How Many Queries Will Resolve Common Randomness?

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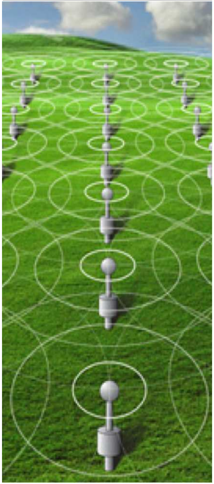
Joint work with Prakash Narayan

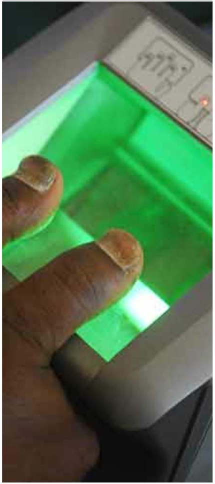
Common Randomness is Shared Bits

Sensor Networks

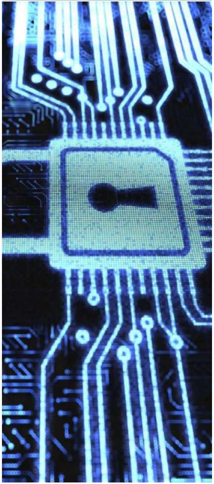


Cloud Computing

Biometric Security



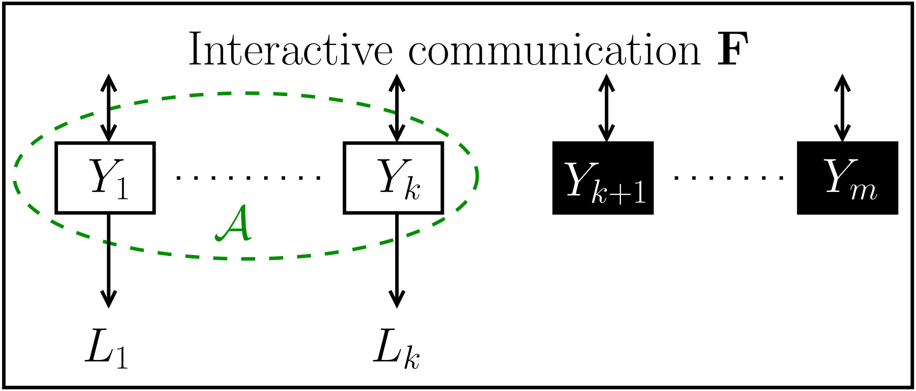
Hardware Security



Outline

1. Formulation and the main result
2. Strong converse for secret key capacity
3. Proof of the direct part
4. Proof of the converse

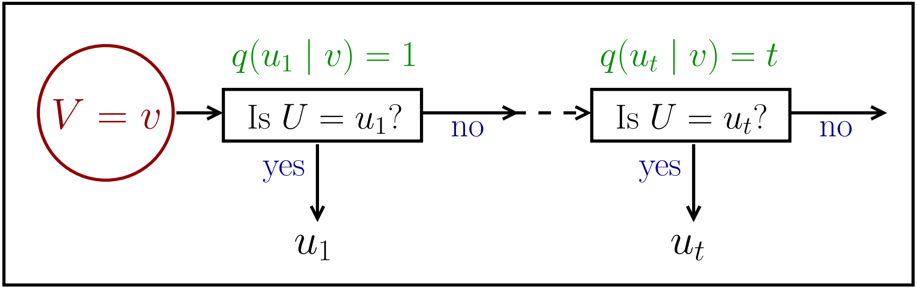
Common Randomness



Definition. L is an ϵ-common randomness for $\mathcal{A}$ from $\mathbf{F}$ if

$$\mathrm{P}\left(L = L_i(Y_i, \mathbf{F}),\ i \in \mathcal{A}\right) \geq 1 - \epsilon$$

Query Strategy



Query strategy for U given V

Massey '94, Arikan '96, Arikan-Merhav '99, Hanawal-Sundaresan '11

Query Strategy

Given rvs U, V with values in the sets $\mathcal{U}, \mathcal{V}$.

