Performance of a Fast, Distributed Multiple Access Based Relay Selection Algorithm Under Imperfect Statistical Knowledge

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***Abstract*—Cooperative wireless systems can exploit spatial diversity by opportunistically selecting the best relay to forward data to a destination. However, determining the best relay is a challenging task and requires a selection algorithm because the relays are geographically separated and only have local channel knowledge. Selecting the best relay is equivalent to finding the relay with the largest metric, where each relay computes its metric using local channel knowledge. We analyze the performance of a fast, distributed, and scalable multiple access based selection algorithm when it assumes incorrect values for two fundamental parameters that it requires to operate efficiently – the number of available relays and the cumulative distribution function (CDF) of the metrics. Such imperfect knowledge will invariably arise in practice. We develop new expressions for the time required to select the best relay as a function of the assumed and actual parameters. We show that imperfect knowledge can significantly slow down the selection algorithm. Further, in a system that uses its observations to update its CDF estimate, we determine the minimum number of observations required to limit the performance degradation. We also develop a minimax formulation that makes the algorithm robust to uncertainties in the number of relays in the system.**

***Index Terms*—Multiple access techniques, relays, cooperative communications, splitting algorithms, diversity, selection, cross-layer optimization, distributed, estimation.**

I. Introduction

RELAY-AIDED cooperative communications exploits spatial diversity by using geographically separated relays to forward information from a source to a destination. It is considered to be a promising technology for future wireless systems, and is likely to be employed in practical standards such as Long Term Evolution Advanced (LTE-A) [1], [2], and IEEE 802.16j and 802.16m WiMAX [3], [4]. When multiple relays are available, selection of a single best relay based on the channel conditions has emerged as a practically appealing solution for harnessing spatial diversity without having to maintain tight synchronization across the relays [5]–[9]. It is being investigated for applications such as wireless sensor networks for industrial automation and for reliable short-range communications [10].

Implementing selection in a multi-relay system requires tackling a new set of challenges because the relays are geographically separated from each other. This is because a relay cannot know whether it is the best relay on the basis of its local channel knowledge. Therefore, a selection algorithm becomes essential to select the best relay. Altogether, two aspects drive the selection process, namely, the criterion that determines which relay is the best and the selection algorithm. We first discuss two common examples below to illustrate criteria based on which a relay is selected.

Example 1: Consider a two-hop cooperative network that uses amplify-and-forward (AF) relays. The signal-to-noise (SNR) ratio at the destination is proportional to the summation of the source-to-destination channel power gain and half the harmonic mean of the source-to-relay (SR) and relay-to-destination (RD) channel power gains [5]. Therefore, the relay with the highest harmonic mean of the SR and RD channel power gains must be selected.

Example 2: Now consider a network that instead uses decode-and-forward (DF) relays. Then, among the relays that have decoded the source's message, the one with the largest RD channel power gain must be selected in order to maximize the transmission rate to the destination or minimize energy consumption by the transmitting relay for a given target data rate [11], [12].

In general, each relay computes a real-valued *metric*, which is a function of its local channel gains and the specific cooperative protocol used for data transmission, and the goal of the selection algorithm is to find the relay with the highest metric. In the AF example above, the metric equals the harmonic mean of the SR and RD channel power gains. Similarly, in the DF example above, the metric equals the RD channel power gain if the relay has decoded the source's message and is zero, otherwise [11]. Note that in addition to the above two examples, other metrics for cooperative communication protocols have been considered in the literature. See, for example, [12], [13] and references therein. After selection, cooperative data transmission takes place using the selected relay. Such selection needs to be carried out once in every coherence interval, within which the channel gains do not change appreciably.

One simple example of a selection algorithm is polling. In

Manuscript received January 26, 2011; revised May 12, 2011; accepted July 22, 2011. The associate editor coordinating the review of this paper and approving it for publication was C.-C. Chong.

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V. Shah and D. Bethanabhotla were at IISc during the course of this work.

A part of this work has appeared in the National Conf. on Communications (NCC), Guwahati, India, Jan. 2009.

This work is partially supported by a research grant from the Advanced Networking Research Consortium (ANRC).

Digital Object Identifier 10.1109/TWC.2011.081011.110183

it, each relay sequentially transmits its metric to a centralized scheduler, which then selects the best relay. While polling is simple, it is not scalable as the time required by it to select the best relay increases linearly with the number of available relays. However, distributed selection algorithms such as the splitting based selection algorithm [14]–[16] and the timer based backoff algorithm [6], [17]–[19] effectively circumvent this problem.

In this paper, we focus on the splitting based selection algorithm, which is based on multiple access principles and is time-slotted. The splitting algorithm is attractive because it is provably fast and scalable. It was proposed for wireless systems in [14] and was subsequently refined and generalized in [15]. In it, only relays whose metrics lie between two thresholds transmit in a slot. At the end of each slot, the sink broadcasts a three-state feedback to all the relays informing them whether zero (idle), one (success), or multiple relays (collision) transmitted. The thresholds are updated based on this feedback. When only one relay transmits in a slot, the design of the algorithm guarantees that it is the best relay. The algorithm then terminates.

The splitting based selection algorithm requires fewer than 2.507 slots, on average, to select the best relay even when the number of available relays tends to infinity [14]. Consequently, a greater portion of the coherence interval can be devoted to relay-aided data transmission, which increases the overall system throughput [11]. Note that the splitting algorithm for selection fundamentally differs from its conventional multiple access control counterparts, e.g., the first come first serve (FCFS) algorithm [20], because the goal of the latter is to enable *all the relays* to access the channel one by one.

The splitting algorithm turns out to be fast and scalable because it updates the thresholds to ensure that about one relay, on average, transmits in a slot [14]–[16]. In order to ensure this, the algorithm requires the knowledge of: (i) the number of relays in the system and (ii) the cumulative distribution function (CDF) of the metrics of the relays. We shall refer to these as the *parameters* of the algorithm. However, in practice, the relay count that the algorithm assumes may be inaccurate because relays may enter or leave the system. Alternately, the relays may go to sleep and become unavailable temporarily.

Similarly, knowing the CDF perfectly is difficult because the wireless channel statistics are different for different environments, and may even change after a sufficiently large time has elapsed [21]. In such a case, the network has two options available to it: (i) Use a 'factory preset' CDF and relay count setting in all the relays. This simplistic approach is preferable in low complexity networks. However, the selection algorithm's performance degrades when the actual parameters encountered in the field are different; it also offers no guarantees about performance. (ii) Estimate the CDF during operation. The more the observations, the more accurate the estimate. However, the time-frequency resources required to collect the observations and to communicate with the relays in the network correspondingly increase.

A. Focus and Contributions

The key question that this paper addresses is the impact of imperfect parameter knowledge on the time required by the splitting based selection algorithm to select the best relay. This is a crucial step in determining the performance of the algorithm in a practical scenario where the statistics are either assumed (*i.e.*, factory preset) or are updated infrequently. We derive exact expressions for the probability distribution of the time required by the algorithm to select the best relay as a function of the assumed and actual values of the metric CDF and relay count. This also leads to an expression for the average time required by the algorithm to select.

We then consider approaches to reduce the performance degradation in a network that can update its statistics on the basis of the metrics observed by its relays. With the help of the analysis, we determine the minimum number of observations that should be used to estimate the CDF in order to ensure that the degradation in performance is within an acceptable limit. Both parametric and non-parametric CDF estimation techniques, which trade-off accuracy for generality, are investigated. For example, we find that with a parametric CDF estimator and Rayleigh fading, just 50 observations ensure that the average selection time is within 3% of the time required using perfect statistical knowledge. With a non-parametric estimator and the same number of observations, the increase in the average selection time is 5%, which is still small. A minimax formulation is also developed to make the algorithm's performance robust to uncertainties in the relay count.

B. Related Work

The performance of selection algorithms under imperfect knowledge has received relatively limited attention in the literature. For example, selection was assumed to be perfect and instantaneous in [5], [7], [22], [23]. The parameters of the splitting algorithm were assumed to be known perfectly in the analyses developed in [14], [15]. While [14] considered a scenario where the metrics change during the selection process, the CDF and the number of users are assumed to be perfectly known. A stochastic approximation algorithm was used in [24] to adjust the parameters of the splitting algorithm. However, it requires several thousand slots to converge and also incurs a marginal performance loss as it uses a simplified parametric model for the algorithm. Instead, our results show that estimating the CDF using the techniques considered in this paper require far fewer observations.

