Lecture-04: Invariant Distribution of Markov Processes

1 Class Properties

Definition 1.1. For a CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ defined on the countable state space $\mathcal{X} \subseteq \mathbb{R}$, we say a state y is **reachable** from state x if $P_{xy}(t) > 0$ for some $t > 0$, and we denote $x \to y$. If two states $x, y \in \mathcal{X}$ are reachable from each other, we say that they **communicate** and denote it by $x \leftrightarrow y$.

Lemma 1.2. *Communication is an equivalence relation.*

Definition 1.3. Communication equivalence relation partitions the state space $\mathcal{X}$ into equivalence classes called **communicating classes**. A CTMC with a single communicating class is called **irreducible**.

Theorem 1.4. *A regular CTMC and its embedded DTMC have the same communicating classes.*

Proof. It suffices to show that $x \to y$ for the regular Markov process iff $x \to y$ in the embedded chain. If $x \to y$ for embedded chain, then there exists a path $x = x_0, x_1, \dots, x_n = y$ such that $p_{x_0x_1}p_{x_1x_2}\dots p_{x_{n-1}x_n} > 0$ and $0 < \nu_{x_0}\nu_{x_1}\dots\nu_{x_{n-1}}$. It follows that S_n is a stopping time and sum of n independent exponential random variables with rates $\nu_{x_0}, \dots, \nu_{x_{n-1}}$, and we can write

$$P_{xy}(t) \geqslant \prod_{k=0}^{n-1} p_{x_kx_{k+1}}\mathbb{E}_{x_0}[P\{T_{n+1} > t - S_n\} \mid \{Z_0 = x_0, \dots Z_n = x_n\}] > 0.$$

Conversely, if the states y is not reachable from state x in embedded chain, then it won't be reachable in the regular CTMC. □

Corollary 1.5. *A regular CTMC is irreducible iff its embedded DTMC is irreducible.*

Remark 1. There is no notion of periodicity in CTMCs since there is no fundamental time-step that can be used as a reference to define such a notion. In fact, for any state $x \in \mathcal{X}$ of a non-instantaneous homogeneous CTMC we have $P_{xx}(t) > e^{-\nu_x t} > 0$ for all $t \geqslant 0$.

1.1 Recurrence and transience

Definition 1.6. For any initial state $X_0 = y \in \mathcal{X}$, we define the first hitting time to state y after leaving state y as

$$\tau_y^+(1) \triangleq \inf\{t > Y_0 : X_t = y\}.$$

The state y is said to be **recurrent** if $P_y\left\{\tau_y^+(1) < \infty\right\} = 1$ and **transient** if $P_y\left\{\tau_y^+(1) < \infty\right\} < 1$. Furthermore, a recurrent state y is said to be **positive recurrent** if $\mathbb{E}_y\tau_y^+(1) < \infty$ and **null recurrent** if $\mathbb{E}_y\tau_y^+(1) = \infty$.

Definition 1.7. For a CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ and its embedded DTMC $Z : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, let the initial state $Z_0 = x \in \mathcal{X}$. We denote the kth return time to state x by $\tau_x^+(k)$, and the number of visits to state y during kth successive visit to state x by $N_{xy}(k)$, the total number of visits to all states during kth successive visit to state x by $N_x(k) \triangleq \sum_{y\in\mathcal{X}} N_{xy}(k)$, and the kth sojourn time in state y by $Y_k^{(y)}$. We observe that $\tau_x^+(k) = \tau_x^+(k-1) + \sum_{y\in\mathcal{X}}\sum_{j=1}^{N_{xy}(k)} Y_j^{(y)}$.

