How Many Queries Will Resolve Common Randomness?

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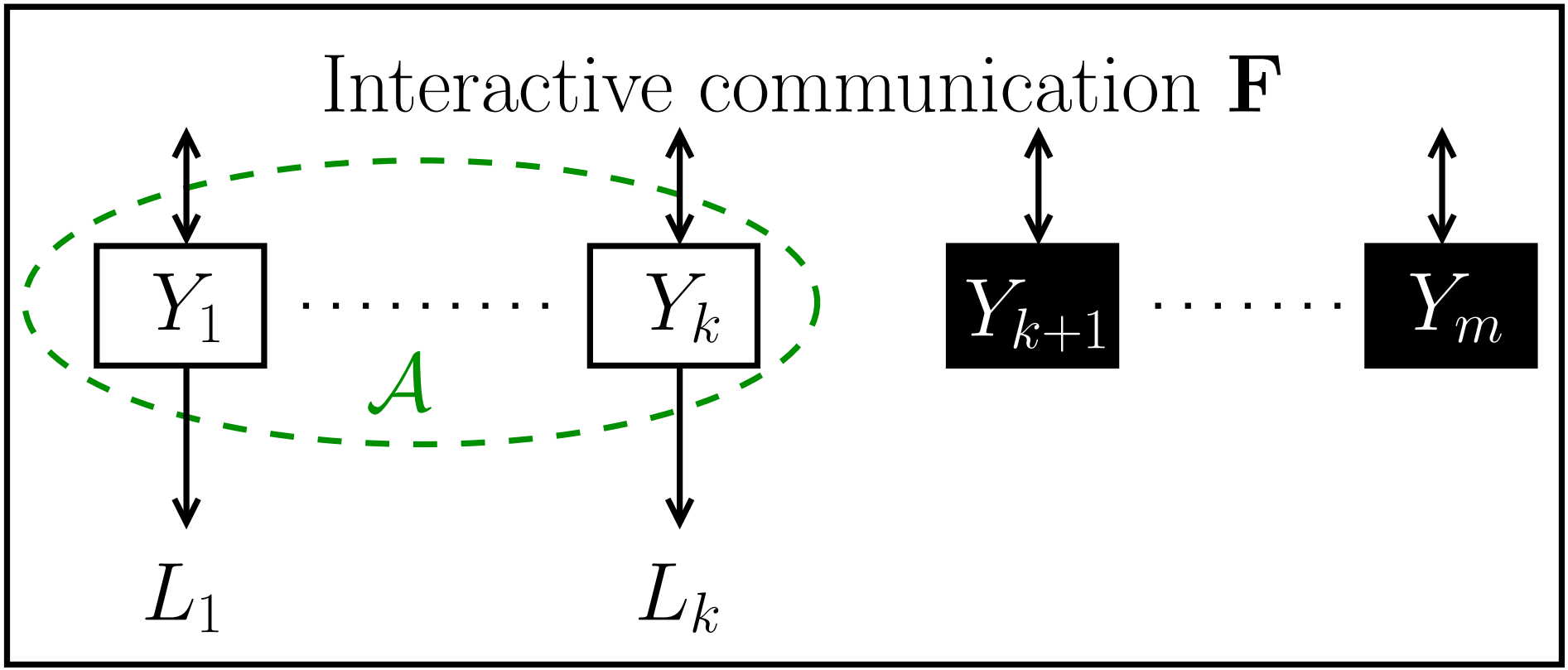
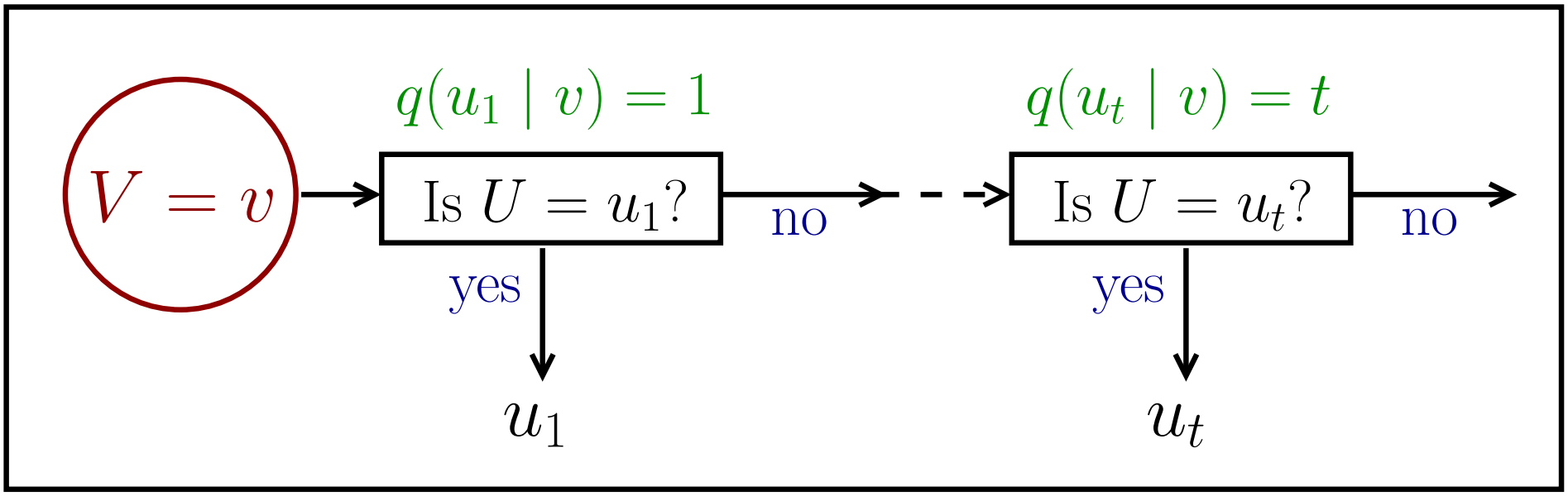
Common Randomness