Remark 1: We do not delve into another reason behind imperfect selection, which is the inaccuracy in the metrics themselves. It occurs because the channel gains, on which the metrics depend, are estimated in the presence of noise and interference and may vary with time [25]. Since the relays' metrics now depend on the estimated channel gains, this can cause a sub-optimal relay – but still the one with the highest metric – to get selected. Its main impact is system-specific, *i.e.*, it affects the performance of cooperative data transmission using the selected relay [26], [27]. It affects the selection algorithm's performance only to the extent that it changes the statistics of the metric, which our general analysis handles.

The rest of the paper is organized as follows. Section II describes the system model and the selection algorithm. Impact of imperfect knowledge is analyzed in Sec. III. CDF estimation techniques and the accuracy required of them for the purposes of the selection algorithm are investigated in Sec. IV. A robust approach to handle imperfect relay counts is developed in Sec. V. Our conclusions follow in Sec. VI. Several mathematical proofs are relegated to the Appendix.

II. Splitting Based Selection Algorithm and System Model

Figure 1 shows a system with n relays and a sink. The sink refers to the source or destination or any other entity in the system that coordinates the selection process and determines the best relay. Each relay i generates a metric $\mu_i \in \mathbb{R}$ as a function of its local knowledge; μ_i is known only to relay i. The goal of the selection algorithm is to select the relay with the highest metric: $\arg\max_{i=1,\ldots,n} \mu_i$. The metrics of different relays are assumed to be independent and identically distributed (i.i.d.) random variables (RVs), as has also been assumed, for example, in [6], [9], [14], [15], [19], [24]. The actual CDF of the metric is denoted by F. Let F_{asm} denote the metric CDF assumed (asm.) by the relays, and let n_{asm} denote the relay count assumed.

The goal of the selection algorithm is to choose the relay with the highest (best) metric. The algorithm is time-slotted and defines two transmission thresholds $H_L[k]$ and $H_H[k]$ in each slot k. A relay transmits in a slot only if its metric lies in between these two thresholds. Also defined is a variable $H_{\min}[k]$, which is the smallest value that the highest metric can possibly take in the k^{th} slot. The thresholds are gradually lowered based on whether zero or multiple relays transmitted in a slot, until exactly one relay transmits.

A quasi-static channel model is assumed. The various channels are assumed to remain constant during the selection process and the subsequent cooperative data transmission phase using the selected relays. Therefore, the metrics are kept fixed during the course of the algorithm.

A. With Perfect Statistical Knowledge

To build intuition, we first consider the case with perfect statistical knowledge. For brevity, we shall refer to this as the *perfect knowledge case*. We define and then explain the selection algorithm when the metrics are uniformly distributed over $[0, 1]$. The case where the metrics follow a general continuous CDF, F, is discussed thereafter.[1]

1) Metrics Uniformly Distributed in $(0, 1)$: The selection algorithm is defined as follows:

Transmission rule: At the beginning of a slot k, a relay transmits if and only if its metric lies in between $H_L[k]$ and $H_H[k]$.

Feedback generation: At the end of each slot, the sink broadcasts to all relays a three-state (two bit) feedback: (i) *idle*, if no relay transmitted in the slot, (ii) *success*, if one relay transmitted and was, therefore, decoded by the sink, or (iii) *collision*, if at least two relays transmitted and, therefore, no transmission could be decoded by the sink.

[1]This model covers several common channel fading models such as Rayleigh, Rician, or Nakagami-m. The case of discontinuous CDFs, which occurs when the metrics take discrete values, can be handled in the same manner using a technique called Proportional Expansion [15]. In it, the users autonomously generate new metrics whose CDF is continuous.

Response to feedback at the end of the k^{th} slot:

- If feedback is an idle and no collisions have occurred thus far, then
$$H_H[k+1] = H_L[k],\ H_{\min}[k+1] = 0,\ \text{and}$$
$$H_L[k+1] = \left(H_L[k] - \frac{\zeta}{n}\right)^+ = \left(1 - \frac{(k+1)\zeta}{n}\right)^+, \quad (1)$$
where $(x)^+ \triangleq \max(x, 0)$.
- If feedback is a collision, then
$$H_H[k+1] = H_H[k],\ H_{\min}[k+1] = H_L[k],\ \text{and}$$
$$H_L[k+1] = \frac{H_L[k] + H_H[k]}{2}. \quad (2)$$
- If feedback is an idle and a collision has occurred in the past, then
$$H_H[k+1] = H_L[k],\ H_{\min}[k+1] = H_{\min}[k],\ \text{and}$$
$$H_L[k+1] = \frac{H_{\min}[k] + H_L[k]}{2}. \quad (3)$$
- If feedback is a success, then terminate.

Initialization ($k = 1$): Set $H_L[1] = 1 - \zeta/n$, $H_H[1] = 1$, and $H_{\min}[1] = 0$.

2) Brief explanation: The algorithm ensures that ζ users on average transmit in every slot until a collision occurs. We, therefore, call ζ as the *contention load parameter*. Intuitively, ζ should be close to 1; a smaller ζ results in too many slots being idle and a large ζ results in too many collisions [15]. In [14], $\zeta = 1$, which greedily maximizes the probability of success in the next slot, was used. However, it was shown in [15] that the optimal value of ζ that minimizes the average time required to select the best relay is 1.088.

At the beginning ($k = 1$), $H_{\min}[1] = 0$ since the highest metric can lie anywhere in between 0 and 1. The transmission thresholds cover an interval of length ζ near the largest possible value of the highest metric. In the case of an idle outcome, the thresholds are lowered. This is because idle outcomes until the k^{th} slot imply that the metrics of all the relays are less than $H_L[k]$. The $(\cdot)^+$ operation in (1) ensures that the lower threshold does not become negative.

In case a collision occurs in the k^{th} slot, it implies that the highest metric lies in the interval $(H_L[k], H_H[k])$. Further, there is at least one other metric that lies in the same interval. Therefore, $H_{\min}[k]$ is updated, the interval is split into two equal halves, and relays in the upper half transmit in the next slot [20, Chap. 4]. If a success occurs, then the one relay that transmitted in that slot is the best relay that is required to be selected. The feedback is assumed to be error-free. This is justifiable given its low payload, and is often assumed in several selection algorithms [14], [19], [20], [28].

We shall call the durations of the algorithm before and after the first non-idle slot as the *idle* and *collision* phases, respectively.

3) Generalization to Metrics with CDF F: The above algorithm can be easily generalized to handle the general case where the metric μ_i is not a uniform RV and has a continuous CDF F. Consider the RV
$$v_i = F(\mu_i). \quad (4)$$

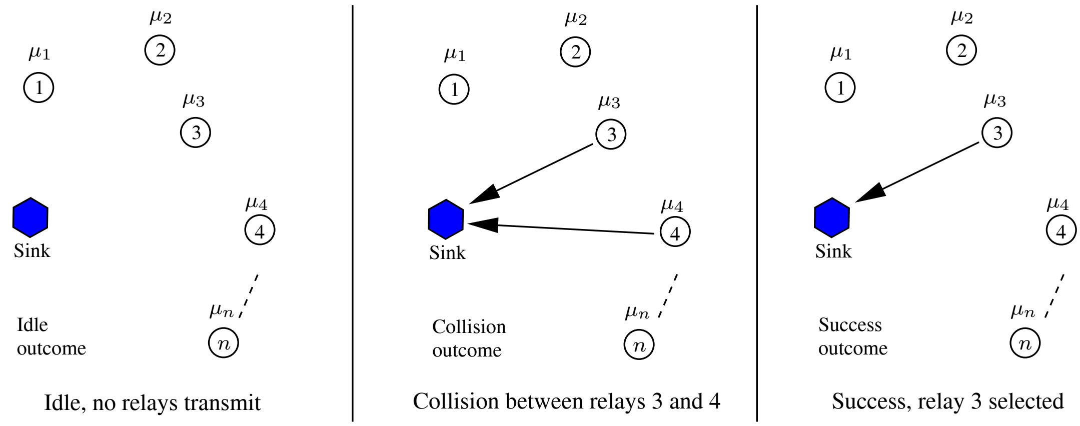

Fig. 1. A system consisting of a sink and n relays, in which the metric of relay i is μ_i. The figure illustrates how the n relays contend in the splitting based selection algorithm, how the three scenarios of idle, success, and collision arise over a duration of three slots, and how the relay with the highest metric gets selected.