Definition. A *query strategy* q for U given $V = v$ is a bijection

$$q(\cdot|v) : \mathcal{U} \to \{1, ..., |\mathcal{U}|\},$$

where the querier, upon observing $V = v$, asks the question

"Is $U = u$?"

in the $q(u|v)^{\text{th}}$ query.

$q(U|V)$: random query number for U upon observing V

Optimum Query Exponent

$Y_i = (X_{i1}, ..., X_{in}) = X_i^n, \quad 1 \leq i \leq m$: i.i.d. observations

Definition. $E \geq 0$ is an ϵ-achievable *query exponent* if

there exists ϵ-CR L_n for $\mathcal{A}$ from $\mathbf{F}_n$ such that

$$\sup_q \mathrm{P}\left(q(L_n \mid \mathbf{F}_n) < 2^{nE}\right) \to 0 \quad \text{as} \quad n \to \infty,$$

where the $\sup$ is over every query strategy for L_n given $\mathbf{F}_n$.

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$$|\{u : q(u \mid v) < \gamma\}| < \gamma$$

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$$E^*(\epsilon) \triangleq \sup\{E : E \text{ is an } \epsilon\text{-achievable query exponent}\}$$

$$E^* \triangleq \inf_{0<\epsilon<1} E^*(\epsilon) : \text{ optimum query exponent}$$

Main Result

Theorem

For $0 < \epsilon < 1$, the optimum query exponent E^* equals

$$E^* = E^*(\epsilon) = H\left(X_{\mathcal{M}}\right) - \max_{\lambda \in \Lambda(\mathcal{A})} \sum_{B \in \mathcal{B}} \lambda_B H\left(X_B \mid X_{B^c}\right).$$

$$\mathcal{B} = \{B \subsetneq \mathcal{M} : B \neq \emptyset, \mathcal{A} \not\subseteq B\}$$

$\Lambda(\mathcal{A})$ = set of all $\{\lambda_B \in [0,1] : B \in \mathcal{B}\}$ such that

$$\sum_{B \in \mathcal{B} : B \ni i} \lambda_B = 1, \quad i \in \mathcal{M}$$

$\lambda \in \Lambda(\mathcal{A})$ is a *fractional partition* of $\mathcal{M}$

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For $m = 2$: The expression on the right $= I\left(X_1 \wedge X_2\right)$

Strong Converse for Secret Key Capacity

Secret Key Capacity

Definition. $C(\epsilon)$ is the supremum over rates of rv $K \in \mathcal{K}$ s.t.

(i) K is an ϵ-CR for $\mathcal{A}$ from $\mathbf{F}$

(ii) K is almost independent of $\mathbf{F}$:

$$n\, \mathrm{s}_{\mathrm{var}}(K; \mathbf{F}) = n \left\| \mathrm{P}_{K,\mathbf{F}} - \mathrm{U}_{\mathcal{K}} \times \mathrm{P}_{\mathbf{F}} \right\|_1 \to 0$$

Secret key capacity C is defined as $\inf_{0<\epsilon<1} C(\epsilon)$

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I. Csiszár and P. Narayan, Secret key capacity for multiple terminals, IEEE Trans. Inform. Theory, 2004.

Optimum Query Exponent and SK Capacity

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Achievability: $E^*(\epsilon) \geq C(\epsilon)$ - Easy

Converse: $E^*(\epsilon) \leq C$ - Main contribution

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Theorem (Strong converse for SK capacity)

For $0 < \epsilon < 1$, the ϵ-SK capacity is given by

$$C(\epsilon) = E^* = C.$$

Proof of Achievability

Query Strategies and Conditional Probabilities

Lemma. The rvs U, V, satisfy

$$\mathrm{P}\left(\left\{(u,v) : \mathrm{P}_{U|V}(u|v) \leq \frac{1}{\gamma}\right\}\right) \approx 1. \quad (*)$$

Then for every query strategy q for U given V,

$$\mathrm{P}\left(q(U|V) \geq \gamma\right) \approx 1.$$

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Also, the converse holds.

