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Optimal Receiver for MPSK Signaling with Imperfect Channel Estimation

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Abstract — **We derive the structure of the optimal receiver for MPSK signaling over a correlated Rayleigh fading channel. The channel is estimated using the minimum mean square error criterion by means of pilot symbols. For the case of high signal-to-noise ratio, an approximate expression for the symbol error probability (SEP) of this scheme is obtained as an integral, and compared with the SEP of a suboptimal receiver which uses maximal-ratio combining. Numerical results show that the performance gap between the optimal and suboptimal receivers increases with increase of the channel correlation and the number of diversity branches, whereas it decreases with increase of pilot-to-signal ratio.**

Keywords – **Correlated Rayleigh fading, MPSK, optimal receiver, symbol error probability.**

I. Introduction

Communication systems using channel estimation at the receiver have gained lots of importance with the advent of third-generation wireless systems. Most of the third generation systems use MPSK and MQAM signaling over a fading channel with diversity. In multipath diversity techniques it is assumed that many independent observations are available for the same signal which do not fade simultaneously. Therefore an improved performance can be expected if the outputs from all diversity branches are used jointly. The important issue here is how the weights of the signals from different fading channels are combined. The combination, ensuring maximum signal-to-noise ratio (SNR) at the output of the combiner, is called maximal-ratio combining (MRC). This construct, though ensures maximum average SNR at the output when the channel is known perfectly, is still suboptimal for correlated channels in the presence of channel estimation errors. As the channel gain in a fading environment is random, the receiver estimates the channel gain after each fixed interval. A minimum mean square error (MMSE) estimation method using pilot symbols is an effective method of obtaining the channel state information.

Analytical expressions for the symbol error probability (SEP) in case of BPSK signaling using practical channel estimation have been derived in [1, 2]. In this paper we find the structure of an optimal receiver for MPSK signaling over a correlated Rayleigh fading channel, when MMSE channel estimation is used by means of pilot symbols. For the case of high SNR, an approximate expression for the SEP of this scheme is obtained as an integral, and compared with the SEP of a suboptimal receiver which uses MRC.

II. System Model

Consider an L-branch diversity reception system using M-PSK signaling in flat fading. The complex baseband sampled received signal vector is given by

$$\mathbf{r} = \mathbf{h}s + \mathbf{n}\,, \tag{1}$$

where s is the complex information bearing symbol with energy $|s|^2 = 2E_s$, $\mathbf{h}$ is the random complex channel gain vector, and $\mathbf{n}$ is the additive white Gaussian noise (AWGN) vector (independent of $\mathbf{h}$), which is a zero-mean complex circular Gaussian random vector distributed as $\mathcal{CN}\left(\mathbf{0}_{L\times 1}, 2N_0\mathbf{I}_L\right)$, $\mathbf{0}_{L\times 1}$ denoting the $L\times 1$ vector of zeros and $\mathbf{I}_L$ the $L \times L$ identity matrix. The channel gain vector $\mathbf{h}$ is a zero-mean complex circular Gaussian random vector distributed as $\mathcal{CN}\left(\mathbf{0}_{L\times 1}, \mathbf{K}_h\right)$, implying correlated Rayleigh fading with *channel covariance matrix* $\mathbf{K}_h$.

A. Channel Estimation

The channel is estimated using pilot symbols over N time indices. Let $\mathbf{s}_p = [s_{p_1}, \ldots, s_{p_N}]^T$ be the $N\times 1$ pilot symbol vector used for channel estimation, where $(\cdot)^T$ denotes transpose. The $L\times N$ received pilot signal matrix $\mathbf{R}_p$ can be written as

$$\mathbf{R}_p = \mathbf{h}\mathbf{s}_p^T + \mathbf{N}_p, \tag{2}$$

where N_p is the $L \times N$ AWGN matrix, with independent and identically distributed (i.i.d.) elements, each having a $\mathcal{CN}(0, 2N_0)$ distribution.