It is uniformly distributed between 0 and 1 [29]. Further, since the CDF is a monotonically increasing function, the relay, say j, with the highest metric (μ_j) among all the relays also has the highest v_j. Thus, the users now participate in the splitting algorithm defined in Sec. II-A1 on the basis of v_i. Alternately, the updating of the thresholds as a function of feedback in the k^{th} slot as per (1), (2), and (3), and their initialization can be recast in terms of the metrics, the CDF, F, and its inverse, F^{inv}, as follows. The inverse exists because F is monotonically increasing and continuous.

- Idle and no collisions have occurred thus far: $H_H[k+1] = H_L[k]$, $H_{\min}[k+1] = 0$, and $H_L[k+1] = F^{\text{inv}}\left(\left(F(H_L[k]) - \frac{\zeta}{n}\right)^+\right) = F^{\text{inv}}\left(\left(1 - \frac{(k+1)\zeta}{n}\right)^+\right)$.
- Collision: $H_H[k+1] = H_H[k]$, $H_{\min}[k+1] = H_L[k]$, and $H_L[k+1] = F^{\text{inv}}\left(\frac{F(H_L[k])+F(H_H[k])}{2}\right)$.
- Idle and a collision has occurred in the past: $H_H[k+1] = H_L[k]$, $H_{\min}[k+1] = H_{\min}[k]$, and $H_L[k+1] = F^{\text{inv}}\left(\frac{F(H_{\min}[k])+F(H_L[k])}{2}\right)$.

The variables are initialized as: $H_{\min}[1] = 0$, $H_L[1] = F^{\text{inv}}\left(1 - \frac{\zeta}{n}\right)$, and $H_H[1] = F^{\text{inv}}(1) = \infty$. The transmission, feedback generation, and termination rules remain unchanged.

The above reformulation again ensures that ζ relays on average transmit in the idle phase. Further, in the event of a collision, half the relays that collided will transmit, on average, in the next slot. Notice that the threshold updates now depend on the CDF, F, and the number of relays, n. The splitting algorithm's variables are summarized in Table I.

TABLE I
KEY NOTATION USED IN THE PAPER

Symbol	*Description*
Splitting algorithm variables	
$H_L[k]$	Lower transmission threshold in k^{th} slot
$H_H[k]$	Upper transmission threshold in k^{th} slot
$H_{\min}[k]$	Minimum possible value of highest metric in k^{th} slot.
Metric statistics and relay count variables	
F	Actual metric CDF
F_{asm}	Assumed metric CDF
F^{inv}	Inverse of actual metric CDF
$F^{\text{inv}}_{\text{asm}}$	Inverse of assumed metric CDF
n	Actual number of relays
n_{asm}	Assumed number of relays
Performance analysis variables	
$C(t; n_{\text{asm}}, F_{\text{asm}}, \zeta)$	Probability that the best relay is selected within t slots when the number of relays is assumed to be n_{asm} and the metric CDF is assumed to be F_{asm}
$m(n_{\text{asm}}, F_{\text{asm}}, \zeta)$	Average number of slots required to select the best relay when the number of relays is assumed to be n_{asm} and the metric CDF is assumed to be F_{asm}

Remark 2: When the metrics of different relays are not statistically identical, all relays would still use the same F_{asm} in order to preserve the ordering of $v_1, \ldots, v_n$. For example, F_{asm} can be set as the identity function or it can be set as the average of the metric CDFs of all the relays. An analysis of this case is considerably more involved and is beyond the scope of this paper. The i.i.d. metrics model ensures analytical tractability. The reader is referred to [30] for a discussion about the problem of selection with heterogeneous users.

B. With Imperfect CDF Knowledge and Relay Count

Let the continuous CDF assumed by the relays be F_{asm} and let the relay count assumed be n_{asm}, when the actual CDF is F and the actual relay count is n. Let $F^{\text{inv}}_{\text{asm}}$ denote the inverse of F_{asm}. In this case, the relays would use $F^{\text{inv}}_{\text{asm}}$ and n_{asm} instead of F^{inv} and n in the splitting algorithm defined in Sec. II-A3. Therefore, $v_i = F_{\text{asm}}(\mu_i)$ is no longer uniformly distributed. Consequently, the algorithm can no longer ensure that ζ relays transmit, on average, in a slot or that half the users that collided transmit, on average, in the next slot.

III. ANALYSIS

It can be verified that even with imperfect knowledge, the splitting based selection algorithm described above is guaranteed to select the best relay so long as the support of F is a subset of the support of F_{asm}. This is a weak requirement on F_{asm}, and can be easily satisfied. Thus, imperfect knowledge does not make the selection algorithm fail catastrophically, which is an attractive attribute of the algorithm. However, it does increase the average number of slots required by the algorithm to select the best relay, as we shall see below.

Notation: The variable $C\left(t; n_{\text{asm}}, F_{\text{asm}}, \zeta\right)$ denotes the probability that the number of slots required to select the best relay is less than or equal to $t \in \mathbb{Z}^{+}$, when the CDF of the metric is assumed to be F_{asm} and the number of relays is assumed to be n_{asm}. Similarly, $m\left(n_{\text{asm}}, F_{\text{asm}}, \zeta\right)$ denotes the average number of slots required to select the best relay. The ceil function is denoted by $\lceil\cdot\rceil$. The indicator function $I_{\{\eta\}}$ is defined as follows: $I_{\{\eta\}} = 1$, if the condition η is true, and is 0, otherwise. The probability of an event A is denoted by $\Pr\left(A\right)$.

The elaborate notation above, which is summarized in Table I, is designed to bring out the role of imperfect statistics. We do not include the actual CDF (F) and the actual relay count (n) in the notation in order to keep it compact, even though they also affect the performance of the algorithm. The contention load parameter, ζ, is included in the notation as it leads to compact expressions in the analysis that follows below. Note that when the statistics are perfectly known, *i.e.*, $F_{\text{asm}} = F$ and $n_{\text{asm}} = n$, the above probability and mean simply become $C\left(t; n, F, \zeta\right)$ and $m\left(n, F, \zeta\right)$, respectively.

A. Reference Case: Perfectly Known CDF ($F_{asm} = F$) and User Count ($n_{asm} = n$)

We first derive an expression for $C\left(t; n_{\text{asm}}, F_{\text{asm}}, \zeta\right)$ when $F_{\text{asm}} = F$ and $n_{\text{asm}} = n$ (perfect knowledge). This is useful for two reasons. First, it serves as a benchmark. Second, its proof generalizes to the more difficult case with imperfect knowledge, which is handled next.

We first prove the following useful lemma about the probability that the algorithm requires exactly b slots to resolve a collision.

Lemma *1:* The probability $p(a, b)$ that exactly $b \geq 2$ more slots are required to select the best relay given that $a \geq 2$ relays collided in a slot is

$$p(a, b) = \frac{1}{2^a} p(a, b-1) + \frac{1}{2^a} \sum_{i=2}^{a} \binom{a}{i} p(i, b-1), \ b \geq 2, \tag{5}$$

where the recursion is initialized by $p(a, 1) = a/2^a$.

Proof: The proof is given in Appendix A. ■

The expression for the probability that the time required by the algorithm to select the best relay is less than or equal to t slots is then as follows.

Theorem *1:* With perfect knowledge of F and n, the probability that the algorithm selects the best relay within $t \in \mathbb{Z}^{+}$ slots is

$$\begin{aligned} C\left(t; n, F, \zeta\right) = & \sum_{i=1}^{\min(t,q)} \zeta \left(1 - \frac{i\zeta}{n}\right)^{n-1} \\ & + I_{\{t > q+1\}} \left(1 - \frac{q\zeta}{n}\right)^{n} \sum_{j=1}^{t-q-1} p(n, j) \\ + & \sum_{i=1}^{\min(t,q)-1} \sum_{k=2}^{n} \binom{n}{k} \left(\frac{\zeta}{n}\right)^{k} \left(1 - \frac{i\zeta}{n}\right)^{n-k} \sum_{j=1}^{t-i} p(k, j), \end{aligned} \tag{6}$$

where $q = \left\lceil \frac{n}{\zeta} \right\rceil - 1$.

