Lecture 19

Isoperimetry and Concentration

A Classical Isoperimetric Theorem

"Among all shapes of the same volume in the Euclidean space, the Euclidean ball has the maximum surface area."

A key tool we will use is the Brunn-Minkowski inequality.

Consider two nonempty, compact sets $A, B \subseteq \mathbb{R}^n$.

Denote by $A+B$ the Minkowski sum $\{x+y \mid x \in A, y \in B\}$.

(Brunn-Minkowski)

$$\boxed{\mathrm{Vol}(A+B)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n}}$$

Equivalent forms:

(1) $\mathrm{Vol}(\lambda A + \bar{\lambda} B)^{1/n} \geq \lambda\, \mathrm{Vol}(A)^{1/n} + \bar{\lambda}\, \mathrm{vol}(B)^{1/n}$ $\quad \lambda \in [0,1]$

(2) $\mathrm{Vol}(A_1 + A_2) \geq \mathrm{vol}(B_1 + B_2)$.

where B_1 and B_2 are Euclidean balls around origin with $\mathrm{vol}(B_i) = \mathrm{vol}(A_i)$, $i = 1, 2$.

(This is true since BM holds with equality for Euclidean balls)

Blow-up and Surface area

ϵ-blowup of a set S

$$S_\epsilon = \{x : d(x,S) \le \epsilon\}$$

Let B denote the unit ball in the Euclidean space.

Then, $\boxed{S_\epsilon = S + \epsilon B}$

Surface area of a set

$$\mathrm{vol}(\partial S) = \lim_{\epsilon \to 0} \frac{\mathrm{vol}(S_\epsilon) - \mathrm{vol}(S)}{\epsilon}$$

$*$ $\boxed{\mathrm{vol}(rB) = c(n) r^n; \quad \mathrm{vol}(\partial B) = n\, c(n)}$

Theorem Given a set $A \subseteq \mathbb{R}^n$, let $r \stackrel{\text{def}}{=} [\mathrm{vol}(A)/c(n)]^{1/n}$.

Then, $\mathrm{vol}(\partial A) \ge n\, c(n) r^{n-1} = \mathrm{vol}(\partial B^r)$

$\hookrightarrow$ ball of radius r

Proof. By Brunn-Minkowski,

$$\begin{aligned}
\mathrm{vol}(A_\epsilon)^{1/n} &= \mathrm{vol}(A + \epsilon B)^{1/n} \\
&\ge \mathrm{vol}(A)^{1/n} + \epsilon\, \mathrm{vol}(B)^{1/n} \\
&= c(n)^{1/n} (r + \epsilon)
\end{aligned}$$

$$\Rightarrow \mathrm{vol}(\partial A) \ge c(n) \lim_{\epsilon \to 0} \frac{(r+\epsilon)^n - r^n}{\epsilon} = n\, c(n) r^{n-1}. \qquad \blacksquare$$

$\boxed{B}$ Connection b/w isoperimetry and concentration (Levy's ineq.)

Isoperimetric inequality roughly captures "how much mass is accumulated near the boundary of a set."

In this sense, the following quantity captures (3)

isoperimetric inequalities: $\boxed{\alpha(t) = \sup_{A \subseteq \mathcal{X} : P(A) \geq 1/2} P(A_t^c)}$

The following results relate concentration and isometry.

Theorem (Lévy's inequality) For any Lipchitz function f,

$$P(f(X) \geq \mathbb{M}f(X) + t) \leq \alpha(t),$$

$$P(f(X) \leq \mathbb{M}f(X) - t) \leq \alpha(t).$$

Proof. Let $A = \{x : f(x) \leq \mathbb{M}f(X)\} \Rightarrow P(A) \geq 1/2$.

Then, $A_t = \{y : \exists x \in A \text{ s.t. } d(x,y) \leq t\}$

$$\subseteq \{y : f(y) \leq \mathbb{M}f(X) + t\}$$

$$\Rightarrow \mathbb{P}(f(X) > \mathbb{M}f(X) + t) \leq \mathbb{P}(A_t^c) \leq \alpha(t).$$ ∎

Theorem (Converse)

Suppose that $\beta(t)$ is s.t. $\forall$ Lipchitz f,

$$\mathbb{P}(f(X) \geq \mathbb{M}f(X) + t) \leq \beta(t).$$

Then, $\beta(t) \geq \alpha(t)$.

Proof. Let $f_A(x) = d(x, A)$. f_A is Lipchitz and $\mathbb{M}f_A = 0$ if $P(A) \geq 1/2$. Thus,

$$P(A_t^c) = \mathbb{P}(f_A(X) \geq t) \leq \beta(t) \Rightarrow \alpha(t) \leq \beta(t).$$ ∎

Thus, we can get good bounds for concentration around the median by deriving bounds for $\alpha(t)$.

Remark:

An exact isoperimetry theorem will have the form:

$\exists\, B \subseteq \mathcal{X}$ with $P(B) \geq 1/2$ s.t. $\forall\, A$ s.t. $P(A) \geq 1/2$,

$$P(A_t) \geq P(B_t).$$

(then, $\alpha(t) \leq P(B_t^c)$).

In some examples, such a set B can be found.

Ex 1. Isoperimetry for (S^{n-1}, d = Geodesic, uniform)

Levy's Isoperimetry Theorem

For $B = \{x \in S^{n-1} : x_1 \geq 0\}$, $P(B) = \frac{1}{2}$ and $\forall\, A$ s.t. $P(A) \geq \frac{1}{2}$, $P(A_\epsilon) \geq P(B_\epsilon)$.

hemi-sphere

$\Rightarrow \alpha(t) \leq P(B_t^c)$

It can be shown that: $P(B_t^c) \leq e^{-(n-1)t^2/2}$

(most of the mass is near the equator)

Ex 2 Gaussian Isoperimetric Theorem

($\mathbb{R}^n$, $\|\cdot\|_2$, Standard normal measure)

$B = \{x \in \mathbb{R}^n : x_1 \geq 0\} \equiv$ half-plane

$P(B) = 1/2$. $P(A) \geq 1/2 \Rightarrow P(A_\epsilon) \geq P(B_\epsilon)$

$\Rightarrow \alpha(t) \leq P(B_t^c) = P(X_1 > t) = Q(t) \leq e^{-t^2/2}$.