We consider the MMSE estimate of the channel, which is given by $\hat{\mathbf{h}} = \mathbf{R}_p\mathbf{a}$, where $\mathbf{a}$ is an $N\times 1$ vector. Using the orthogonality principle to minimize $\|\hat{\mathbf{h}} - \mathbf{h}\|^2$, the MMSE estimate can be expressed as

$$\hat{\mathbf{h}} \;=\; \mathbf{h}\mathbf{s}_p^T\mathbf{a} + \mathbf{N}_p\mathbf{a}\,, \tag{3}$$

where

$$\mathbf{a} = \frac{\mathbf{E}\left[\|\mathbf{h}\|^2\right]}{(\|\mathbf{s}_p\|^2\mathbf{E}\left[\|\mathbf{h}\|^2\right] + 2N_0L)}\mathbf{s}_p^*, \quad (4)$$

$\mathbf{E}[\cdot]$ denoting the expectation operator. Substituting (4) in (3) we get

$$\hat{\mathbf{h}} = \alpha\mathbf{h} + \mathbf{e}, \quad (5)$$

where

$$\alpha = \frac{\|\mathbf{s}_p\|^2\mathbf{E}\left[\|\mathbf{h}\|^2\right]}{\|\mathbf{s}_p\|^2\mathbf{E}\left[\|\mathbf{h}\|^2\right] + 2N_0L}, \quad (6)$$

and $\mathbf{e}$ is the estimation error vector which is distributed as $\mathcal{CN}\left(\mathbf{0}_{L\times 1}, \sigma_e^2\mathbf{I}_L\right)$, with

$$\sigma_e^2 = 2N_0\|\mathbf{a}\|^2. \quad (7)$$

B. Optimal Receiver Structure

Let $\mathcal{S}$ denote the data symbol set for MPSK signaling, given by

$$\mathcal{S} = \left\{\sqrt{2E_s}e^{\frac{\jmath 2\pi(k-1)}{M}},\ k = 1,\ldots,M\right\}, \quad (8)$$

where $\jmath = \sqrt{-1}$. Assuming equiprobable data symbol transmission following the pilot symbol transmissions, the optimal receiver (optimal in the maximum likelihood sense) uses the decision rule

$$\hat{s} = \text{argmax}_{s\in\mathcal{S}} f(\mathbf{r}|\hat{\mathbf{h}}, s), \quad (9)$$

where $f(\mathbf{r}|\hat{\mathbf{h}}, s)$ is the conditional probability density function (p.d.f.) of $\mathbf{r}$, conditioned on $\hat{\mathbf{h}}$ and s.

We find from (1) and (5) that for the zero-mean jointly complex circular Gaussian random vectors $\mathbf{r}$ and $\hat{\mathbf{h}}$, the covariance matrix of $\mathbf{r}$, the covariance matrix of $\hat{\mathbf{h}}$, and the cross-covariance matrix of $\mathbf{r}$ and $\hat{\mathbf{h}}$, are given respectively as (noting that $|s|^2 = 2E_s$)

$$\begin{aligned}\mathbf{K}_r &= 2E_s\mathbf{K}_h + 2N_0\mathbf{I}_L, \\ \mathbf{K}_{\hat{h}} &= \alpha^2\mathbf{K}_h + \sigma_e^2\mathbf{I}_L, \\ \mathbf{K}_{r\hat{h}} &= \alpha s\mathbf{K}_h. \end{aligned} \quad (10)$$