Proof: The proof is given in Appendix B. ■

Remark 3: Hitherto, only an expression for the average number of slots required to select the best relay has been derived for the perfect knowledge case in [14], [15]. The above result is more powerful because it derives the probability distribution of the time required by the algorithm itself. From it, other measures such as mean and variance can also be easily computed.

Corollary *1:* With perfect knowledge of F and n, the average number of slots required to select the best relay is

$$m\left(n, F, \zeta\right) = 1 + \sum_{t=1}^{\infty} \left(1 - C\left(t; n, F, \zeta\right)\right), \tag{7}$$

where $C\left(t; n, F, \zeta\right)$ is given by (6).

Proof: The proof is given in Appendix C. ■

Substituting (6) in (7) gives an equivalent unwrapped version of the recursive equation in [15, (1)].

B. Impact of Imperfectly Known Relay Count n_{asm} (With Correct CDF, $F_{asm} = F$)

We start with the simple case in which the relay count used by the algorithm is incorrect. The CDF of the metric is assumed to be perfectly known ($F_{\text{asm}} = F$) in this section. The expressions for $C\left(t; n_{\text{asm}}, F, \zeta\right)$ and $m\left(n_{\text{asm}}, F, \zeta\right)$ are as follows.

Theorem *2:* The probability that the best relay is selected within t slots, $C\left(t; n_{\text{asm}}, F, \zeta\right)$, when the number of relays is assumed to be n_{asm} and the actual value is n, is

$$C\left(t; n_{\text{asm}}, F, \zeta\right) = C\left(t; n, F, \frac{n}{n_{\text{asm}}}\zeta\right). \tag{8}$$

Proof: The proof is given in Appendix D. ■

Corollary *2:* The average number of slots required to select the best relay, when the assumed number of relays is n_{asm} and the actual value is n, is

$$m\left(n_{\text{asm}}, F, \zeta\right) = m\left(n, F, \frac{n}{n_{\text{asm}}}\zeta\right). \tag{9}$$

Proof: The result directly follows from Theorem 2 and Corollary 1. ■

Thus, the CDF and average of the selection duration with imperfect relay count are the same as those for perfect relay count (Sec. III-A), but with a different contention load parameter.

C. Impact of Imperfectly Known CDF and Relay Count

We now consider the general case in which both the CDF and the relay count are not known correctly ($F_{\text{asm}} \neq F$ and $n_{\text{asm}} \neq n$). Analogous to the notation used in Sec. III-A, let $p(a, b; \ell, u)$ be the probability that exactly b more slots are required given that a relays transmitted in a slot. Further, let

$$\ell \triangleq F_{\text{asm}}\left(H_L\right) \quad \text{and} \quad u \triangleq F_{\text{asm}}\left(H_H\right), \tag{10}$$

where H_L and H_H are the lower and upper transmission thresholds in a slot. Note the inclusion of the transmission thresholds in $p(a, b; \ell, u)$ for the imperfectly known CDF case.

The following lemma characterizes $p(a, b; \ell, u)$ recursively.

Lemma *2:*

$$p(a,b;\ell,u) = p\left(a, b-1, \ell, \frac{u+\ell}{2}\right)(1-\beta(\ell,u))^a + \sum_{i=2}^{a}\binom{a}{i}(\beta(\ell,u))^i(1-\beta(\ell,u))^{a-i} p\left(i, b-1; \frac{u+\ell}{2}, u\right), \quad (11)$$

where $p(a,1;\ell,u) = a\beta(\ell,u)(1-\beta(\ell,u))^{a-1}$ and $\beta(\ell,u) = \frac{F^{\text{eq}}(u)-F^{\text{eq}}\left(\frac{\ell+u}{2}\right)}{F^{\text{eq}}(u)-F^{\text{eq}}(\ell)}$. Here, $F^{\text{eq}} = F \circ F_{\text{asm}}^{\text{inv}}$ is a composite function that is defined by $F^{\text{eq}}(x) = F(F_{\text{asm}}^{\text{inv}}(x))$.

Proof: The proof is relegated to Appendix E. ■

Thus, $p(a,1;\ell,u)$ depends on both the actual and estimated CDFs of the metric. Note that with perfect statistical knowledge, F^{eq} reduces to the identity function: $F^{\text{eq}}(x) = x$.

Theorem *3:* The probability that the algorithm selects the best relay within t slots when the CDF of the metric is assumed to be F_{asm} and the actual CDF is F, and the relay count is assumed to be n_{asm} when the actual relay count is n, is

$$\begin{aligned} C(t; n_{\text{asm}}, F_{\text{asm}}, \zeta) = & \sum_{i=1}^{\min(t,q')-1}\sum_{k=2}^{n}\left[\binom{n}{k}\left(F^{\text{eq}}\left(1-\frac{i\zeta}{n_{\text{asm}}}\right)\right)^{n-k}\right. \\ & \times\left(F^{\text{eq}}\left(1-\frac{(i-1)\zeta}{n_{\text{asm}}}\right) - F^{\text{eq}}\left(1-\frac{i\zeta}{n_{\text{asm}}}\right)\right)^k \\ & \left.\times\sum_{j=1}^{t-i} p\left(k, j, 1-\frac{i\zeta}{n_{\text{asm}}}, 1-\frac{(i-1)\zeta}{n_{\text{asm}}}\right)\right] \\ & + I_{\{t>q'+1\}}\left[F^{\text{eq}}\left(1-\frac{q'\zeta}{n_{\text{asm}}}\right)\right]^n \sum_{j=1}^{t-q'-1} p\left(n, j, 0, 1-\frac{q'\zeta}{n_{\text{asm}}}\right) \\ & + \sum_{i=1}^{\min(t,q')} n\left(F^{\text{eq}}\left(1-\frac{i\zeta}{n_{\text{asm}}}\right)\right)^{n-1} \\ & \times\left[F^{\text{eq}}\left(1-\frac{(i-1)\zeta}{n_{\text{asm}}}\right) - F^{\text{eq}}\left(1-\frac{i\zeta}{n_{\text{asm}}}\right)\right], \quad (12) \end{aligned}$$

where $q' = \left\lceil \frac{n_{\text{asm}}}{\zeta} \right\rceil - 1$ and F^{eq} is defined in Lemma 2.

Proof: The proof is given in Appendix F. ■

The average number of slots required to select the best relay then follows easily.

Corollary *3:* The average number of slots required by the selection algorithm when the CDF of the metric that is assumed by the relays is F_{asm} and the actual CDF is F is given by

$$m(n_{\text{asm}}, F_{\text{asm}}, \zeta) = 1 + \sum_{t=1}^{\infty}(1 - C(t; n_{\text{asm}}, F_{\text{asm}}, \zeta)), \quad (13)$$

where $C(t; n_{\text{asm}}, F_{\text{asm}}, \zeta)$ is given by (12).

Proof: The proof is similar to that in Appendix C, and is not repeated here. ■

Note that the variance and higher moments can also be easily evaluated from Theorem 3.

The following corollary, which directly follows from Theorem 3, shows how the problem with imperfect relay count and imperfect CDF is equivalent to a simpler problem with imperfect CDF but correct relay count.

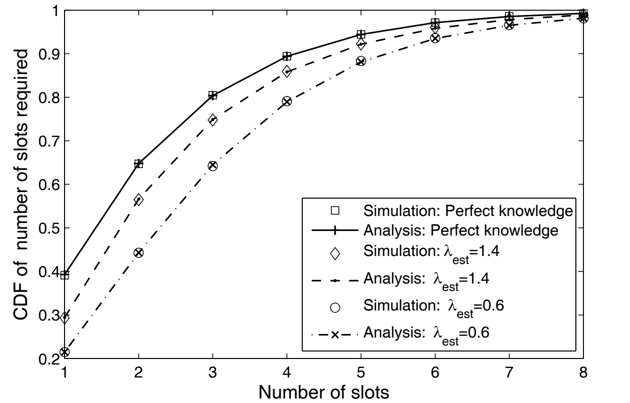

Fig. 2. CDF of the number of slots required to select the best relay given imperfect CDF knowledge ($\zeta = 1$ and $n = 8$).

