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A simple criterion on degree sequences of graphs

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ABSTRACT

A finite sequence of nonnegative integers is called graphic if the terms in the sequence can be realized as the degrees of vertices of a finite simple graph. We present two new characterizations of graphic sequences. The first of these is similar to a result of Havel-Hakimi, and the second equivalent to a result of Erdős & Gallai, thus providing a short proof of the latter result. We also show how some known results concerning degree sets and degree sequences follow from our results.

1. Introduction

A finite sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers is said to be *graphic* if there exists a finite simple graph G with vertices $v_1, v_2, \ldots, v_n$ such that each v_i has degree d_i. Two obvious necessary conditions for such a sequence to be graphic are: (1) $d_i < n$ for each i, and (2) $\sum_{i=1}^{n} d_i$ is *even*. However, these two conditions together do not ensure that a sequence will be graphic. Necessary and sufficient conditions for a sequence of nonnegative integers to be graphic are well known. Sierksma and Hoogeveen [10] list seven such characterizations. Two of the most well known characterizations of graphic sequences are due to Havel [7], and independently, Hakimi [5], and jointly due to Erdős and Gallai [4]. The result of Havel–Hakimi has been extended by Kleitman and Wang [9], and that of Erdős and Gallai by Eggleton [3], and by Tripathi and Vijay [11].

We give a simple criterion to characterize graphic sequences. Our characterization relies on repeated applications of the Havel–Hakimi theorem to Theorem 1, which we prove below. We recall the Havel–Hakimi theorem:

Theorem H–H (*Havel [7]; Hakimi [5]*). *Let $\{a_1, a_2, \ldots, a_n\}$ be a sequence of positive integers such that $n-1 \geq a_1 \geq a_2 \geq \cdots \geq a_n$. Then the sequence $\{a_1, a_2, \ldots, a_n\}$ is graphic if and only if the sequence $\{a_2-1, a_3-1, \ldots, a_{a_1+1}-1, a_{a_1+2}, \ldots, a_n\}$ is graphic.*

The following extension of Theorem H–H, due to Kleitman and Wang, is sometimes more useful:

Theorem KW (*Kleitman and Wang [9]*). *Let $s := \{a_1, a_2, \ldots, a_n\}$ be a list of positive integers. Then the list s' obtained from s by deleting any term a_i and subtracting 1 from the a_i largest terms remaining in the list is graphic if and only if the list s is graphic.*

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2. Main results

We say that two sequences s_1 and s_2, of nonnegative integers, are *graphically equivalent* provided s_1 is graphic if and only if s_2 is graphic. Theorem 1 provides an instance of graphically equivalent sequences, much like Theorems H–H and KW. However, in this case, for any $d \geq 0$, we replace the largest term Δ in the sequence by $\Delta + d$ and also adjoin d 1's.

Theorem 1. *Let $\{a_1, a_2, a_3, \ldots, a_n\}$ be a sequence of positive integers such that $n-1 \geq a_1 \geq a_2 \geq \cdots \geq a_n$, and let $N \geq a_1+1$. Let 1_r denote r occurrences of 1. Then the sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ is graphic if and only if the sequence $\{N, a_2, a_3, \ldots, a_n, 1_{N-a_1}\}$ is graphic.*

Proof. Suppose $s := \{a_1, a_2, a_3, \ldots, a_n\}$ is a graphic sequence. To a graph G with degree sequence s, add $N - a_1$ vertices and join each of these to the vertex of degree a_1. This gives a graph with degree sequence $\{N, a_2, a_3, \ldots, a_n, 1_{N-a_1}\}$.

Conversely, suppose $s' := \{N, a_2, a_3, \ldots, a_n, 1_{N-a_1}\}$ is a graphic sequence. Applying Theorem KW $N - a_1$ times, with $a_i = 1$ each time, results in the sequence s, which must be graphic. This completes the proof. □

Theorem 1 is similar in nature to Theorems H–H and KW, but gives a graphically equivalent sequence with a larger number of terms. The following definitions are central to our results, and repeatedly used in what follows.

