(1) The Cramér transform

The aim of this exercise is to show that the supremum in the definition of the Cramér transform can be taken over all real numbers, instead of just over the nonnegative reals.

Consider a random variable X with finite mean. Show that for  $t \geq \mathbb{E}[X]$ ,

$$\sup_{\lambda \ge 0} \left( \lambda t - \log \mathbb{E} \left[ e^{\lambda X} \right] \right) = \sup_{\lambda \in \mathbb{R}} \left( \lambda t - \log \mathbb{E} \left[ e^{\lambda X} \right] \right).$$

(An analogous statement holds when  $t \leq \mathbb{E}[X]$  as well.)

*Hint:* What is the sign of the expression being optimized when  $\lambda < 0$ ?

- (2) Chernoff bounds for well-known distributions
  - (a) Compute the Cramér transform for the Bernoulli(p) distribution, *i.e.*, find an explicit form for the function  $\Lambda^* : [0, \infty) \to \mathbb{R}$ , defined as

$$\Lambda^*(t) = \sup_{\lambda} \left( \lambda t - \log \mathbb{E} \left[ e^{\lambda X} \right] \right),$$

where X is a Bernoulli(p) random variable, 0 .

- (b) Let Z be a Binomial(n, p) random variable, and t > 0 a positive real. Either using part (a) or otherwise, write down the optimum Chernoff bound for the tail probability  $\mathbb{P}[Z > \mathbb{E}[Z] + t]$  for t > 0.
- (c) Compute a useful lower bound for the exponential random variable with mean  $\theta$ , *i.e.*, X with pdf  $f_X(x) = \theta e^{-\theta x}$  for x > 0.
- (3) Moment bounds vs. Chernoff bounds Let X be a nonnegative random variable, and t > 0 be a positive real.
  - (a) Show that for any positive integer q, an upper bound on the tail probability  $\mathbb{P}[X > t]$  is  $\mathbb{E}[X^q]t^{-q}$ . (This is called a moment bound.)
  - (b) Consider the best possible moment bound on the tail probability,  $\min_q \mathbb{E}[X^q] t^{-q}$ , where the minimum is taken over all positive integers. Show that this is at least as good as the (best) Chernoff bound, i.e., show that

$$\min_{q} \mathbb{E} \left[ X^{q} \right] t^{-q} \leq \inf_{\lambda > 0} e^{-\lambda t} \mathbb{E} \left[ e^{\lambda X} \right].$$

Hint: Consider the Taylor series expansion of the right hand side.

(4) Medians & means

Let  $\mathbb{M}Z$  be a median of the square-integrable random variable Z (i.e.,  $\mathbb{P}[Z \ge \mathbb{M}Z] \ge 1/2$  and  $\mathbb{P}[Z \le \mathbb{M}Z] \ge 1/2$ ). Show that

$$|\mathbb{M}Z - \mathbb{E}[Z]| \le \sqrt{\operatorname{Var}(Z)}.$$

(5) Elementary inequalities

Prove the following basic inequalities used in class:

- (a)  $h(x) := (1+x)\log(1+x) x \ge \frac{x^2}{2+2x/3}$  for  $x \ge 0$ . (This is used to derive tail bounds for Poisson rvs and Bernstein's inequality.)
- (b)  $h_1(x) := -\log(1+x) + x \ge (\sqrt{1+x}-1)^2$  for  $x \ge 0$ . (This is used to derive tail bounds for Chi-squared rv.)
- (c) For any positive integer  $m, m! < e\sqrt{m}(m/e)^m$ .
- (6) Balls and bins

Suppose n balls are dropped independently and uniformly into n bins. Let Z denote the maximum number of balls in a bin. Find an upper bound for Z which holds with large probability.