- $U =$ SK of rate R, $V = \mathbf{F} \Rightarrow (*)$ holds with $\gamma \approx 2^{nR}$

Proof of $C(\epsilon) \leq E^*(\epsilon)$

For an ϵ-SK K for $\mathcal{A}$ from $\mathbf{F}$ of rate $R = (1/n)\log|\mathcal{K}|$:

$$\mathrm{P}\left(\left\{(k, \mathbf{i}) : \mathrm{P}_{K|\mathbf{F}}\left(k \mid \mathbf{i}\right) > \tfrac{2}{\exp(nR)}\right\}\right)$$

$$\leq \mathbb{E}\left\{\left| \log|\mathcal{K}|\mathrm{P}_{K|\mathbf{F}}\left(K \mid \mathbf{F}\right)\right|\right\}$$

$$\leq \mathrm{s}_{\mathrm{var}}(K;\mathbf{F}) \log \tfrac{|\mathcal{K}|^2}{\mathrm{s}_{\mathrm{var}}(K;\mathbf{F})} \approx 0 \qquad [\because\ n\,\mathrm{s}_{\mathrm{var}}(K;\mathbf{F}) \to 0\ ]$$

For every query strategy q for K given $\mathbf{F}$

$$\mathrm{P}\left(q(K \mid \mathbf{F}) \geq 2^{nR}\right) \approx 1 \quad \Rightarrow \quad R \leq E^*(\epsilon)$$

Proof of Converse

Proof of Converse for $\mathcal{A} = \mathcal{M}$

Alternative Expression for C when $\mathcal{A} = \mathcal{M}$

[Csiszár-Narayan '04] observed that for $\mathcal{A} = \mathcal{M}$

$$C \leq \frac{1}{k-1} D\left(\mathrm{P}_{X_{\mathcal{M}}} \middle\| \prod_{i=1}^{k} \mathrm{P}_{X_{\pi_i}}\right),$$

for every partition $\pi = \{\pi_1, ..., \pi_k\}$ of $\mathcal{M}$.

Alternative Expression for C when $\mathcal{A} = \mathcal{M}$

[Chan-Zheng '10] showed that for $\mathcal{A} = \mathcal{M}$

$$C = \min_{\pi} \frac{1}{|\pi| - 1} D\left(\mathrm{P}_{X_{\mathcal{M}}} \middle\| \prod_{i=1}^{|\pi|} \mathrm{P}_{X_{\pi_i}} \right).$$

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We shall show

$$E^*(\epsilon) \leq E_\pi = \frac{1}{|\pi| - 1} D\left(\mathrm{P}_{X_{\mathcal{M}}} \middle\| \prod_{i=1}^{|\pi|} \mathrm{P}_{X_{\pi_i}} \right), \quad \text{for every } \pi.$$

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Roughly: For an ϵ-CR L for $\mathcal{M}$ from $\mathbf{F}$, there exists q_0 s.t.

$$\mathrm{P}\left(q_0(L \mid \mathbf{F}) \leq 2^{nE_\pi} \right) > 0, \quad \text{for every } \pi$$

A General Converse

For rvs $Y_1, \ldots, Y_k$, let L be an ϵ-CR for $\{1, \ldots, k\}$ from $\mathbf{F}$.

Theorem

Let θ be such that

$$\mathrm{P}\left(\left\{(y_1, \ldots, y_k) : \frac{\mathrm{P}_{Y_1,\ldots,Y_k}(y_1, \ldots, y_k)}{\prod_{i=1}^k \mathrm{P}_{Y_i}(y_i)} \leq \theta\right\}\right) \approx 1.$$

Then, there exists a query strategy q_0 for L given $\mathbf{F}$ such that

$$\mathrm{P}\left(q_0(L \mid \mathbf{F}) \lesssim \theta^{\frac{1}{k-1}}\right) \geq (1 - \sqrt{\epsilon})^2 > 0.$$

Proof of $E^*(\epsilon) \leq E_\pi$

Choose $Y_i = X^n_{\pi_i}$ for $i \in \{1, ..., k = |\pi|\}$.