The received vector $\mathbf{r}$, conditioned on $\hat{\mathbf{h}}$ and s, is therefore distributed as $\mathcal{CN}(\mathbf{m}, \mathbf{K})$, where

$$\begin{aligned}\mathbf{m} &= \mathbf{K}_{r\hat{h}}\mathbf{K}_{\hat{h}}^{-1}\hat{\mathbf{h}} \\ &= \alpha s\mathbf{K}_h(\alpha^2\mathbf{K}_h + \sigma_e^2\mathbf{I}_L)^{-1}\hat{\mathbf{h}}, \quad (11a)\\ \mathbf{K} &= \mathbf{K}_r - \mathbf{K}_{r\hat{h}}\mathbf{K}_{\hat{h}}^{-1}\mathbf{K}_{r\hat{h}}^H \\ &= 2E_s\mathbf{K}_h + 2N_0\mathbf{I}_L \\ &\quad -2\alpha^2E_s\mathbf{K}_h(\alpha^2\mathbf{K}_h + \sigma_e^2\mathbf{I}_L)^{-1}\mathbf{K}_h, \quad (11b)\end{aligned}$$

$(\cdot)^H$ denoting the Hermitian (conjugate transpose) operator, with a conditional p.d.f.

$$f(\mathbf{r}|\hat{\mathbf{h}}, s) = \frac{1}{\pi^L\det(\mathbf{K})}\exp\left\{-(\mathbf{r}-\mathbf{m})^H\mathbf{K}^{-1}(\mathbf{r}-\mathbf{m})\right. . \quad (12)$$

Since $\mathbf{K}$ does not depend on s, substituting (12) in (9) and taking the logarithm results in the decision rule

$$\begin{aligned}\hat{s} &= \text{argmax}_{s\in\mathcal{S}}\text{Re}\left\{\mathbf{m}^H\mathbf{K}^{-1}\mathbf{r}\right\} \\ &= \text{argmax}_{s\in\mathcal{S}}\text{Re}\left\{s^*\hat{\mathbf{h}}^H(\alpha^2\mathbf{K}_h + \sigma_e^2\mathbf{I})^{-1}\mathbf{K}_h \right.\\ &\qquad \left.\times\mathbf{K}^{-1}\mathbf{r}\right\}, \end{aligned} \quad (13)$$

where $\mathbf{K}$ is given by (11b).

III. Error Analysis

Let the matrix $\mathbf{A}$ be defined as

$$\mathbf{A} \triangleq (\alpha^2\mathbf{K}_h + \sigma_e^2\mathbf{I})^{-1}\mathbf{K}_h\mathbf{K}^{-1}. \quad (14)$$

This can be simplified as

$$\mathbf{A} = \left(2\left[E_s\sigma_e^2 + \alpha^2N_0\right]\mathbf{I}_L + 2N_0\sigma_e^2\mathbf{K}_h^{-1}\right)^{-1}. \quad (15)$$

Note that $\mathbf{A}$ is a Hermitian matrix.

Define the *average SNR per branch* Γ_s and the *average pilot-to-noise ratio (PNR) per branch* Γ_p as

$$\begin{aligned}\Gamma_s &\triangleq \frac{E_s\mathbf{E}\left[\|\mathbf{h}\|^2\right]}{N_0L}, \\ \Gamma_p &\triangleq \frac{\alpha^2\mathbf{E}\left[\|\mathbf{h}\|^2\right]}{\sigma_e^2L} = \frac{\|\mathbf{s}_p\|^2\mathbf{E}\left[\|\mathbf{h}\|^2\right]}{2N_0L}, \end{aligned} \quad (16)$$

This gives (from (6))

$$\alpha = \frac{\Gamma_p}{\Gamma_p + 1}. \quad (17)$$

Without loss of generality, consider the case when symbol $s = \sqrt{2E_s}$ is transmitted. The decision variable in (13) can now be written as

$$\begin{aligned}z &= \hat{\mathbf{h}}^H\mathbf{A}(\sqrt{2E_s}\mathbf{h} + \mathbf{n}) \\ &= \alpha\sqrt{2E_s}\mathbf{h}^H\mathbf{A}\mathbf{h} + \sqrt{2E_s}\mathbf{e}^H\mathbf{A}\mathbf{h} + \alpha\mathbf{h}^H\mathbf{A}\mathbf{n} \\ &\quad +\mathbf{e}^H\mathbf{A}\mathbf{n}. \end{aligned} \quad (18)$$

The probability of correct decision is the probability that the phase of z lies between $-\pi/M$ and π/M, which, subtracted from unity, gives the SEP.