Definition. L is an ϵ-common randomness for $\mathcal{A}$ from $\mathbf{F}$ if

$$\mathrm{P}\left(L = L_i(Y_i, \mathbf{F}),\ i \in \mathcal{A}\right) \geq 1 - \epsilon$$

Query Strategy



Query strategy for U given V

Massey '94, Arikan '96, Arikan-Merhav '99, Hanawal-Sundaresan '11

Query Strategy

Given rvs U, V with values in the sets $\mathcal{U}, \mathcal{V}$.

Definition. A *query strategy* q for U given $V = v$ is a bijection

$$q(\cdot|v) : \mathcal{U} \to \{1, ..., |\mathcal{U}|\},$$

where the querier, upon observing $V = v$, asks the question

$$\text{“Is } U = u\text{?”}$$

in the $q(u|v)^{\text{th}}$ query.

$q(U|V)$: random query number for U upon observing V

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$$|\{u : q(u \mid v) < \gamma\}| < \gamma$$

Optimum Query Exponent

$Y_i = (X_{i1}, ..., X_{in}) = X_i^n, \quad 1 \le i \le m$: i.i.d. observations

Definition. $E \ge 0$ is an ϵ-achievable *query exponent* if there exists ϵ-CR L_n for $\mathcal{A}$ from $\mathbf{F}_n$ such that

$$\sup_q \mathrm{P}\left(q(L_n \mid \mathbf{F}_n) < 2^{nE}\right) \to 0 \quad \text{as} \quad n \to \infty,$$

where the sup is over every query strategy for L_n given $\mathbf{F}_n$.

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$$E^*(\epsilon) \triangleq \sup\{E : E \text{ is an } \epsilon\text{-achievable query exponent}\}$$

$$E^* \triangleq \inf_{0<\epsilon<1} E^*(\epsilon) : \text{ optimum query exponent}$$

Main Result

Theorem

For $0 < \epsilon < 1$, *the optimum query exponent* E^* *equals*

$$E^* = E^*(\epsilon) = H\left(X_{\mathcal{M}}\right) - \max_{\lambda \in \Lambda(\mathcal{A})} \sum_{B \in \mathcal{B}} \lambda_B H\left(X_B \mid X_{B^c}\right).$$

$$\mathcal{B} = \{B \subsetneq \mathcal{M} : B \neq \emptyset, \mathcal{A} \nsubseteq B\}$$

$$\Lambda(\mathcal{A}) = \text{set of all } \{\lambda_B \in [0,1] : B \in \mathcal{B}\} \text{ such that}$$

$$\sum_{B \in \mathcal{B}: B \ni i} \lambda_B = 1, \quad i \in \mathcal{M}$$

$\lambda \in \Lambda(\mathcal{A})$ is a *fractional partition* of $\mathcal{M}$

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For $m = 2$: The expression on the right $= I\left(X_1 \wedge X_2\right)$

Strong Converse for Secret Key Capacity

Secret Key Capacity

Definition. $C(\epsilon)$ is the supremum over rates of rv $K \in \mathcal{K}$ s.t.