Corollary *4:*

$$C(t; n_{\text{asm}}, F_{\text{asm}}, \zeta) = C\left(t; n, F_{\text{asm}}, \frac{n}{n_{\text{asm}}}\zeta\right), \quad (14)$$

$$m(n_{\text{asm}}, F_{\text{asm}}, \zeta) = m\left(n, F_{\text{asm}}, \frac{n}{n_{\text{asm}}}\zeta\right). \quad (15)$$

D. Numerical Results

We now graphically illustrate the impact of imperfect CDF on the selection algorithm's performance. As an example, we take the metric to have an exponential distribution, as is the case for the channel power gain of a frequency-flat Rayleigh fading channel. Its mean is taken to be unity. Therefore, the actual CDF is $F(t) = 1 - e^{-t}$, $t \geq 0$. Note that our analysis applies to other distributions as well. Let the assumed CDF in this example be $F_{\text{asm}}(t) = 1 - e^{-\lambda_{\text{est}} t}$, $t \geq 0$. The closer λ_{est} is to 1, the more accurate is the assumed CDF.

Figure 2 plots $C(t; n, F_{\text{asm}}, \zeta)$ for $n = 8$ relays. Observe that the curve shifts downwards compared to the one for the perfect knowledge case, which implies a degradation in performance. For example, the median time to select increases by 24% when $\lambda_{\text{est}} = 1.4$ and by 60% when $\lambda_{\text{est}} = 0.6$. Thus, the performance degradation can be significant. The corresponding average number of slots required to select the best relay is plotted in Fig. 3 as a function of the number of relays in the system for $\lambda_{\text{est}} = 1.2$. On account of imperfect CDF, once n exceeds 11, we observe that the algorithm requires more than 2.507 slots to select the best relay; this was the upper bound derived in [14] for the perfect knowledge case.

Figure 4 plots $C(t; n_{\text{asm}}, F, \zeta)$ when $n = 20$ relays are present in the system, but the number of relays is assumed to be either 10 or 30. The CDF of the metric, (which is assumed to be perfectly known in this example) is $F_{\text{asm}}(t) = F(t) = 1 - e^{-t}$, $t \geq 0$. Again, when $n_{\text{asm}} \neq n$, one observes a degradation in performance. For example, the median time required to select the best relay increases by 10% and 26% when n_{asm} is 30 and 10, respectively.

Figure 5 plots the average number of slots to select the best relay as a function of n, for $n_{\text{asm}} = (1 + 0.4)n$, which

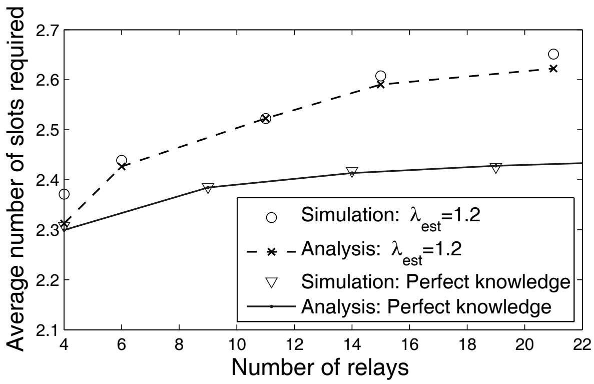

Fig. 3. Zoomed-in view of the average number of slots required to select the best relay given imperfect CDF knowledge ($\zeta = 1$).

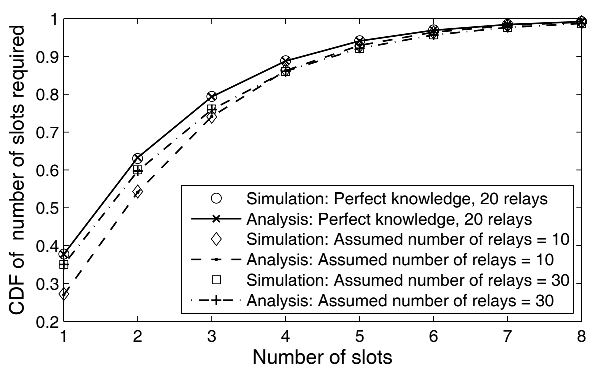

Fig. 4. CDF of the number of slots required to select the best relay when the assumed number of relays is incorrect ($\zeta = 1$ and $n = 20$).

is shown in the figure as '$+40\%$ error in n', and $n_{\text{asm}} = (1 - 0.4)n$, which is shown as '$-40\%$ error in n'. While the average number of slots increases, it saturates for larger n; this is unlike the case with imperfect CDF knowledge in Fig. 3. Note also that the analytical results match the Monte Carlo simulation results well in both the figures.

IV. On the Accuracy of CDF Estimate for Splitting Based Selection

The previous section derived a general expression for the time required to select the best relay as a function of any assumed CDF F_{asm}. This result is useful in a network in which F_{asm} is pre-specified. An alternate approach is for a network to update its estimate of the CDF, F_{asm}, on the basis of the realizations of the metric observed by the sink or relays or both. The sink then communicates F_{asm} to all the relays.[2] Increasing the number of observations improves the accuracy of the CDF estimate. However, it also increases the time required for estimation. In the following, we apply the analysis developed in Sec. III to determine the number of observations that are required to ensure that performance degradation of the algorithm lies within a pre-specified limit. This number also clearly depends on the CDF estimation technique used.

[2]Over a finite bandwidth broadcast channel, the sink can only communicate a discretized version of F_{asm} to all the relays. One solution that enables this is the following. The sink communicates the quantized versions of the observations to all the relays, which then use the same technique to estimate the CDF. We have observed that with 4 or more bits per observation, the effect of quantization is negligible. A detailed discussion and results are omitted due to space constraints.

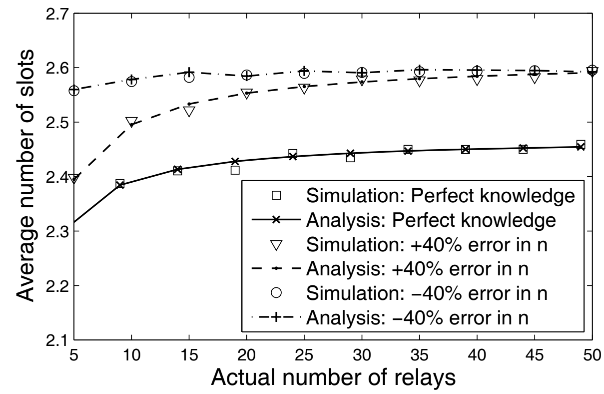

Fig. 5. Zoomed-in view of the average number of slots required to select the best relay when the assumed number of relays is incorrect ($\zeta = 1$). A '$+40\%$ error in n' means $n_{\text{asm}} = 1.4n$, while a '-40% error in n' means $n_{\text{asm}} = 0.6n$.

In general, the estimation techniques fall into two broad classes:

- In *parametric estimation*, the functional form of the CDF is assumed to be known a priori at the sink and the relays.
- In *non-parametric estimation*, no prior information about the CDF of the metric is assumed; the estimated CDF is a function of only the observations collected. Histogram, empirical CDF, kernel estimators, orthogonal series estimators, and restricted maximum likelihood density estimators are examples of non-parametric estimation methods [31]–[33].

So long as the assumed form of the CDF is correct, parametric estimation requires fewer observations than non-parametric estimation to estimate the CDF. However, non-parametric estimation is more general as it requires no prior information about the CDF.

We separately investigate parametric and non-parametric CDF estimation techniques below, and quantify their impact on the selection algorithm's performance. Given the vast literature on estimation, we focus on some common techniques.

A. Parametric Estimation

This is best understood by means of a case study. As before, let the metrics be i.i.d. exponentially distributed RVs. Thus, the actual CDF of the metrics is $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$, where λ^{-1} is the mean that is unknown and needs to be estimated. Let the CDF assumed be $F_{\text{asm}}(x) = 1 - e^{-\lambda_{\text{est}} x}$, $x \geq 0$. The maximum likelihood estimate, λ_{est}, of λ is the inverse of the empirical mean of the collected observations [34, Chap. IV].