When the condition

$$\mathbf{E}\left[\left|\sqrt{2E_s}\mathbf{e}^H\mathbf{A}\mathbf{h} + \alpha\mathbf{h}^H\mathbf{A}\mathbf{n}\right|^2\right] \gg \mathbf{E}\left[\left|\mathbf{e}^H\mathbf{A}\mathbf{n}\right|^2\right] \quad (19)$$

is satisfied, the double noise term $\mathbf{e}^H\mathbf{An}$ in (18) can be neglected. Noting that $\mathbf{E}\left[\|\mathbf{h}\|^2\right] = \mathrm{tr}\left(\mathbf{K}_h\right)$, we can rewrite the condition (19) using the statistics of $\mathbf{h}$, $\mathbf{e}$, and $\mathbf{n}$ as

$$\Gamma_s + \Gamma_p \gg \frac{\mathrm{tr}\left(\mathbf{K}_h\right)\mathrm{tr}\left(\mathbf{A}^2\right)}{L\,\mathrm{tr}\left(\mathbf{K}_h\mathbf{A}^2\right)}\,. \tag{20}$$

which is a high SNR condition.

Under the condition (20), the decision variable in (18) can be approximated as

$$z \approx \alpha\sqrt{2E_s}\mathbf{h}^H\mathbf{Ah} + \sqrt{2E_s}\mathbf{e}^H\mathbf{Ah} + \alpha\mathbf{h}^H\mathbf{An}\,. \tag{21}$$

The instantaneous SNR (SNR conditioned on $\mathbf{h}$) for this decision variable can be written as

$$\begin{aligned}\gamma_{opt} &= \frac{\left(\alpha\sqrt{2E_s}\mathbf{h}^H\mathbf{Ah}\right)^2}{\mathbf{E}\left[\left|\sqrt{2E_s}\mathbf{e}^H\mathbf{Ah} + \alpha\mathbf{h}^H\mathbf{An}\right|^2\middle|\,\mathbf{h}\right]}\\ &= \frac{\alpha^2E_s\left(\mathbf{h}^H\mathbf{Ah}\right)^2}{\left(E_s\sigma_e^2+\alpha^2N_0\right)\left(\mathbf{h}^H\mathbf{A}^2\mathbf{h}\right)}\,.\end{aligned} \tag{22}$$

Let $\Psi_{\gamma_{opt}}(\jmath\omega) = \mathbf{E}\left[e^{\jmath\omega\gamma_{opt}}\right]$ denote the characteristic function (c.f.) of γ_{opt}. The SEP is approximated in terms of this c.f. as [3]

$$P_e \approx \frac{1}{\pi}\int_0^{\frac{\pi(M-1)}{M}}\Psi_{\gamma_{opt}}\left(-\frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\theta}\right)d\theta\,. \tag{23}$$

However, obtaining the exact c.f. of γ_{opt} is a difficult task. We therefore obtain an approximate c.f. by applying the condition (20).

Denoting

$$\mathrm{tr}\left(\mathbf{K}_h\right) = \Omega L\,, \tag{24}$$

we can rewrite (15) as

$$\mathbf{A} = \frac{1}{2\left[E_s\sigma_e^2+\alpha^2N_0\right]}\left(\mathbf{I}_L + \frac{1}{\left(\Gamma_s+\Gamma_p\right)}\Omega\mathbf{K}_h^{-1}\right)^{-1}\,. \tag{25}$$

Further, denoting the Hermitian matrix $\mathbf{F}$ as

$$\mathbf{F} = \frac{1}{\left(\Gamma_s+\Gamma_p\right)}\Omega\mathbf{K}_h^{-1}\,, \tag{26}$$

the instantaneous SNR can be written as

$$\gamma_{opt} = \frac{\alpha^2E_s\left(\mathbf{h}^H\left(\mathbf{I}_L+\mathbf{F}\right)^{-1}\mathbf{h}\right)^2}{\left(E_s\sigma_e^2+\alpha^2N_0\right)\left(\mathbf{h}^H\left(\mathbf{I}_L+\mathbf{F}\right)^{-2}\mathbf{h}\right)}\,. \tag{27}$$