(i) K is an ϵ-CR for $\mathcal{A}$ from $\mathbf{F}$

(ii) K is almost independent of $\mathbf{F}$:

$$n\, \mathrm{s}_{\mathrm{var}}(K; \mathbf{F}) = n \left\| \mathrm{P}_{K,\mathbf{F}} - \mathrm{U}_{\mathcal{K}} \times \mathrm{P}_{\mathbf{F}} \right\|_1 \to 0$$

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I. Csiszár and P. Narayan, Secret key capacity for multiple terminals, IEEE Trans. Inform. Theory, 2004.

Optimum Query Exponent and SK Capacity

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Achievability: $E^*(\epsilon) \geq C(\epsilon)$ - Easy

Converse: $E^*(\epsilon) \leq C$ - Main contribution

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Theorem (Strong converse for SK capacity)

For $0 < \epsilon < 1$, the ϵ-SK capacity is given by

$$C(\epsilon) = E^* = C.$$

Proof of Achievability

Query Strategies and Conditional Probabilities

Lemma. The rvs U, V, satisfy

$$\mathrm{P}\left(\left\{(u,v) : \mathrm{P}_{U|V}\left(u|v\right) \leq \frac{1}{\gamma}\right\}\right) \approx 1. \quad (*)$$

Then for every query strategy q for U given V,

$$\mathrm{P}\left(q(U|V) \geq \gamma\right) \approx 1.$$

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- Proof of $C(\epsilon) \leq E^*(\epsilon)$

  $U =$ SK of rate R, $V = \mathbf{F} \Rightarrow (*)$ holds with $\gamma \approx 2^{nR}$

Proof of Converse

Proof of Converse for $\mathcal{A} = \mathcal{M}$

Alternative Expression for C when $\mathcal{A} = \mathcal{M}$

[Csiszár-Narayan '04] observed that for $\mathcal{A} = \mathcal{M}$

$$C \leq \frac{1}{k-1} D\left(\mathrm{P}_{X_{\mathcal{M}}} \middle\| \prod_{i=1}^{k} \mathrm{P}_{X_{\pi_i}} \right),$$

for every partition $\pi = \{\pi_1, ..., \pi_k\}$ of $\mathcal{M}$.

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We shall show

$$E^*(\epsilon) \leq E_\pi = \frac{1}{|\pi| - 1} D\left(\mathrm{P}_{X_{\mathcal{M}}} \middle\| \prod_{i=1}^{|\pi|} \mathrm{P}_{X_{\pi_i}} \right), \quad \text{for every } \pi.$$

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Roughly: For an ϵ-CR L for $\mathcal{M}$ from $\mathbf{F}$, there exists q_0 s.t.

$$\mathrm{P}\left(q_0(L \mid \mathbf{F}) \leq 2^{nE_\pi} \right) > 0, \quad \text{for every } \pi$$

A General Converse

For rvs $Y_1, ..., Y_k$, let L be an ϵ-CR for $\{1, ..., k\}$ from $\mathbf{F}$.

Theorem

Let θ be such that

$$\mathrm{P}\left(\left\{(y_1, ..., y_k) : \frac{\mathrm{P}_{Y_1,...,Y_k}(y_1, ..., y_k)}{\prod_{i=1}^{k} \mathrm{P}_{Y_i}(y_i)} \leq \theta\right\}\right) \approx 1.$$

Then, there exists a query strategy q_0 for L given $\mathbf{F}$ such that

$$\mathrm{P}\left(q_0(L \mid \mathbf{F}) \lesssim \theta^{\frac{1}{k-1}}\right) \geq (1 - \sqrt{\epsilon})^2 > 0.$$

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- Proof of $E^*(\epsilon) \leq E_\pi$

  Choose $Y_i = X^n_{\pi_i}$ for $i \in \{1, ..., k = |\pi|\}$.

