD}}}) = \alpha, \tag{18}$$

with $Q_M(\cdot,\cdot)$ representing the Marcum-Q function of order M.

The mΨ^2 statistic and the mΨSD (1/2)

- It was observed through simulations that under some cases, only first few ZCs were different under H_0 and H_1
- The modified Ψ^2 statistic is defined as

$$
{}_{\mathrm{m}}\Psi_M^2 \triangleq \sum_{j=1}^{k} e^{-(j-1)} \frac{(\Delta_{j,M} - \mu_{j,M})^2}{\mu_{j,M}} \tag{19}
$$

- The mΨSD has a critical region

$$
\left\{ Y_i, i \in \mathcal{M} : {}_{\mathrm{m}}\Psi_M^2 > \tau_{\mathrm{m}\Psi\mathrm{SD}} \right\}, \tag{20}
$$

The mψ^2 statistic and the mψSD (2/2)

- Under previously stated conditions, we have observed that ${}_{\mathrm{m}}\Psi^2_M$ statistic follows an F-distribution with parameters 17.5 and 7 respectively. Therefore, $\tau_{\mathrm{m\Psi SD}}$ satisfies

$$1 - \mathcal{I}_{\left(\frac{17.5\tau_{\mathrm{m\Psi SD}}}{17.5\tau_{\mathrm{m\Psi SD}} + 7}\right)}(8.75, 3.5) = \alpha, \qquad (21)$$

with $\mathcal{I}_x(a, b)$ representing the regularized incomplete beta function with parameters x, a and b respectively.

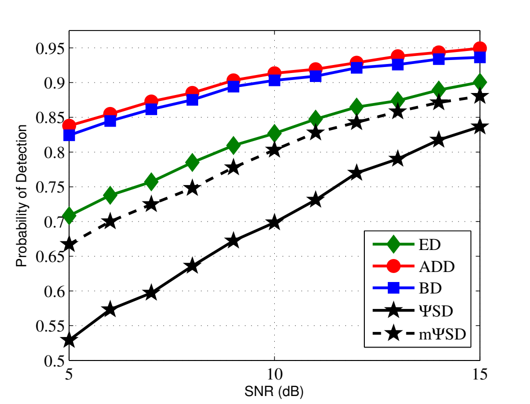

Figure: Detection performance with SNR for known signal case, under Rayleigh fading and Gaussian noise. $M = 32, \alpha = 0.05$.

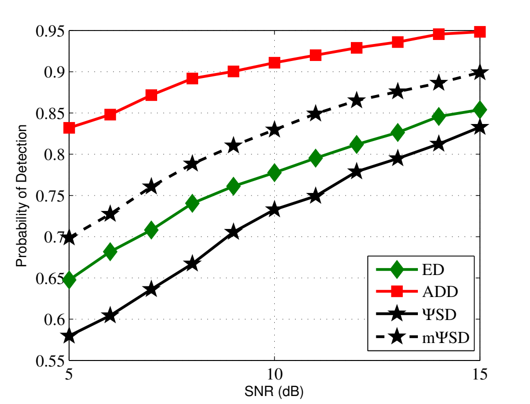

Figure: Detection performance with SNR for known signal case, under Rayleigh fading and Laplacian noise. $M = 32, \alpha = 0.05$.

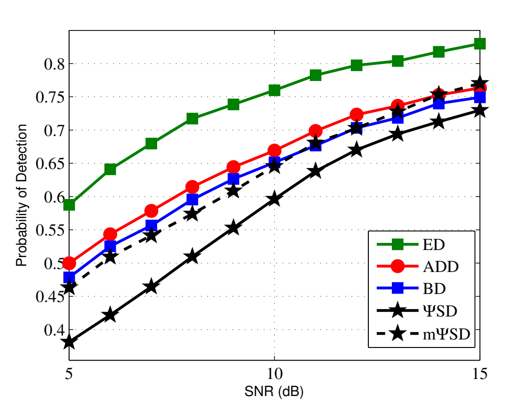

Figure: Detection of 4kHz sinusoidal signal, under Rayleigh fading and Gaussian noise. $M = 32, \alpha = 0.05$.

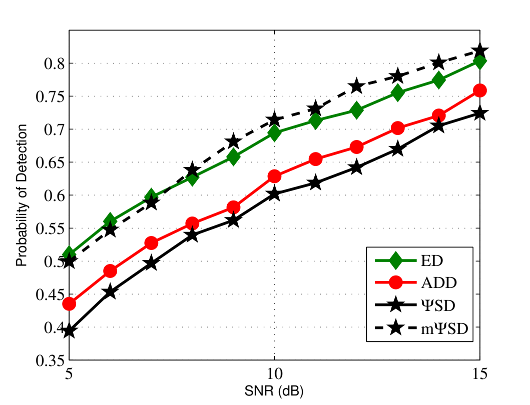

Figure: Detection of 4kHz sinusoidal signal, under Rayleigh fading and Laplacian noise. $M = 32, \alpha = 0.05$.

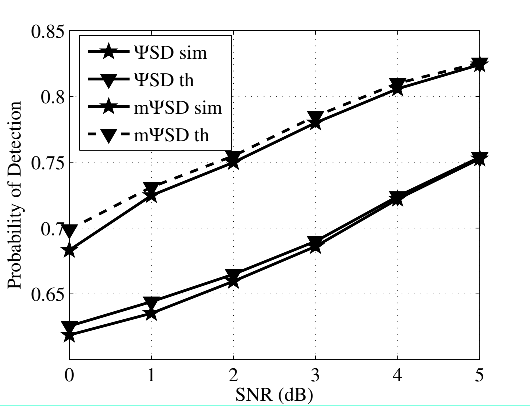

Figure: Comparision of theoretical and simulated p_d values. $M = 300, \alpha = 0.05$.

Conclusions

- Proposed a modified GoFT based on ZCs, which is robust under noise uncertainty
- It can be readily used under Gaussian and Laplacian noise environments