Using the first order approximation

$$\left(\mathbf{I}_L+\mathbf{F}\right)^{-k} \approx \mathbf{I}_L - k\mathbf{F}\,, \tag{28}$$

which holds for the high SNR condition (20), and the definitions of Γ_s and Γ_p in (16) along with (24), we can express γ_{opt} approximately as

$$\gamma_{opt} \approx \frac{\Gamma_s}{\left(\frac{\Gamma_s}{\Gamma_p}+1\right)}\frac{1}{\Omega}\mathbf{h}^H\left(\mathbf{I}_L+2\mathbf{F}\right)\mathbf{h}\,. \tag{29}$$

Since the right-hand side of (29) is a Hermitian quadratic form in $\mathbf{h}$, the c.f. of γ_{opt} can be approximated as [4]

$$\Psi_{\gamma_{opt}}(\jmath\omega) \approx \frac{1}{\det\left(\mathbf{I}_L - \jmath\omega\frac{\Gamma_s}{\left(\frac{\Gamma_s}{\Gamma_p}+1\right)}\frac{1}{\Omega}\mathbf{K}_h\left(\mathbf{I}_L+2\mathbf{F}\right)\right)}\,. \tag{30}$$

Substituting (30) in (23), an *approximate expression for the SEP*, valid for the high SNR condition (20), is given by

$$P_e \approx \frac{1}{\pi}\int_0^{\frac{\pi(M-1)}{M}} \frac{d\theta}{\det\left(\mathbf{I}_L + \frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\theta}\frac{\Gamma_s}{\left(\frac{\Gamma_s}{\Gamma_p}+1\right)}\frac{1}{\Omega}\mathbf{K}_h\left(\mathbf{I}_L+2\mathbf{F}\right)\right)}\,. \tag{31}$$

Let $\lambda_1,\ldots,\lambda_K$ denote the K distinct eigenvalues of the *normalized channel covariance matrix* $\frac{1}{\Omega}\mathbf{K}_h$, λ_i having multiplicity q_i for $i = 1,\ldots,K$. The SEP can be expressed in terms of these eigenvalues as

$$P_e \approx \frac{1}{\pi}\int_0^{\frac{\pi(M-1)}{M}} \prod_{i=1}^{K}\left(\frac{\sin^2\theta}{\sin^2\theta + \sin^2\left(\frac{\pi}{M}\right)\frac{\Gamma_s\Gamma_p}{\left(\Gamma_s+\Gamma_p\right)^2} \times\left[\left(\Gamma_s+\Gamma_p\right)\lambda_i+2\right]}\right)^{q_i} d\theta\,. \tag{32}$$

The integral given by (32) can be evaluated in closed-form.

In case of the suboptimal receiver which uses MRC, the decision variable z_{subopt} is given by putting $\mathbf{A} = \mathbf{I}_L$ in (18).

Under the high SNR approximation

$$\Gamma_s + \Gamma_p \gg 1 \tag{33}$$

obtained by putting $\mathbf{A} = \mathbf{I}_L$ in (20), we get from (27)

$$\gamma_{subopt} = \frac{\Gamma_s}{\left(\frac{\Gamma_s}{\Gamma_p}+1\right)}\frac{1}{\Omega}\|\mathbf{h}\|^2\,, \tag{34}$$

and from (31) and (32) the approximate SEP expressions

$$P_{e,subopt} \approx \frac{1}{\pi}\int_0^{\frac{\pi(M-1)}{M}} \frac{d\theta}{\det\left(\mathbf{I}_L + \frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\theta}\frac{\Gamma_s}{\left(\frac{\Gamma_s}{\Gamma_p}+1\right)}\frac{1}{\Omega}\mathbf{K}_h\right)} \tag{35}$$