Therefore, we get $F^{\text{eq}}(x) = F(F^{\text{inv}}_{\text{asm}}(x)) = 1 - (1-x)^{\frac{\lambda}{\lambda_{\text{est}}}}$, $0 \leq x \leq 1$. Substituting this in Theorem 3 yields the following analytical expression for the probability that the best relay is selected within t slots:

$$C\left(t; n, F_{\text{asm}}, \zeta\right) = \sum_{i=1}^{\min(t,q)} n \left(1 - \left(\frac{i\zeta}{n}\right)^{\frac{\lambda}{\lambda_{\text{est}}}}\right)^{n-1}$$

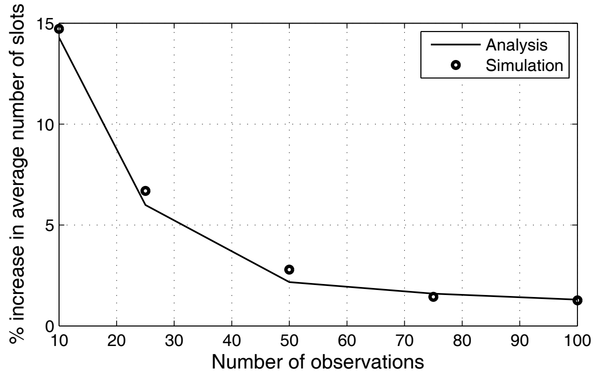

Fig. 6. Percentage increase in the average time required to select the best relay as a function of the number of observations used to estimate the CDF. Parametric estimation is used and the metrics are exponentially distributed RVs with unit mean ($\zeta = 1$ and $n = 10$).

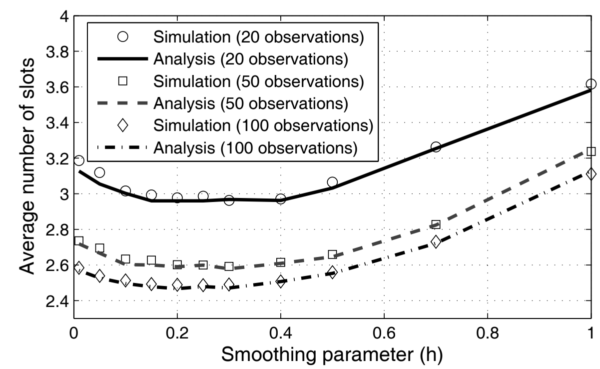

Fig. 7. Zoomed-in view of the average number of slots required to select the best relay as a function of the smoothing parameter h used for non-parametric estimation. Gaussian kernel estimator is used and the metrics are exponentially distributed random variables with unit mean ($\zeta = 1$ and $n = 10$).

$$\times \left[\left(\frac{i\zeta}{n}\right)^{\frac{\lambda}{\lambda_{\text{est}}}} - \left(\frac{(i-1)\zeta}{n}\right)^{\frac{\lambda}{\lambda_{\text{est}}}} \right]$$
$$+ \sum_{i=1}^{\min(t,q)-1} \sum_{k=2}^{n} \left[\binom{n}{k} \left(\left(\frac{i\zeta}{n}\right)^{\frac{\lambda}{\lambda_{\text{est}}}} - \left(\frac{(i-1)\zeta}{n}\right)^{\frac{\lambda}{\lambda_{\text{est}}}} \right)^k \right.$$
$$\left. \times \left(1 - \left(\frac{i\zeta}{n}\right)^{\frac{\lambda}{\lambda_{\text{est}}}}\right)^{n-k} \sum_{j=1}^{t-i} p\left(k, j, 1 - \frac{i\zeta}{n}, 1 - \frac{(i-1)\zeta}{n}\right) \right]$$
$$+ I_{\{t>q+1\}} \left(1 - \left(\frac{q\zeta}{n}\right)^{\frac{\lambda}{\lambda_{\text{est}}}}\right)^n \sum_{j=1}^{t-q-1} p\left(n, j, 0, 1 - \frac{q\zeta}{n}\right), \tag{16}$$

where $p(a, b, \ell, u)$ is given by Lemma 2 (with $\beta(\ell, u) = \frac{u^{\frac{\lambda}{\lambda_{\text{est}}}} - \left(\frac{\ell+u}{2}\right)^{\frac{\lambda}{\lambda_{\text{est}}}}}{u^{\frac{\lambda}{\lambda_{\text{est}}}} - \ell^{\frac{\lambda}{\lambda_{\text{est}}}}}$ in (11)). Note that corresponding expressions can also be derived when the metrics follow other probability distributions.

Figure 6 plots the percentage increase in the average number of slots as a function of the number of observations used to parametrically estimate the CDF. As the number of observations increases, the average number of slots required to select converges to that of the perfect knowledge case since λ_{est} converges almost surely to λ. With just 25 observations, the average number of slots required to select increases by 6% compared to the perfect knowledge case. The increase is just 3% when 50 observations are used for CDF estimation. Also, note the close match between the analytical and Monte Carlo simulation results.

B. Non-parametric Estimation

Given the many non-parametric estimation methods developed in the literature, we focus on the classical Kernel estimators. Let $x_1, x_2, \ldots, x_s$ be s independent observations of the metric with an unknown CDF $F(x)$. These are used to come up with an estimate $F_{\text{asm}}(x)$ of the CDF. This estimate is then used by the selection algorithm, as per Sec. II-B. Let f denote the (unknown) probability density function (PDF) of the metric.

The kernel estimate, $f_{\text{asm}}(x)$, of PDF $f(x)$ is given as

$$f_{\text{asm}}(x) = \frac{1}{sh} \sum_{i=1}^{s} K\left(\frac{x - x_i}{h}\right), \tag{17}$$

where the function K is called the *Kernel*, h is a smoothing parameter [32, Chap. 7], and f_{asm} denotes the assumed (estimated) PDF. Hence, the CDF estimate is given by

$$F_{\text{asm}}(x) = \int_{-\infty}^{x} f_{\text{asm}}(y)\, dy = \frac{1}{sh} \sum_{i=1}^{s} \int_{-\infty}^{x} K\left(\frac{y - x_i}{h}\right) dy. \tag{18}$$

The kernel function $K(y)$ is typically a smooth unimodal symmetric function with a peak at $y = 0$. As the number of observations increases, the estimated PDF converges to the actual PDF. One common example is a zero-mean unit-variance Gaussian kernel: $K(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$, $y \in \mathbb{R}$. A second example is the Epanechnikov kernel, which is defined as $K(y) = \frac{3}{4}(1 - y^2) I_{\{|y| \leq 1\}}$. It gives the smallest asymptotic integrated mean squared error (IMSE) [31]. A third example is the empirical CDF, which is obtained when $K(y) = \delta(y)$, where $\delta(y)$ denotes the Dirac-delta function.

An important point to note here is that for the Kernel estimator to work efficiently, the smoothing parameter h needs to be judiciously chosen, as is the case in several problems involving non-parametric estimation [33]. This becomes evident from Fig. 7, which plots the average number of slots required by the splitting algorithm to select the best relay as a function of h, when the Gaussian kernel is used. We choose the Gaussian kernel since it has an infinite support and yet performs almost as well as the Epanechnikov kernel [31]. We observe that the optimal value of h lies in between 0.2 and 0.4 regardless of the number of observations used. For $h = 0.3$ and 20 observations, the average time to select increases by 21.6% (compared to the perfect knowledge case). With 50 observations, the increase is just 5.1%. When the 10 relays make these observations in parallel, the 50 observations can be obtained in just 5 coherence intervals. Thus, updating the CDF is easily feasible.

Remark 4: When the linearly interpolated empirical CDF is used, the increase in the average number of slots is greater than for the Gaussian kernel estimator. For example, with

25 observations, the increase is $25.6\% > 21.6\%$. However, the empirical CDF does not face the challenging problem of having to judiciously choose a smoothing parameter [31].

V. On Accuracy of the Relay Count Estimate and Determining a Robust ζ

The analysis in Sec. III showed that the performance of the selection algorithm depends only on the ratio $\frac{n}{n_{\text{asm}}}$ and ζ. Even though the system may not know the actual relay count, it can be designed to handle n_{asm} lying within a range of the actual value, *i.e.*, $\frac{n}{n_{\text{asm}}} \in [\Delta_{\min}, \Delta_{\max}]$. In general, the larger the range, the more the uncertainty about the actual relay count. Clearly, $\Delta_{\min} = \Delta_{\max}$ corresponds to perfectly known relay count. It is, therefore, of interest to see if the selection algorithm's parameters can be chosen so as to make the algorithm's performance robust to this uncertainty.

