Secret Key Generation and Secure Computing

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Secure Computing of a Function of Data

Correlated data is collected and stored at separated locations.

Examples include:

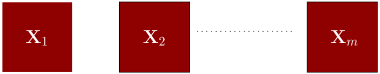
- Data grids and data centers,
- Distributed video coding,
- Sensor networks, etc.

Each location wants to know the value of a function of the data.

- using a communication that keeps the value of the function "secure".

Does there exist a communication protocol to do that?

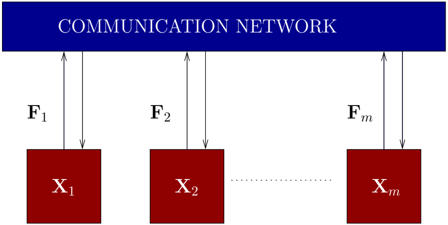
Multiterminal Source Model



Observed data: Correlated rvs $\mathbf{X}_{\mathcal{M}} = (\mathbf{X}_1, ..., \mathbf{X}_m)$.

- Probability distribution of the data is known.

Interactive Communication Protocol



- Terminals communicate over an available network.
- Multiple rounds of interactive communication are allowed.
- Interactive communication: $\mathbf{F} = \mathbf{F}_1, ..., \mathbf{F}_m$.

Assumptions

Assumption on the data

1. *Abundance of data:* Accumulated data grows with time n.
   - $\mathbf{X}_i = X_i^n = (X_{i1}, ..., X_{in})$
   - Data observed at time instance t: $X_{\mathcal{M}t} = (X_{1t}, ..., X_{mt})$.
2. Observations are i.i.d. across time:
   - $X_{\mathcal{M}1}, ..., X_{\mathcal{M}n}$ are i.i.d. rvs.
3. Observations are discrete valued.

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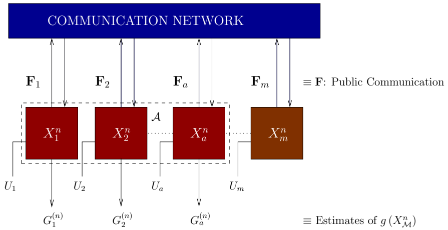
Assumptions on the protocol

1. Each terminal has access to all the communication.
2. Transmission depends on local data and previous communication.
   - interactive communication over multiple rounds.

Secure Computing of Functions



Secure computability of g by $\mathcal{A}$:

$$\Pr\left(G_i^{(n)} = g\left(X_{\mathcal{M}}^n\right), i \in \mathcal{A}\right) \approx 1 : \quad \text{Recoverability}$$

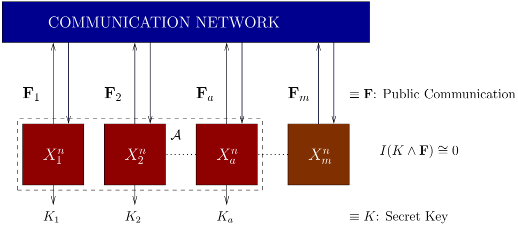
$$I\left(g\left(X_{\mathcal{M}}^n\right) \wedge \mathbf{F}\right) \approx 0 : \quad \text{Secrecy}$$

- Single-letter function: $g\left(X_{\mathcal{M}}^n\right) = \left(g\left(X_{\mathcal{M}1}\right), \ldots, g\left(X_{\mathcal{M}n}\right)\right)$.
- Notation: $G = g\left(X_{\mathcal{M}}\right), \quad G^n = g\left(X_{\mathcal{M}}^n\right)$.

When is a given function g securely computable?

Secret Key Generation

- [Maurer 1993, Ahlswede-Csiszár '93, Csiszár-Narayan '04]
  Agreeing on secret bits using public communication.