Definition 1. For a given sequence $\{a_1, a_2, \ldots, a_n\}$ of positive integers such that $a_1 \geq a_2 \geq \cdots \geq a_n$, we denote by λ the largest positive integer i for which $a_i \geq i$ and by R_i the number of occurrences of i in the sequence.

Our next result is derived by successive applications of Theorems 1 and H–H.

Theorem 2. *The sequence $\{a_1, a_2, \ldots, a_n\}$, with $a_1 \geq a_2 \geq \cdots \geq a_n$, is graphic if and only if for $1 \leq s \leq \lambda - 1$*

$$f(s) := \sum_{i=1}^{s} [(n-1) - a_i - (s-i)R_i] \geq 0, \tag{1}$$

and $f(\lambda)$ is even.

Proof. Starting with the given sequence s_0, we first apply Theorem 1 and then the *Havel–Hakimi* condition to obtain a graphically equivalent sequence s_1:

$$\begin{aligned} s_0 &:= \{a_1, a_2, \ldots, a_n\}; \\ &\quad \{n-1, a_2, \ldots, a_n, 1_{n-1-a_1}\}; \\ s_1 &:= \{a_2 - 1, a_3 - 1, a_4 - 1, \ldots, a_n - 1, 1_{n-1-a_1}\}. \end{aligned}$$

Observe that this procedure transforms a sequence s_0 to a graphically equivalent sequence s_1 with a *smaller* maximum term. Thus this procedure must eventually lead to a sequence of 1's, after discarding all 0's at each step. However, to apply Theorem 1, we must ensure that none of the terms in the sequence are 0 and also that the maximum term is at least one less than the number of positive terms. So we must keep track of the number of positive terms at each stage in the procedure. Therefore we may equivalently assume that the sequence s_1 contains only positive terms. To do this, we may remove the R_1 (in s_0) terms in s_1 that are 0 and assume that the transformed sequence has only positive terms.

Let N_i denote the number of positive terms in the sequence s_i obtained from s_0 after i iterations; thus $N_0 = n$ and $N_1 = (n-1-R_1) + (n-1-a_1) = 2(n-1) - a_1 - R_1$. If we apply Theorem 1 followed by the Havel–Hakimi condition to the sequence s_1, finally removing the zero terms, we get

$$\begin{aligned} s_1 &:= \{a_2 - 1, a_3 - 1, a_4 - 1, \ldots, a_n - 1, 1_{n-1-a_1}\} \quad \text{with } N_1 \text{ positive terms;} \\ &\quad \{N_1 - 1, a_3 - 1, \ldots, a_n - 1, 1_{n-1-a_1}, 1_{N_1-a_2}\}; \\ s_2 &:= \{a_3 - 2, a_4 - 2, \ldots, a_n - 2, 1_{N_1-a_2}\} \quad \text{with } N_2 \text{ positive terms,} \end{aligned}$$

where $N_2 = (n-2-R_1-R_2) + (N_1 - a_2) = 3n - 4 - (a_1 + a_2) - (2R_1 + R_2)$. Continuing in this manner, after i steps, removing the terms that are 0 at each stage, we arrive at

$$s_i := \{a_{i+1} - i, a_{i+2} - i, \ldots, a_n - i, 1_{N_{i-1}-a_i+(i-2)}\} \tag{2}$$

with

$$N_i = (i+1)n - 2i - (a_1 + a_2 + \cdots + a_i) - (R_i + 2R_{i-1} + \cdots + iR_1) \tag{3}$$

positive terms. In order to continue this procedure at each stage, it is not only necessary that the maximum term in s_i be at most $N_i - 1$ (otherwise we cannot apply Theorem 1), but also sufficient (otherwise s_i cannot be graphic). Thus we need

$$a_{i+1} - i \leq N_i - 1 = (i+1)n - 2i - (a_1 + a_2 + \cdots + a_i) - (R_i + 2R_{i-1} + \cdots + iR_1) - 1,$$

or

$$(i+1)(n-1) \geq (a_1 + a_2 + \cdots + a_{i+1}) + (R_i + 2R_{i-1} + \cdots + iR_1). \tag{4}$$