We do this as follows. Using the analysis developed in Sec. III, we find a robust value of ζ that minimizes the worst case impact of the uncertainty on the average selection time. Mathematically, the problem can be formulated as follows:

$$\begin{aligned}\zeta^* &= \arg\min_{\zeta>0} \left[\max_{\frac{n}{n_{\text{asm}}} \in [\Delta_{\min}, \Delta_{\max}]} m\left(n_{\text{asm}}, F, \zeta\right) \right], \\ &= \arg\min_{\zeta>0} \left[\max_{\frac{n}{n_{\text{asm}}} \in [\Delta_{\min}, \Delta_{\max}]} m\left(n, F, \frac{n}{n_{\text{asm}}}\zeta\right) \right]. \quad (19)\end{aligned}$$

Here, (19) follows from Corollary 2. The expression for $m\left(n, F, \frac{n\zeta}{n_{\text{asm}}}\right)$ is given in (7).[3]

As can be seen in [15, Fig. 2], $m\left(n, F, \zeta\right)$ has a minimum at $\zeta = 1.088$. For $\zeta < 1.088$, $m\left(n, F, \zeta\right)$ increases as ζ decreases since more slots are wasted as idle slots. However, once ζ exceeds 1.088, $m\left(n, F, \zeta\right)$ increases as ζ increases because the number of collisions increases and the algorithm wastes more time resolving them. This observation implies that ζ^* is the solution of:

$$m\left(n, F, \Delta_{\min}\zeta^*\right) = m\left(n, F, \Delta_{\max}\zeta^*\right). \quad (20)$$

Intuitively, the worst case error occurs at the two extreme points, $\Delta_{\min}$ and $\Delta_{\max}$, of the uncertainty range. When $\zeta = \zeta^*$, the penalty becomes the same for the two extreme points. Any deviation from ζ^* increases the penalty for at least one of the extreme points and is sub-optimal. Substituting (7) in (20) and numerically solving (20) yields ζ^*.

We shall henceforth refer to ζ^* as the 'robust ζ'. Figure 8 plots the maximum increase in the average number of slots as a function of the maximum percentage error in the relay count when $n = 25$. Consider, for example, the case when the maximum error in n is $\pm 20\%$, *i.e.*, $\Delta_{\min} = 1/1.2 = 0.83$ and $\Delta_{\max} = 1/0.8 = 1.25$. In this case, the maximum increase in the average number of slots with ζ^* is 1.0%. If instead ζ is set as 1.088, which is the optimal value if n is known perfectly and is large, the maximum increase is 1.7%. Similarly, for a $\pm 50\%$ uncertainty range for n, the maximum increase in the average number of slots is 10.5% for $\zeta = \zeta^*$ as compared to 15.0% for $\zeta = 1.088$.

[3]The problem can be simplified even further by approximating $m\left(n, F, \frac{n}{n_{\text{asm}}}\zeta\right)$ with its asymptote $m\left(\infty, F, \frac{n}{n_{\text{asm}}}\zeta\right)$. This approximation is accurate for n as small as 10 [15].

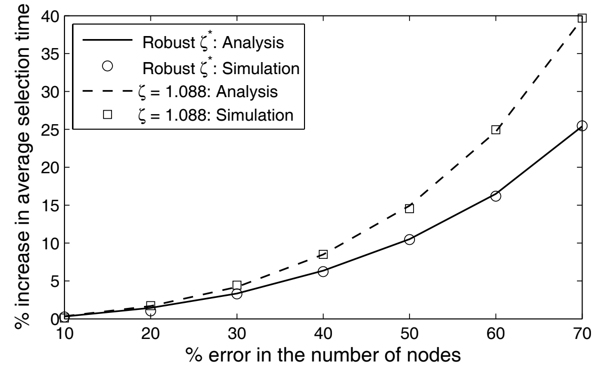

Fig. 8. Maximum percentage increase in the average time required to select the best relay as a function of the maximum percentage error in the relay count when the robust ζ^* is used ($n = 25$). Also shown is the degraded performance when the value of ζ is set to 1.088, which is the optimal value for perfect knowledge of relay count.

VI. Conclusions

Selection algorithms are essential in multi-relay wireless systems such as cooperative communication systems, in which the relays are geographically distributed and possess only local channel knowledge. The splitting based selection algorithm, which is based on multiple access principles, is appealing because it is fast and scalable. In this paper, we analyzed the impact of imperfect knowledge of two fundamental parameters required by the selection algorithm. The analysis led to expressions for the probability that the algorithm selects the best relay within a pre-specified time and, consequently, the average time required to select the best relay.

We also studied a system that updates its statistical knowledge, and developed a direct correspondence between the number of observations used to estimate the CDF of the metric and the degradation in the algorithm's performance. We saw that the selection time increases by just 3% for parametric estimation and by just 5% for non-parametric estimation with only 50 observations for exponentially distributed metrics. Thus, two orders of magnitude fewer samples are required than what has been considered possible using the stochastic approximation based parameter tuning approach pursued in [24]. Even with an uncertainty range of 50% in the number of relays, the worst case increase in the average selection time of the robust version of the algorithm was just 10.5%. Thus, the splitting based selection can work satisfactorily in a practical deployment, and ensures that a significant portion of the coherence time is available for cooperative data transmission using the selected relay.

The results in this paper motivate a further study of the performance of other selection algorithms such as the timer backoff based selection algorithm in cooperative systems. Interesting problems include a generalization of the techniques developed in this paper to include feedback errors and the performance evaluation of splitting algorithms for multiple access control with imperfect parameter knowledge. While the paper focused on cooperative systems, selection arises in several other wireless systems such as cellular systems, wireless local area networks, and wireless sensor networks. A specialized performance evaluation of the impact of imperfect knowledge on the overall performance of these wireless systems is another interesting avenue for future work.

Appendix

A. Proof of Lemma 1

If $a \geq 2$ relays are involved in a collision, then the probability that i relays, which have i.i.d. uniformly distributed metrics, transmit in the next slot is equal to $\binom{a}{i}\frac{1}{2^a}$. This is because the interval is split into lower and upper half intervals, and only relays in the upper half interval transmit. Thus, $p(a,1)$, which is the probability that exactly one relay transmits in the next slot, equals $a/2^a$.

For $b > 1$, the following three mutually exclusive cases arise in calculating $p(a,b)$:

1) The next slot is idle: This happens when no relay lies in the upper half interval. The probability of its occurrence is $\binom{a}{0}/2^a$. Further, it also implies that all the a relays must lie in the lower half interval, which is split in subsequent slots. In such a case, the best relay must be found from the a relays in *exactly* $b-1$ slots, which occurs with probability $p(a,b-1)$.
2) The next slot is a collision among i relays: This occurs with probability $\binom{a}{i}/2^a$. The best relay needs to be found from among the i relays in exactly $b-1$ slots, which happens with probability $p(i,b-1)$.
3) The next slot is a success: In this case, the algorithm has already terminated in fewer than b slots. This case, therefore, does not contribute to $p(a,b)$.

Using the law of total probability, we get the desired result in (5).

B. Proof of Theorem 1

Since the thresholds are lowered by ζ/n after every idle slot, the idle phase duration can never exceed $q+1 = \left\lceil \frac{n}{\zeta} \right\rceil$ slots. Consider first the case where $t \geq q+1$. Thus, $\min(t,q) = q$. The probability that the first non-idle slot is the i^{th} slot and a success occurs in it (*i.e.*, one node transmits) is $\zeta\left(1-\frac{i\zeta}{n}\right)^{n-1}$. If $k \geq 2$ relays collide instead in the first non-idle slot (i^{th} slot and $i \leq q$), the probability that the collision is resolved in any of the remaining $t-i$ slots is $\sum_{j=1}^{t-i} p(k,j)$. This is because $p(a,b)$, by definition, is the probability that exactly b slots are required to resolve a collision among a relays. The probability that the first non-idle slot is the i^{th} slot and k relays collided in it is $\binom{n}{k}\left(\frac{\zeta}{n}\right)^k\left(1-\frac{i\zeta}{n}\right)^{n-k}$, for $i \leq q$.

In the $(q+1)^{\text{th}}$ slot, all the n relays in the system must necessarily transmit since the lower threshold, $H_L[q+1]$, becomes 0 (as per (1)). Further, no relay must have transmitted in an earlier slot. Therefore, the probability that the $(q+1)^{\text{th}}$ slot is the first non-idle slot is $\left(1-\frac{q\zeta}{n}\right)^n$. In this case, the best relay must be found within $t-(q+1)$ slots, which occurs with probability $\sum_{j=1}^{t-q-1} p(n,j)$.