- Terminals in $\mathcal{A}$ form estimates of the key.
  - Recoverability:

$$\mathtt{Pr}\left(K_1 = K_2 = \ldots = K_a = K\right) \approx 1.$$

  - Security:

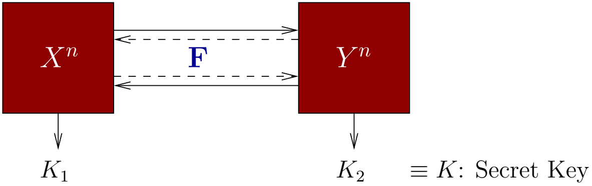
$$I(K \wedge \mathbf{F}) \approx 0.$$

Secret Key Capacity

Rate of the secret key $= \frac{1}{n} H(K)$.

Secret key capacity $C(\mathcal{A})$ = maximum achievable rate of a secret key.

For two terminals



[Maurer '93, Ahlswede-Csiszár '93]

$$C = I(X \wedge Y).$$

Optimum Rate SK for Two Terminals

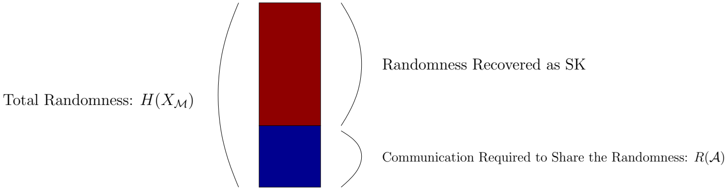
- Maurer-Ahlswede-Csiszár
  - *Common randomness* (CR) generated: X^n or Y^n.
  - Rate of communication required $= \min\{H(X|Y); H(Y|X)\}$.
  - Decomposition:
    $H(X) = H(X \mid Y) + I(X \wedge Y)$,
    $H(Y) = H(Y \mid X) + I(X \wedge Y)$.
- Csiszár-Narayan
  - *Common randomness* generated: X^n, Y^n.
  - Rate of communication required $= H(X|Y) + H(Y|X)$.
  - Decomposition:
    $H(X,Y) = H(X \mid Y) + H(Y \mid X) + I(X \wedge Y)$.
- A generalized decomposition: [Tyagi ISIT '11]

Secret Key Capacity

[Csiszár-Narayan '04]

Omniscience: Having an access to all the randomness.

$R(\mathcal{A}) \equiv$ Communication for omniscience at $\mathcal{A}$.

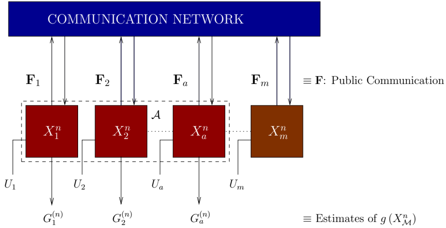


SK capacity:

$$C = H(X_{\mathcal{M}}) - R(\mathcal{A}).$$

$R_{CO}(\mathcal{A})$ can be characterized in a *single-letter-form*.

Secure Computing of Functions



Secure computability of g by $\mathcal{A}$:

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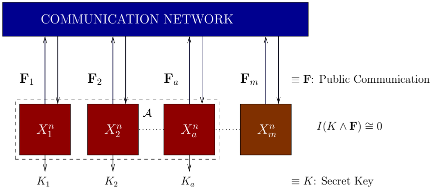
A Necessary Condition

Secret Key Generation



[Csiszár-Narayan ‘04]

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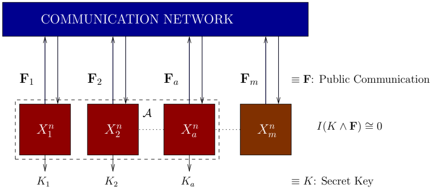

[Csiszár-Narayan '04]

$$C(\mathcal{A}) = H\left(X_{\mathcal{M}}\right) - R(\mathcal{A}),$$

If g is securely computable by $\mathcal{A}$,

$$H(G) \leq C(\mathcal{A}).$$

Is $H(G) < C(\mathcal{A})$ sufficient?

All terminals wish to compute: $\mathcal{A} = \mathcal{M}$

[TNG '10]

If $H(G) < C(\mathcal{M}) \Rightarrow$ a protocol for SC of g by $\mathcal{M}$ exists.