With $i = \lambda$, we arrive at

$$s_\lambda := \{a_{\lambda+1} - \lambda, a_{\lambda+2} - \lambda, \ldots, a_n - \lambda, 1_{N_{\lambda-1}-a_\lambda+(\lambda-2)}\}, \tag{5}$$

where $N_{\lambda-1} = \lambda n - 2(\lambda - 1) - (a_1 + a_2 + \cdots + a_{\lambda-1}) - (R_{\lambda-1} + 2R_{\lambda-2} + \cdots + (\lambda - 1)R_1)$. The condition on λ implies $a_{\lambda+1} \le \lambda \le a_\lambda$. Thus $s_\lambda = \{1_r\}$, where $r = N_{\lambda-1} - a_\lambda + (\lambda - 2)$, is graphic if and only if

$$r = \lambda(n-1) - (a_1 + a_2 + \cdots + a_\lambda) - (R_{\lambda-1} + 2R_{\lambda-2} + \cdots + (\lambda-1)R_1) = f(\lambda) \equiv 0 \pmod 2. \tag{6}$$

This completes the proof of our theorem. □

A theorem of Erdős and Gallai [4] gives another well known characterization of graphic sequences. Unlike the Havel–Hakimi condition, this characterization requires verification of n inequalities, where n denotes the number of terms in the sequences. For the sake of reference, we state this result below:

Theorem EG. *[Erdős and Gallai, [4]] Let $\{a_1, a_2, \ldots, a_n\}$ be a sequence of positive integers such that $a_1 \ge a_2 \ge \cdots \ge a_n$. Then the sequence $\{a_1, a_2, \ldots, a_n\}$ is graphic if and only if $\sum_{i=1}^n a_i$ is even and the inequalities*

$$\sum_{i=1}^{s} a_i \le s(s-1) + \sum_{i=s+1}^{n} \min\{s, a_i\} \tag{7}$$

hold for each s with $1 \le s \le n$.

It is interesting to note that inequalities (7) need not be verified for all s, but only for those s with $a_s > a_{s+1}$ and $a_s \ge s$ in the notation of Theorem EG; see [11]. Since Theorems 2 and EG both require checking inequalities as a characterization for graphic sequences, it stands to reason that there is some relation between the inequalities (1) and (7), where in (7) we check only those s for which both $a_s > a_{s+1}$ and $a_s \ge s$. Indeed, as we shall presently show, they are algebraically the same. Fix s with $1 \le s \le \lambda - 1$. Then inequality (1) is the same as

$$\sum_{i=1}^{s} a_i \le \sum_{i=1}^{s} [(n-1) - (s-i)R_i] = s(s-1) + s(n-s) - \sum_{i=1}^{s} (s-i)R_i.$$

Now for $s < \lambda$, $i \le s$ implies $a_i \ge a_s \ge s$. Therefore

$$\sum_{i=s+1}^{n} \min\{s, a_i\} = s(n-s) - \sum_{\substack{i=1\\ a_i<s}}^{n} (s-a_i) = s(n-s) - \sum_{j=1}^{s-1} (s-j)R_j = s(n-s) - \sum_{i=1}^{s} (s-i)R_i.$$

We have thus shown

$$f(s) := \sum_{i=1}^{s} [(n-1) - a_i - (s-i)R_i] = s(s-1) + \sum_{i=s+1}^{n} \min\{s, a_i\} - \sum_{i=1}^{s} a_i \tag{8}$$

for $1 \le s \le \lambda - 1$. This proves the equivalence of (1) and (7). In particular, Eq. (8) shows that $f(\lambda)$ is even if and only if $\sum_{i=1}^n a_i$ is even since $\min\{\lambda, a_i\} = a_i$ for $\lambda + 1 \le i \le n$. This proves the desired equivalence between Theorem 2 and the stronger version of Theorem EG. We summarize these comments in the following.

Remark 1. Theorem 2 can be strengthened to verifying (1) only for those s with both $a_s > a_{s+1}$ and $a_s \ge s$.