When $t < q+1$, only the cases considered above in which the first non-idle slot occurs on or before the $\min(t,q)$ slot need to be considered. Hence, in either case, the desired result, which is written in a compact form using the indicator function, follows from the law of total probability.

C. Proof of Corollary 1

Consider a non-negative integer-valued random variable X. We know that

$$X = \sum_{t=0}^{\infty} I_{\{X>t\}}. \tag{21}$$

Taking expectation on both sides yields

$$\mathbf{E}[X] = \sum_{t=0}^{\infty} \mathbf{E}\left[I_{\{X>t\}}\right] = \sum_{t=0}^{\infty}(1-\Pr(X \leq t)).$$

In our problem, X is the number of slots required by the algorithm to select the best relay. Substituting the expression for $\Pr(X \leq t)$, which is derived in (6), in (21) gives the desired result.

D. Proof of Theorem 2

During the idle phase, the lower thresholds for the first i idle slots are given by $H_L[1] = F^{\text{inv}}\left(1-\frac{\zeta}{n_{\text{asm}}}\right)$, $H_L[2] = F^{\text{inv}}\left(1-\frac{2\zeta}{n_{\text{asm}}}\right), \ldots, H_L[i] = F^{\text{inv}}\left(1-\frac{i\zeta}{n_{\text{asm}}}\right)$. Given that the first non-idle slot is the i^{th} slot and k relays collide in it, the probability that the best relay gets selected in any of the remaining $t-i$ slots is $\sum_{j=1}^{t-i} p(k,j)$. The probability that the first non-idle slot is the i^{th} slot and k relays collide in it is $\binom{n}{k}\left(\frac{\zeta}{n_{\text{asm}}}\right)^k\left(1-\frac{i\zeta}{n_{\text{asm}}}\right)^{n-k}$, for $i \leq q'$. Here, $q' = \left\lceil\frac{n_{\text{asm}}}{\zeta}\right\rceil - 1$. The probability that the first non-idle slot is the $(q'+1)^{\text{th}}$ slot and k relays transmit in it is $\left(1-\frac{q'\zeta}{n_{\text{asm}}}\right)^n$, for $k=n$, and is 0, otherwise. Hence, the expression is the same as that in Theorem 1, with ζ replaced by $\frac{n}{n_{\text{asm}}}\zeta$.

E. Proof of Lemma 2

The threshold update rules in Sec. II-B ensure that, among all the relays that collide in the interval $(H_L, H_H) = \left(F_{\text{asm}}^{\text{inv}}(\ell), F_{\text{asm}}^{\text{inv}}(u)\right)$, only those relays whose metrics lie in the interval $\left(F_{\text{asm}}^{\text{inv}}\left(\frac{F_{\text{asm}}(H_L)+F_{\text{asm}}(H_H)}{2}\right), H_H\right) = \left(F_{\text{asm}}^{\text{inv}}\left(\frac{\ell+u}{2}\right), F_{\text{asm}}^{\text{inv}}(u)\right)$ transmit in the next slot. Let $\beta(\ell,u)$ denote the probability that a relay i transmits in the next slot, given that it was just involved in a collision. It is given by

$$\beta(\ell,u) = \frac{\Pr\left(\mu_i \in \left(F_{\text{asm}}^{\text{inv}}\left(\frac{\ell+u}{2}\right), F_{\text{asm}}^{\text{inv}}(u)\right)\right)}{\Pr\left(\mu_i \in \left(F_{\text{asm}}^{\text{inv}}(\ell), F_{\text{asm}}^{\text{inv}}(u)\right)\right)}. \tag{22}$$

Since the actual CDF of μ_i is F, (22) simplifies to

$$\beta(\ell,u) = \frac{F(F_{\text{asm}}^{\text{inv}}(u)) - F(F_{\text{asm}}^{\text{inv}}(\frac{\ell+u}{2}))}{F(F_{\text{asm}}^{\text{inv}}(u)) - F(F_{\text{asm}}^{\text{inv}}(\ell))},$$
$$= \frac{F^{\text{eq}}(u) - F^{\text{eq}}\left(\frac{\ell+u}{2}\right)}{F^{\text{eq}}(u) - F^{\text{eq}}(\ell)}, \tag{23}$$

where $F^{\text{eq}} = F \circ F_{\text{asm}}^{\text{inv}}$. Since the metrics are i.i.d., the probability that $0 \leq i \leq a$ relays transmit in the next slot is $\binom{a}{i}\beta(\ell,u)^i(1-\beta(\ell,u))^{a-i}$.

For $b=1$, $p(a,1,\ell,u)$ is simply the probability that one relay out of a relays transmits in the next slot. Therefore,

$$p(a,1,\ell,u) = a\beta(\ell,u)(1-\beta(\ell,u))^{a-1}.$$

Now, consider $b > 1$. Given that a relays collided, the following three mutually exclusive events are possible in the next slot:

- An idle occurs with probability $(1-\beta(\ell,u))^a$. In this case, the best relay is to be selected from among the a relays, which lie in the interval $\left(F_{\text{asm}}^{\text{inv}}(\ell), F_{\text{asm}}^{\text{inv}}\left(\frac{\ell+u}{2}\right)\right)$, in exactly $(b-1)$ slots. This occurs with probability $p\left(a, b-1, \ell, \frac{\ell+u}{2}\right)$.
- A success occurs, in which case the probability that a success occurs again exactly $b-1$ slots later is clearly 0.
- A collision occurs among $i \geq 2$ relays in the next slot, in which case the best relay is to be selected from among the i relays that lie in the interval $\left(F_{\text{asm}}^{\text{inv}}\left(\frac{\ell+u}{2}\right), F_{\text{asm}}^{\text{inv}}(u)\right)$ in $b-1$ slots. This occurs with probability $p\left(i, b-1, \frac{\ell+u}{2}, u\right)$.

Hence, the desired result follows from the law of total probability.

F. Proof of Theorem 3

Note that the probability that a given relay's metric lies in the interval (τ_1, τ_2) is $F(\tau_2) - F(\tau_1)$. Let $t \geq q' + 1$. For $i \leq q' < t$, let the first non-idle slot be the i^{th} slot and let $k \geq 1$ relays transmit in it. Then the metrics of the k relays lie in the interval $\left(F_{\text{asm}}^{\text{inv}}\left(1-\frac{i\zeta}{n_{\text{asm}}}\right), F_{\text{asm}}^{\text{inv}}\left(1-\frac{(i-1)\zeta}{n_{\text{asm}}}\right)\right)$ and the rest of the relays' metrics lie below $F_{\text{asm}}^{\text{inv}}\left(1-\frac{i\zeta}{n_{\text{asm}}}\right)$. Since the metrics are i.i.d., the probability of this event happening for $i \leq q'$ equals

$$\binom{n}{k}\left[F^{\text{eq}}\left(1-\frac{(i-1)\zeta}{n_{\text{asm}}}\right) - F^{\text{eq}}\left(1-\frac{i\zeta}{n_{\text{asm}}}\right)\right]^k \times \left(F^{\text{eq}}\left(1-\frac{i\zeta}{n_{\text{asm}}}\right)\right)^{n-k},$$

where $F^{\text{eq}} = F \circ F_{\text{asm}}^{\text{inv}}$, as defined before. The probability that after the i^{th} slot, in which a collision occurs, the best relay is found in the remaining $t-i$ slots is $\sum_{j=1}^{t-i} p\left(k, j, 1-\frac{i\zeta}{n_{\text{asm}}}, 1-\frac{(i-1)\zeta}{n_{\text{asm}}}\right)$.

Also, the first non-idle slot is the $(q'+1)^{\text{th}}$ slot if all the metrics of the n relays lie in $\left(0, F_{\text{asm}}^{\text{inv}}\left(1-\frac{q'\zeta}{n_{\text{asm}}}\right)\right)$. The probability of this equals $\left(F^{\text{eq}}\left(1-\frac{q'\zeta}{n_{\text{asm}}}\right)\right)^n$. In this case, the probability that the best relay is found in any of the $t-(q'+1)$ slots that remain is $\sum_{j=1}^{t-q'-1} p\left(n, j, 0, 1-\frac{q'\zeta}{n_{\text{asm}}}\right)$.

When $t < q'+1$, the best relay will get selected if the first non-idle slot occurs on or before $\min(t, q')$ slots. Hence, in either case, the desired result follows from the law of total probability.

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