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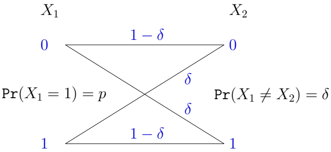
An Example for $m = 2$



- $g(x_1, x_2) = x_1 + x_2 \mod 2 \Rightarrow H(G) = h(\delta)$.
- $C(\{1,2\}) = I(X_1 \wedge X_2) = h((1-p)\delta + p(1-\delta)) - h(\delta)$.
- g is securely computable if

$$2h(\delta) < h((1-p)\delta + p(1-\delta)).$$

Example: Secure Computation of Parity

Binary Symmetric Sources: $p = \frac{1}{2}$

- Secure computability condition: $h(\delta) < 1 - h(\delta)$.
- $\mathbf{P}$: parity check matrix of a *linear* SW code for X_1 given X_2.
- $I(G^n \wedge X_1^n) = 0 \Rightarrow I(G^n \wedge F_1) = 0$.
- K: location of X_1^n in the coset of the standard array (for $\mathbf{P}$).
- Rate of $K = 1 - h(\delta)$.
- $I(K \wedge F_1) = 0$.
- Can show: $I(K \wedge F_1, G^n) = 0$.
- $I(G^n \wedge F_2, F_1) = I(G^n \wedge F_2 \mid F_1) \tilde{\leq} I(K \wedge F_1, G^n) = 0$.

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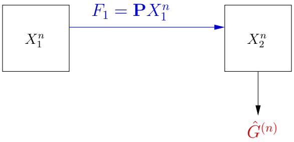

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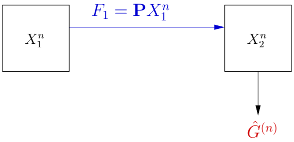

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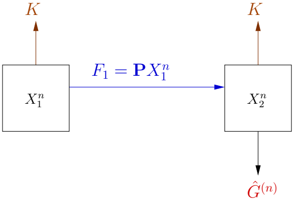

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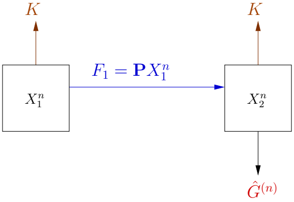

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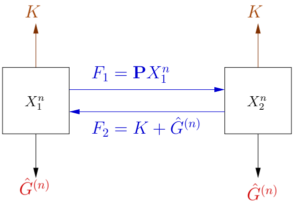
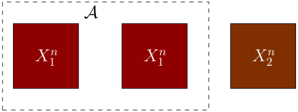


Is $H(G) < C(\mathcal{A})$ sufficient?

All terminals wish to compute: $\mathcal{A} = \mathcal{M}$ [TNG '10]

If $H(G) < C(\mathcal{M}) \Rightarrow$ a protocol for SC of g by $\mathcal{M}$ exists.

Counterexample for $\mathcal{A} \subsetneq \mathcal{M}$



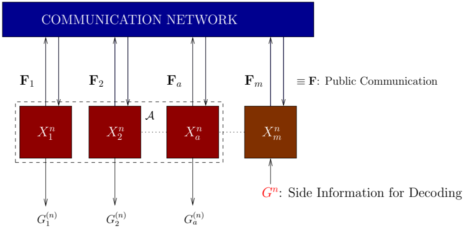
$X_1 \perp\!\!\!\perp X_2$

- $g(x_1, x_1, x_2) = x_2$.
- Let $H(X_2) < H(X_1) = C(\mathcal{A})$.$\rightarrow$ $H(G) < C(\mathcal{A})$ is satisfied.

However, g is clearly **not securely computable**.

A New Necessary Condition

If G^n is securely computable by $\mathcal{A}$:



Provide G^n as side information to terminals in $\mathcal{A}^c$.

- Available *only for decoding* but *not for communicating*.

G^n forms a secret key for all terminals, termed an aided secret key.

- Let $C_{g,\mathcal{A}}(\mathcal{M})$ be that largest achievable rate of such a key.