Then, for n large it holds that

$$\mathrm{P}\left(\left\{(y_1, ..., y_k) : \frac{\mathrm{P}_{Y_1,...,Y_k}(y_1, ..., y_k)}{\prod_{i=1}^{k} \mathrm{P}_{Y_i}(y_i)} \leq \theta_n\right\}\right) \approx 1$$

with

$$(1/n)\log\theta_n \approx D\left(\mathrm{P}_{X_\mathcal{M}} \middle\| \mathrm{P}_{X_{\pi_1}} \times ... \times \mathrm{P}_{X_{\pi_k}}\right)$$

$$\Rightarrow\ \mathrm{P}\left(q_0(L \mid \mathbf{F}) \leq \theta_n^{\frac{1}{k-1}}\right) = \mathrm{P}\left(q_0(L \mid \mathbf{F}) \leq 2^{nE_\pi}\right) > 0$$

Using this for an ϵ-CR L that achieves a query exponent E:

$$E \leq E_\pi$$

Proof Outline for the General Converse

For rvs $Y_1, ..., Y_k$, let L be an ϵ-CR for $\{1, ..., k\}$ from $\mathbf{F}$.

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$$\mathrm{P}\left(q_0(L \mid \mathbf{F}) \lesssim \theta^{\frac{1}{k-1}}\right) > 0.$$

Small Cardinality Sets with Large Probabilities

We show: $\exists$ a subset $\mathcal{I}_0$ of values of $\mathbf{F}$ and sets $\mathcal{L}(\mathbf{i}) \subseteq \mathcal{L}$ s.t.

$$|\mathcal{L}(\mathbf{i})| \lesssim \theta^{\frac{1}{k-1}} \quad \text{and} \quad \mathrm{P}_{L|\mathbf{F}}\left(\mathcal{L}(\mathbf{i}) \mid \mathbf{i}\right) > 0, \quad \mathbf{i} \in \mathcal{I}_0$$

$$\mathrm{P}_{\mathbf{F}}\left(\mathcal{I}_0\right) > 0$$

Lossless Data Compression:

Find small cardinality sets with large $\mathrm{P}_{L|\mathbf{F}}$ probabilities

Small Cardinality Sets with Large Probabilities

Rényi entropy of order α of a probability measure μ on $\mathcal{U}$:

$$H_\alpha(\mu) \triangleq \frac{1}{1-\alpha} \log \sum_{u \in \mathcal{U}} \mu(u)^\alpha, \quad 0 \leq \alpha \neq 1$$

Lemma. There exists a set $\mathcal{U}_\delta \subseteq \mathcal{U}$ with $\mu\left(\mathcal{U}_\delta\right) \geq 1 - \delta$ s.t.

$$|\mathcal{U}_\delta| \lesssim \exp\left(H_\alpha(\mu)\right), \qquad 0 \leq \alpha < 1.$$

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Conversely, for any set $\mathcal{U}_\delta \subseteq \mathcal{U}$ with $\mu(\mathcal{U}_\delta) \geq 1 - \delta$,

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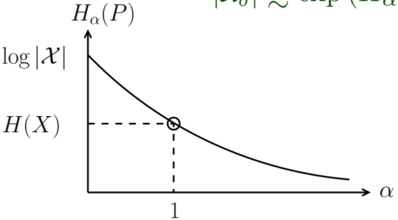

To Complete the Proof

$$\mathcal{E} \triangleq \left\{ (y_1, \ldots, y_k) : \frac{\mathrm{P}_{Y_1,\ldots,Y_k}(y_1, \ldots, y_k)}{\prod_{i=1}^{k} \mathrm{P}_{Y_i}(y_i)} \leq \theta \right\} \bigcap \{ \text{ no errors} \}$$

$$\mu(l) \triangleq \mathrm{P}\left(L = l, (Y_1, \ldots, Y_k) \in \mathcal{E} \mid \mathbf{F} = \mathbf{i}\right)$$