Remark 2. We note that the above discussion shows that Theorem 2 gives an alternate proof of Theorem EG. The original proof of Theorem EG was simplified by Choudum in [2]. Sierksma and Hoogeveen showed that Hässelbarth's criterion [6] is equivalent to Theorem EG in [10]. However, Theorem 2 shows that Theorem EG can be reformulated solely in terms of the frequency of the items in the list, and in this manner, proved in a simpler though indirect manner.

3. Applications

In this section, we apply the results in the previous section, mainly Theorem 2, to obtain some results on degree sequences and degree sets. An easy consequence of Theorem 2 is the following well known result:

Proposition 1. *Let n, r be integers, with $n \ge 1$ and $0 \le r \le n - 1$. Then the sequence $\{r_n\}$ is graphic if and only if nr is even. In other words, there exists a r-regular graph of order n if and only if nr is even.*

Proof. Using the notation in Theorem 2, $\lambda = r$ and the only nonzero R_i is $R_r = n$. Therefore the inequality in Eq. (1) is met, and the sequence $\{r_n\}$ is graphic if and only if

$$(n-1)r - r^2 = nr - r(r+1) \equiv nr \equiv 0 \pmod 2. \quad \square$$

Following Behzad and Chartrand in [1], we say that a graph has a *perfect* degree sequence if the sequence consists of distinct integers. Therefore the only perfect degree sequence of a graph of order n must be $\{0, 1, 2, \dots, n-1\}$. Perfect degree sequences cannot exist since both $d(x) = 0$ and $d(y) = n - 1$ cannot hold in any (simple) graph of order n. Moreover, we say that a degree sequence is *quasi-perfect* if it has exactly one repeated entry. Behzad and Chartrand also showed that there are exactly two quasi-perfect graphic sequences of order n for each $n > 1$, and that the corresponding graphs are complements of each other. We explicitly show that there are exactly two quasi-perfect graphic sequences of order n for each $n > 1$. We use the following lemma, whose proof is trivial and omitted:

Lemma 1. *Let $\{a_1, a_2, \dots, a_n\}$ be a sequence of nonnegative integers. Then the sequence $\{a_1, a_2, \dots, a_n\}$ is graphic if and only if the sequence $\{n-1-a_1, n-1-a_2, \dots, n-1-a_n\}$ is graphic.*

Theorem 3 (*Behzad and Chartrand [1]*). *For each $n > 1$, there are exactly two quasi-perfect graphic sequences of order n:*

$$n-1, n-2, \dots, \lfloor n/2 \rfloor, \lfloor n/2 \rfloor, \dots, 2, 1; \quad n-2, n-3, \dots, \lfloor (n-1)/2 \rfloor, \lfloor (n-1)/2 \rfloor, \dots, 1, 0.$$

Proof. Let $n > 1$. Since a quasi-perfect degree sequence of order n must have exactly one of $0, n-1$, the two possibilities are $0, 1, \dots, n-2$ with some r appearing twice, and $1, 2, \dots, n-1$, again with some r appearing twice. We prove this result only in the case when $n = 2m$ is even, the other case being similar.

We prove that there is exactly one value of r in each case. Consider the sequence $\{2m-1, 2m-2, \dots, r+1, r, r, r-1, \dots, 1\}$, where $1 \le r \le 2m-1$. With the notation of Theorem 2, we may denote this sequence by $\{a_1, a_2, \dots, a_{2m}\}$. Recall that λ denotes the largest positive integer i such that $a_i \ge i$ and R_i the number of occurrences of i in the sequence. The requirement on parity of the sum yields $r \equiv m \pmod 2$. Moreover $a_{m+1} \le m \le a_m$ implies $\lambda = m$. Observe that

$$f(s) = [(n-1) - a_1 - (s-1)R_1] + \cdots + [(n-1) - a_{s-1} - R_{s-1}] + [(n-1) - a_s]$$