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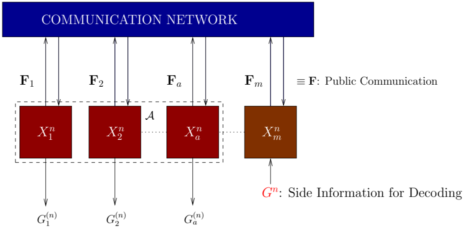

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For a g securely computable by $\mathcal{A}$,

$$H(G) \leq C_{g,\mathcal{A}}(\mathcal{M})$$

Aided Secret Key Capacity

Theorem

The aided secret key capacity is

$$C_{g,\mathcal{A}}(\mathcal{M}) = H(X_{\mathcal{M}}) - R_{g,\mathcal{A}}(\mathcal{M}),$$

where

$R_{g,\mathcal{A}}(\mathcal{M}) =$ *min. sum rate of communication for omniscience at* $\mathcal{M}$
when G^n *is available as side information for decoding to terminals in* $\mathcal{A}^c$*.*

Characterization of Securely Computable Functions

Theorem

If g is securely computable by $\mathcal{A}$: $H(G) \leq C_{g,\mathcal{A}}(\mathcal{M})$.

Conversely, g is securely computable by $\mathcal{A}$ if: $H(G) < C_{g,\mathcal{A}}(\mathcal{M})$.

For securely computable function g:

- *Omniscience can be obtained at $\mathcal{A}$ using $\mathbf{F} \underset{\sim}{\perp\!\!\!\perp} G^n$.*
- *Noninteractive communication suffices.*
- *Randomization is not needed.*

Sketch of the Proof

Consider random binning *of appropriate rate* at each terminal:

- To allow omniscience at $\mathcal{M}$,
  with G^n *given to the terminals in* $\mathcal{A}^c$ *for decoding.*
- To keep bin indices independent of G^n.

Sketch of the Proof

1. $H(G) < C_{g,\mathcal{A}}(\mathcal{M}) = H(X_{\mathcal{M}}) - R_{g,\mathcal{A}}(\mathcal{M})$
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2. Generate random mappings $F_i = F_i(X_i^n)$ of rate R_i:
   $\sum_i R_i \approx R_{g,\mathcal{A}}(\mathcal{M})$ with $(R_1, ..., R_m)$ s.t.
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- Shall show:
  For almost all $(\mathbf{y}, \mathbf{z})$:
  $$F_i \mid \{G^n = \mathbf{y}, F_{\mathcal{M}\setminus\{i\}} = \mathbf{z}\} \approx \text{uniform}.$$

- Family of distributions on X_i^n : $\left\{P_{X_i^n \mid \{G^n = \mathbf{y}, F_{\mathcal{M}\setminus\{i\}} = \mathbf{z}\}}\right\}$.

- Seek conditions for random mappings to be uniformly distributed
  - w.r.t. a given family of distributions.

- **Balanced Coloring Lemma:**
  [R. Ahlswede-I. Csiszár, '98], [I. Csiszár-P.N., '04]

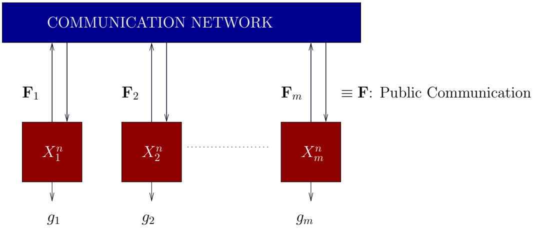
  Given a family of distributions with probabilities uniformly bounded above,

  $$\texttt{Pr}\left(\text{random coloring} \approx \text{uniform, w.r.t. all pmfs in the family}\right) \geq q,$$

  where q depends on the size of the family, the uniform bound and the rate of coloring.

- For the case at hand: a slightly generalized version is applied.
  - $q = q(n)$ grows to 1 super-exponentially in n.

Secure Computability of Multiple Functions



Secrecy Condition: $I\left(\mathbf{F} \wedge G_1^n, ..., G_m^n\right) \approx 0.$

$\boxed{\text{Which functions } g_1, ..., g_m \text{ are securely computable?}}$

Omniscience is not allowed in general

- For $m = 2$: $X_1 \perp\!\!\!\perp X_2 \quad g_i(x_1, x_2) = x_i.$