There exists $\mathcal{L}(\mathbf{i}) \subseteq \mathcal{L}$ with $\mu(\mathcal{L}(\mathbf{i})) \geq \mu(\mathcal{L}) - \delta$ and

$$|\mathcal{L}(\mathbf{i})| \lesssim \exp\left(H_{\frac{1}{k}}(\mu)\right) = \left(\sum_{l} \mu(l)^{\frac{1}{k}}\right)^{\frac{k}{k-1}}$$

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To Complete the Proof

Proof is completed using:

1. A change of measure argument
2. Structural properties of a CR L and interactive $\mathbf{F}$
3. Hölder's inequality

$$|\mathcal{L}(\mathbf{i})| \lesssim \exp\left(H_{\frac{1}{k}}(\mu)\right) = \left(\sum_l \mu(l)^{\frac{1}{k}}\right)^{\frac{k}{k-1}} \lesssim \theta^{\frac{1}{k-1}} \quad : \text{To show}$$

Abstract Alphabet and Communication

Let θ be such that

$$\mathrm{P}\left(\left\{y^k : \frac{d\,\mathrm{P}_{Y_1,\ldots,Y_k}}{d\,\prod_{i=1}^k \mathrm{P}_{Y_i}}(y^k) \leq \theta\right\}\right) \approx 1.$$

Then, there exists a query strategy q_0 for L given $\mathbf{F}$ such that

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Abstract Alphabet and Communication

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- Upper bound on $E^*(\epsilon)$ for jointly Gaussian rvs
- Strong converse for Gaussian secret key capacity

Small Cardinality Sets with Large Probabilities

Let μ be a probability measure on $\mathcal{U}$.

Lemma. There exists a set $\mathcal{U}_\delta \subseteq \mathcal{U}$ with $\mu\left(\mathcal{U}_\delta\right) \geq 1-\delta$ s.t.

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Conversely, for any set $\mathcal{U}_\delta \subseteq \mathcal{U}$ with $\mu\left(\mathcal{U}_\delta\right) \geq 1-\delta$,

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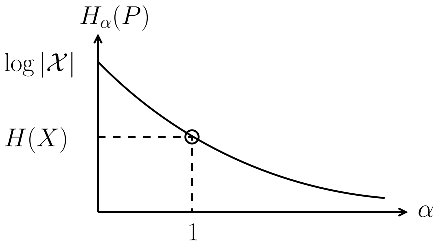

Lossless Source Coding

Given probability measures μ_n on finite sets $\mathcal{U}_n$, $n \geq 1$.

$$R^*(\delta) \triangleq \inf\{R : \ \mu_n(\mathcal{V}_n) \geq 1-\delta, \ \limsup (1/n) \log |\mathcal{V}_n| \leq R\}$$

Proposition. For each $0 < \delta < 1$,

$$\lim_{\alpha \downarrow 1} \ \limsup_n \frac{1}{n} H_\alpha(\mu_n) \ \leq \ R^*(\delta) \ \leq \ \lim_{\alpha \uparrow 1} \ \limsup_n \frac{1}{n} H_\alpha(\mu_n).$$

If μ_n is an i.i.d. probability measure on $\mathcal{U}_n = \mathcal{U}^n$, then

$$R^*(\delta) = H(\mu_1), \qquad 0 < \delta < 1.$$

Summary

Main Result: $E^* = E^*(\epsilon) = C(\epsilon) = C$

- Largest rate SK makes the task of querying eavesdropper the most onerous.
- We proved a strong converse for the SK capacity,
- And a converse for general alphabet and communication
- Rényi entropy can be interpreted as an answer to a lossless source coding problem.

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H. Tyagi and P. Narayan, How many queries will resolve common randomness?, Accepted, IEEE Transactions on Information Theory, 2013.