is the sum of an s-term arithmetic progression with first term $1 - s$ and common difference 2 if $1 \le s \le r$, unless the repeated term r is greater than m. Thus, if $1 \le s \le r \le m$, $f(s) = 0$ and $f(r+1) = -1$ since $R_r = 2$, while $f(2m - r + 1) = -1$ if $r > m$. Therefore, the inequality in (1) fails if $r \ne m$, and by Theorem 2, the given sequence is not graphic in these cases. On the other hand, if $r = m$, $f(s) = 0$ for $1 \le s \le m$, and so the sequence is graphic only in this case. By Lemma 1, the sequence $\{2m-2, 2m-3, \dots, r+1, r, r, r-1, \dots, 1, 0\}$ is graphic if and only if $\{2m-1, 2m-2, \dots, 2m-r, 2m-1-r, 2m-1-r, \dots, 2, 1\}$ is graphic. Therefore there is exactly one value of r for which quasi-perfect sequences with a 0 are graphic, with the repeated value r given by $2m-1-r = m$, so that $r = m-1$. This completes the assertion. □

The *degree set* of a graph is the set of (distinct) degrees of its vertices. A natural question that arises in the context of degree sets is to determine the least order among graphs with a given degree set. This was answered by Kapoor, Polimeni and Wall in [8], and a short proof of this was given by Tripathi and Vijay in [12]. While both these proofs construct the required graph by *inducting* on the size of the set, the following result *explicitly* provides a sequence, which we prove to be graphic by using Theorem 2.

Theorem 4 (*Kapoor, Polimeni and Wall [8]*). *Let S be a set of positive integers with maximum element Δ. Then there exists a $(\Delta + 1)$-term graphic sequence with set of distinct terms S. Moreover, there cannot be such a graphic sequence, with degree set S, having fewer terms.*

Proof. Let $S = \{a_1, a_2, \dots, a_n\}$, where $a_1 > a_2 > \cdots > a_n \ge 1$. We construct a graphic sequence of order $a_1 + 1$ with degree set S. Consider the sequence

$$(a_1)_{m_1}, (a_2)_{m_2}, \dots, (a_n)_{m_n},$$

with $m_1 = a_n$, $m_i = a_{n+1-i} - a_{n+2-i}$ for $2 \le i \le n$, $i \ne r$, and $m_r = a_{\lceil (n+1)/2 \rceil} - a_{\lceil (n+1)/2 \rceil + 1} + 1$, where $r = \lfloor (n+1)/2 \rfloor$. We verify that this sequence is graphic by using the strong version of Theorem 2, stated at the end of Section 2. We show this only in the case n is odd, the case when n is even being similar.

Let $\sigma_i = m_1 + m_2 + \cdots + m_i$ for $1 \le i \le n$. We need to show that the sequence has $a_1 + 1$ terms, that $\lambda = \sigma_r - 1$, that $f(\sigma_i) \ge 0$ for $1 \le i \le r-1$, and that $f(\lambda)$ is even.

We observe that there are

$$a_n + \sum_{i=2}^{n} (a_{n+1-i} - a_{n+2-i}) + 1 = a_1 + 1$$

terms in the sequence.

A simple calculation shows that σ_i equals a_{n+1-i} for $1 \le i \le r-1$ and $a_{n+1-i}+1$ for $r \le i \le n$. Since $a_{\sigma_r - 1} = a_r = a_{\sigma_r}$ and $\sigma_r - 1 = a_{n+1-r} = a_r$, $a_\lambda = \lambda = a_{\lambda+1}$ for $\lambda = \sigma_r - 1$. Let $1 \le j \le r-1$. Then

$$\begin{aligned} f(\sigma_j) &= \sum_{i=1}^{\sigma_j} [a_1 - a_i - (\sigma_j - i)R_i] \\ &= \sum_{i=1}^{j} m_i(a_1 - a_i) - \sum_{i=1}^{j} (\sigma_j - a_i) R_{a_i} \\ &= \sum_{i=1}^{j} m_i(a_1 - \sigma_j) = \sigma_j(a_1 - \sigma_j) \ge 0. \end{aligned} \tag{9}$$

A calculation similar to the one above shows that $f(\lambda) = f(\sigma_r - 1) = a_r(a_1 - a_r)$. This must be even since a_1 is odd.

That there cannot be a graphic sequence with fewer than $\Delta + 1$ terms and with degree set S is obvious. This completes the assertion. $\square$

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