Proof Outline for the General Converse

We show: $\exists$ a subset $\mathcal{I}_0$ of values of $\mathbf{F}$ and sets $\mathcal{L}(\mathbf{i}) \subseteq \mathcal{L}$ s.t.

$$|\mathcal{L}(\mathbf{i})| \lesssim \theta^{\frac{1}{k-1}} \quad \text{and} \quad \mathrm{P}_{L|\mathbf{F}}\left(\mathcal{L}(\mathbf{i}) \mid \mathbf{i}\right) > 0, \quad \mathbf{i} \in \mathcal{I}_0$$

$$\mathrm{P}_{\mathbf{F}}\left(\mathcal{I}_0\right) > 0$$

Lossless Data Compression:

Find small cardinality sets with large $\mathrm{P}_{L|\mathbf{F}}$ probabilities

Small Cardinality Sets with Large Probabilities

Rényi entropy of order α of a probability measure μ on $\mathcal{U}$:

$$H_\alpha(\mu) \triangleq \frac{1}{1-\alpha} \log \sum_{u \in \mathcal{U}} \mu(u)^\alpha, \quad 0 \leq \alpha \neq 1$$

Lemma. There exists a set $\mathcal{U}_\delta \subseteq \mathcal{U}$ with $\mu(\mathcal{U}_\delta) \geq 1 - \delta$ s.t.

$$|\mathcal{U}_\delta| \lesssim \exp\left(H_\alpha(\mu)\right), \qquad 0 \leq \alpha < 1.$$

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Conversely, for any set $\mathcal{U}_\delta \subseteq \mathcal{U}$ with $\mu(\mathcal{U}_\delta) \geq 1 - \delta$,

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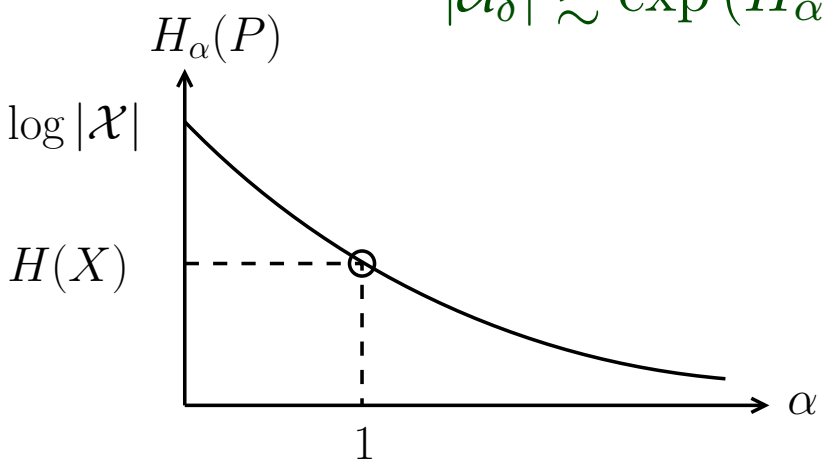

Digression: Lossless Source Coding

Given probability measures μ_n on finite sets $\mathcal{U}_n$, $n \geq 1$.

$$R^*(\delta) \triangleq \inf\{R : \ \mu_n(\mathcal{V}_n) \geq 1-\delta, \ \limsup(1/n)\log|\mathcal{V}_n| \leq R\}$$

Proposition. For each $0 < \delta < 1$,

$$\lim_{\alpha\downarrow 1} \limsup_n \frac{1}{n} H_\alpha(\mu_n) \ \leq \ R^*(\delta) \ \leq \ \lim_{\alpha\uparrow 1} \limsup_n \frac{1}{n} H_\alpha(\mu_n).$$

If μ_n is an i.i.d. probability measure on $\mathcal{U}_n = \mathcal{U}^n$, then

$$R^*(\delta) = H(\mu_1), \qquad 0 < \delta < 1.$$

Abstract Alphabet and Communication

Let θ be such that

$$\mathrm{P}\left(\left\{y^k : \frac{d\,\mathrm{P}_{Y_1,\ldots,Y_k}}{d\prod_{i=1}^{k}\mathrm{P}_{Y_i}}(y^k) \leq \theta\right\}\right) \approx 1.$$

Then, there exists a query strategy q_0 for L given $\mathbf{F}$ such that

$$\mathrm{P}\left(q_0(L \mid \mathbf{F}) \lesssim \theta^{\frac{1}{k-1}}\right) > 0.$$

- Upper bound on $E^*(\epsilon)$ for jointly Gaussian rvs
- Strong converse for Gaussian secret key capacity

Summary

Main Result: $E^* = E^*(\epsilon) = C(\epsilon) = C$

- Largest rate SK makes the task of querying eavesdropper the most onerous.
- We proved a strong converse for the SK capacity,
- And a converse for general alphabet and communication.
- Rényi entropy can be interpreted as an answer to a lossless source coding problem.