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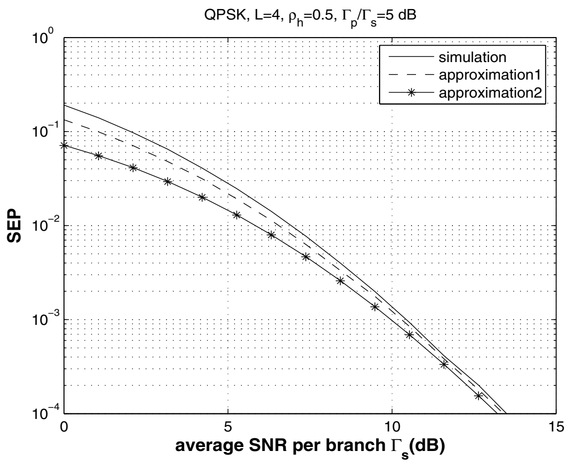

Fig. 1. SEP versus Γ_s for optimal receiver obtained by simulation and approximations when $L = 4$, $\rho_h = 0.5$, and $\Gamma_p/\Gamma_s = 5$ dB

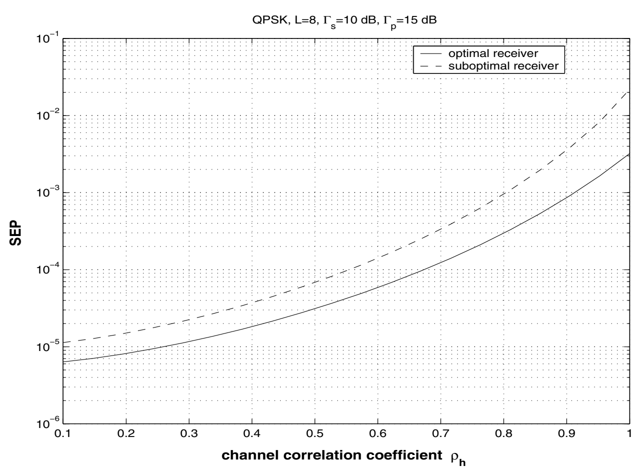

Fig. 2. Variation of performance of optimal and suboptimal receivers with ρ_h when $L = 8$, $\Gamma_s = 10$ dB, and $\Gamma_p = 15$dB

and

$$P_{e,subopt} \approx \frac{1}{\pi}\int_0^{\frac{\pi(M-1)}{M}} \prod_{i=1}^{K}\left(\frac{\sin^2\theta}{\sin^2\theta+\sin^2\left(\frac{\pi}{M}\right)\frac{\Gamma_s\Gamma_p}{(\Gamma_s+\Gamma_p)}\lambda_i}\right)^{q_i} d\theta, \quad (36)$$

respectively.

IV. Numerical Results and Conclusion

For a uniformly correlated Rayleigh fading channel having *channel correlation coefficient* ρ_h, which represents, for example, a space diversity system with closely spaced receive antennas, the SEP in the case of QPSK signaling is computed using the approximate expressions (23) and

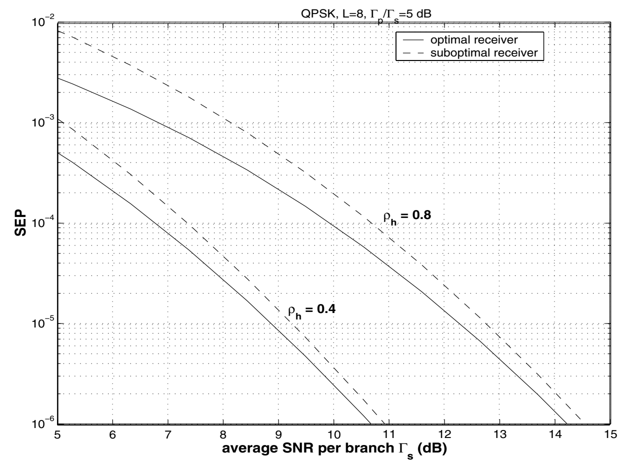

Fig. 3. SEP versus Γ_s with varying ρ_h when $L = 8$ and $\Gamma_p/\Gamma_s = 5$ dB