Common Randomness Principles For Secrecy

- Secure function computation (with public discussion)
- Interactive common information and secret keys
- Querying eavesdroppers and secret keys

Extra Slides

Proof Outline: Remaining Steps

$$\mathcal{E}_{\mathbf{i},l} \triangleq \left\{ \frac{\mathrm{P}_{Y^k}\left(Y^k\right)}{\prod_{i=1}^{k} \mathrm{P}_{Y_i}\left(Y_i\right)} \leq \theta \right\} \bigcap \left\{ \text{no errors, } \mathbf{F} = \mathbf{i}, L = l \right\}$$

Step 1. Change of measure

Let $\tilde{\mathrm{P}}_{Y_1 \ldots, Y_k}\left(y_1, .., y_k\right) \triangleq \prod_{i=1}^{k} \mathrm{P}_{Y_i}\left(y_i\right)$. For $y^k \in \mathcal{E}_{\mathbf{i},l}$

$$\mathrm{P}_{Y^k|\mathbf{F}}\left(y^k \mid \mathbf{i}\right) \leq \frac{\theta \tilde{\mathrm{P}}_{Y^k}\left(y^k\right)}{\mathrm{P}_{\mathbf{F}}\left(\mathbf{i}\right)} < \frac{\theta \tilde{\mathrm{P}}_{Y^k|\mathbf{F}}\left(y^k \mid \mathbf{i}\right)}{\delta}$$

where the last inequality is valid for $\mathbf{i}$ in the set in

$$\mathrm{P}_{\mathbf{F}}\left(\left\{\mathbf{i} : \mathrm{P}_{\mathbf{F}}\left(\mathbf{i}\right) > \delta \tilde{\mathrm{P}}_{\mathbf{F}}\left(\mathbf{i}\right)\right\}\right) \geq 1 - \delta$$

Proof Outline: Remaining Steps

Step 2. Property of interactive $\mathbf{F}$

$$\tilde{\mathrm{P}}_{Y^k|\mathbf{F}}\left(y^k \mid \mathbf{i}\right) = \prod_{j=1}^{k} \tilde{\mathrm{P}}_{Y_j|\mathbf{F}}\left(y_j \mid \mathbf{i}\right)$$

Therefore,

$$\begin{aligned}
\mu(l) &\triangleq \mathrm{P}_{Y^k|\mathbf{F}}\left(\mathcal{E}_{\mathbf{i},l} \mid \mathbf{i}\right) \\
&\leq \frac{\theta}{\delta} \tilde{\mathrm{P}}_{Y^k|\mathbf{F}}\left(\mathcal{E}_{\mathbf{i},l} \mid \mathbf{i}\right) = \frac{\theta}{\delta} \sum_{y^k \in \mathcal{E}_{\mathbf{i},l}} \prod_{j=1}^{k} \tilde{\mathrm{P}}_{Y_i|\mathbf{F}}\left(y_i \mid \mathbf{i}\right) \\
&\leq \frac{\theta}{\delta} \prod_{j=1}^{k} \tilde{\mathrm{P}}_{Y_j|\mathbf{F}}\left(\mathcal{E}_{\mathbf{i},l}^{j} \mid \mathbf{i}\right)
\end{aligned}$$

Proof Outline: Remaining Steps

Then, by Hölder's inequality

$$\left(\sum_l \mu(l)^{\frac{1}{k}}\right)^k \leq \frac{\theta}{\delta}\left(\sum_l \prod_{j=1}^k \tilde{\mathrm{P}}_{Y_j|\mathbf{F}}\left(\mathcal{E}^j_{\mathbf{i},l} \mid \mathbf{i}\right)^{\frac{1}{k}}\right)^k$$

$$\leq \frac{\theta}{\delta}\prod_{j=1}^k\left(\sum_l \tilde{\mathrm{P}}_{Y_j|\mathbf{F}}\left(\mathcal{E}^j_{\mathbf{i},l} \mid \mathbf{i}\right)\right)$$

Step 3. Property of L

The sets $\mathcal{E}^j_{\mathbf{i},l}$ are disjoint for different l and fixed $\mathbf{i}$

Hence, the term on the right above is less than (θ/δ)