Lecture 8 (Contd.) Proof sketch of the claim 1

[Efron-Stein inequality]

$$\mathbb{E}_i\left[\mathbb{E}_{(i)}[Z]\right] = \mathbb{E}\Big[\underbrace{\mathbb{E}[Z \mid X^{-i}]}_{\text{function only of } X^{-i}} \Big| X^i\Big]$$

Let $Z = f(A,B,C)$, A,B,C indep.

$$\mathbb{E}\Big[\mathbb{E}[Z \mid AC] \Big| AB\Big] = \mathbb{E}\Big[\mathbb{E}[Z \mid A] \Big| AB\Big]$$

$$= \mathbb{E}[Z \mid A]$$ ∎

Other equivalent forms of Efron-Stein

Let $v \overset{\text{def}}{=} \sum_{i=1}^{n} \mathbb{E}\left[\mathrm{Var}_{(i)}(Z)\right]$.

Efron-Stein implies that v can be treated as the effective variance of Z. The good thing is that v can be expressed as the sum of individual contributions across $i = 1, 2, \ldots, n$. The next result gives two useful alternative forms of v

Lemma. (a) Let $X' = (X_1', \ldots, X_n')$ be an indep. copy of X. Then,

$$v = \frac{1}{2}\sum_{i=1}^{n} \mathbb{E}\left[(Z - Z_i')^2\right]$$

where

$$Z_i' = f(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n).$$

(b) $$v = \inf_{\{Z_i\}} \mathbb{E}\left[(Z - Z_i)^2\right],$$

where the inf. is over all square integrable func. of

$(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$.

Proof Follows from the following 2 simple facts about variances:

(i) X and Y be iid

$$\mathrm{Var}(X) = \frac{1}{2}\, \mathbb{E}\left[(X-Y)^2\right]$$

(ii)
$$\mathrm{Var}(X|Y) \equiv \mathbb{E}\left[\left(X - \mathbb{E}[X|Y]\right)^2\right]$$
$$= \inf_{Z} \mathbb{E}\left[(X-Z)^2\right] \equiv \mathrm{MMSE}(X|Z)$$

where the infimum is over all functions Z of Y.

Applying these observations to $\mathrm{Var}_{(i)}(Z)$ gives the result. $\blacksquare$

Corollary. f satisfies BDP with $(c_1, \dots, c_n)$

Then, $\mathrm{Var}\left(f(X_1, \dots, X_n)\right) \le \sum_{i=1}^{n} \frac{c_i^2}{4}$

Pf. Use (b) with

$$Z_i' = \frac{1}{2}\left[\sup_{x_i' \in \mathcal{X}_i} f(x^{i-1}, x_i', x_{i+1}^n) + \inf_{x_i' \in \mathcal{X}_i} f(x^{i-1}, x_i', x_{i+1}^n)\right]$$

$\blacksquare$

Example: Z = length of the longest inc. subseq. of $X_1, \dots, X_n$.

$$\mathrm{Var}(Z) \le n \Rightarrow \mathbb{P}\left(|Z - \mathbb{E}Z| > \sqrt{\frac{n}{\delta}}\right) \le \delta.$$

$\boxed{B}$ Variance and tail bounds

The first and the second moment methods

→ Let X be a nonnegative integer-valued rv.

$$\underbrace{\mathbb{P}(X \neq 0) = \mathbb{P}(X \geq 1) \leq \mathbb{E}[X]}_{\text{The "first-moment method"}}$$

Example: Consider an n-uniform hypergraph $\mathcal{H}$ with m edges. If $m < 2^{n-1}$ then $\mathcal{H}$ is 2-colorable, i.e., there exists a coloring of vertices using 2 colors s.t. no hyperedge is monochromatic.

Indeed, consider a random 2-coloring of $\mathcal{H}$ where each vertex is independently and uniformly colored using red/blue colors. Let X denote the # of monochromatic edges. Then,

$$\mathbb{P}(X \neq 0) \leq \mathbb{E}[X] = \sum_{e \in \mathcal{E}} \mathbb{P}(e \text{ is monochromatic})$$

$$= m \cdot 2^{-n+1} < 1.$$

→ For a rv X with $\mathbb{E}[X] \neq 0$,

$$\mathbb{P}(X \leq 0) \leq \mathbb{P}(|X - \mathbb{E}[X]| > \mathbb{E}[X]) \leq \frac{\mathrm{Var}(X)}{\mathbb{E}[X]^2}$$

Improvement: $\mathbb{E}[X]^2 = \mathbb{E}[X \mathbb{1}_{\{X \neq 0\}}]^2$

$$\leq \mathbb{E}[X^2](1 - \mathbb{P}(X=0))$$

$$\Leftrightarrow \underbrace{\mathbb{P}(X=0) \leq \frac{\mathrm{Var}(X)}{\mathbb{E}[X]^2}}_{\text{The "second moment method"}} \quad \text{(Shepp's bound)}$$

→ The bounds above give bounds for the tail $\mathbb{P}(X>0)$ for nonnegative r.v.s.

Concentration around median using Efron-Stein

Quantiles of a distribution:

$Q_\alpha = \inf\{z : \mathbb{P}(Z \leq z) \geq \alpha\}$: α^{th} quantile of Z

$\mathbb{M}Z \equiv$ median of $Z = Q_{1/2}$.

→ Let q_k denote $Q_{1-2^{-k}}$, i.e., $\mathbb{P}(Z > q_k) \leq 2^{-k}$.

(i) $\lim_{k\to\infty} q_k = \operatorname{ess\,sup} Z$

(ii) Suppose $q_{k+1} - q_k \leq c$ for every $k \in \mathbb{N}$.

Then,
$$\boxed{\mathbb{P}(Z > \mathbb{M}[Z] + t) \leq 2^{-\frac{t}{c}}}$$

Proof: Let k_t denote the k s.t.

$$q_{k_t} \leq t + q_1 \leq q_{k_t+1}$$

Then, $q_{k_t+1} = q_1 + \sum_{i=1}^{k_t} (q_{i+1} - q_i) \leq q_1 + c k_t$

$\Rightarrow \quad t \leq c k_t$.

⑤

Thus, $$\mathbb{P}(Z > \mathbb{M}[Z] + t) \leq \mathbb{P}(Z > q_{k_t})$$
$$\leq 2^{-k_t} \leq 2^{-t/c}. \quad \blacksquare$$