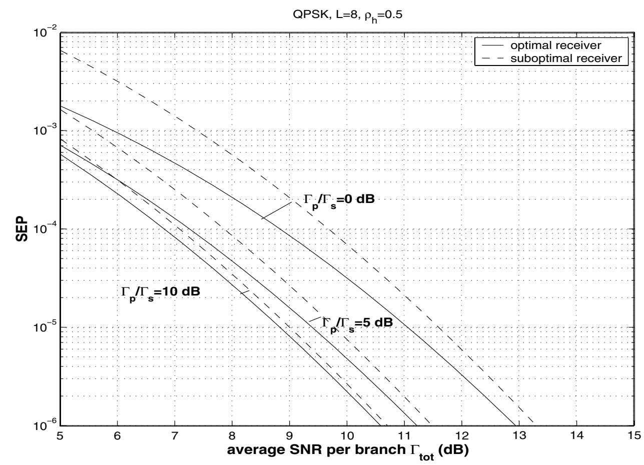

Fig. 4. SEP versus Γ_s with varying Γ_p/Γ_s when $L = 8$ and $\rho_h = 0.5$

(35). The channel covariance matrix is given by

$$\mathbf{K}_h = \Omega \begin{bmatrix} 1 & \rho_h & \dots & \rho_h \\ \rho_h & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho_h \\ \rho_h & \dots & \rho_h & 1 \end{bmatrix}_{L\times L}, \quad (37)$$

where $-1/(L-1) < \rho_h < 1$.

Fig. 1 compares the SEP of the optimal receiver computed by simulation using 10^6 runs with that computed by Monte Carlo simulation of the *approximation1* expression (23) and the *approximation2* expression (35). We find that as the average SNR per branch Γ_s and the average PNR per branch Γ_p increase, the two curves tend to converge, showing the validity of the approximate SEP expression for high values of $\Gamma_s + \Gamma_p$. For further evaluation we have used the approximate expression (35) due

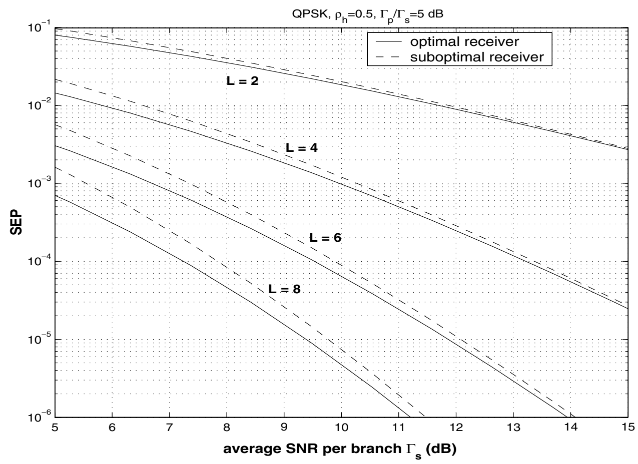

Fig. 5. SEP versus Γ_s with varying L when $\rho_h = 0.5$ and $\Gamma_p/\Gamma_s = 5$ dB

to its computational simplicity.

The variation of the error performance of the optimal and suboptimal receivers with ρ_h is shown in Fig. 2. We find that the optimal receiver has superior performance, with the performance gap increasing with increase of ρ_h.

Figs. 3, 4, and 5 give plots of the SEP versus Γ_s with channel correlation coefficient ρ_h, average pilot-to-signal ratio per branch Γ_p/Γ_s, and number of branches L, respectively. We find that the performance gap between the optimal and suboptimal receivers increases with increase of ρ_h and L, but decreases with increase of Γ_p/Γ_s.

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