Theorem 1.8. *An irreducible pure jump CTMC is recurrent iff its embedded DTMC is recurrent.*

Proof. There is nothing to prove for $|\mathcal{X}| = 1$. Hence, we assume $|\mathcal{X}| \geqslant 2$ without loss of generality. Suppose that the embedded Markov chain $Z : \Omega \to \mathcal{X}^{\mathbb{N}}$ is recurrent. Since the embedded chain is irreducible and recurrent, CTMC has no absorbing states. This implies $N_{xy}(1)$ and $N_x(1)$ are finite almost surely, and the random sequence $Y^{(y)} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is *i.i.d.* exponential with rate $\nu_y \in (0, \infty)$, and sequences $Y^{(y)}$ are independent for each state $y \in \mathcal{X}$. Since the recurrence time $\tau_x^+(1)$ as an a.s. finite sum of finite random variables, it follows that $\tau_x^+(1)$ is finite almost surely.

Conversely, if the embedded Markov chain is not recurrent, it has a transient state $x \in \mathcal{X}$ for which $P_x\{N_x = \infty\} > 0$. By the same argument, $P_x\{\tau_x^+ = \infty\} > 0$ and hence the CTMC is not recurrent. □

Corollary 1.9. *Recurrence is a class property.*

Remark 2. Consider an irreducible positive recurrent DTMC $Z : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with transition probability matrix $p : \mathcal{X} \to \mathcal{M}(\mathcal{X})$ and invariant distribution $u \in \mathcal{M}(\mathcal{X})$. Let $Z_0 = x$. Recall that number of visits to all states before returning to state x is the inter-return time to state x. We denote by $S_x(k) = \sum_{j=1}^{k} N_x(j)$ as the kth return time to state x. It follows that the number of visits to state y between two successive visits to state x is $N_{xy}(k) = \sum_{n=S_x(k-1)+1}^{S_x(k-1)+N_x(k)} \mathbb{1}_{\{X_n=y\}}$. We observe that $S_x^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$ is a renewal process with reward $N_{xy}(k)$ in the kth renewal duration. From the renewal reward theorem, we get

$$u_y = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{X_n=y\}} = \frac{\mathbb{E}_x N_{xy}(k)}{\mathbb{E}_x N_x(k)} = u_x \mathbb{E}_x N_{xy}(k).$$

Theorem 1.10. *Consider an irreducible recurrent CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with sojourn time rates $\nu \in \mathbb{R}_+^{\mathcal{X}}$ and transition matrix $p \to \mathcal{X} \to \mathcal{M}(\mathcal{X})$ for the embedded Markov chain. Let $u \in \mathbb{R}_+^{\mathcal{X}}$ be any strictly positive solution of $u = up$, then*

$$\mathbb{E}_x \tau_x^+(1) = \frac{1}{u_x} \sum_{y\in\mathcal{X}} \frac{u_y}{\nu_y}, \quad x \in \mathcal{X}.$$

Proof. Let $X_0 = x \in \mathcal{X}$, and $N_{xy}(k)$ be the number of visits to state $y \in \mathcal{X}$ between kth successive visits to state x in the embedded Markov chain. From the recurrence of the embedded Markov chain, we know that for any strictly positive solution to $u = up$ we have $\mathbb{E}_x N_{xy}(k) = \frac{u_y}{u_x}$. Let $Y_k^{(x)}$ denote the sojourn time of the CTMC X in state x during the kth visit. The random sequence $Y^{(x)} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is *i.i.d.* exponential with rate ν_x. Therefore, we can write $\tau_x^+(1) = \sum_{y\in\mathcal{X}} \sum_{k=1}^{N_{xy}(1)} Y_k^{(y)}$. We recall that jump chain and sojourn times are independent given the initial state, and hence $N_{xy}(1)$ and $Y^{(y)}$ sequences are independent for each state $y \neq x$. Result follows from taking expectations on both sides, exchanging summation and expectations for positive random variables, to get $\mathbb{E}_x \tau_x^+(1) = \sum_{y\in\mathcal{X}} \mathbb{E} Y_k^{(y)} \mathbb{E}_x N_{xy}$. □

Remark 3. An irreducible regular CTMC maybe null recurrent where embedded Markov chain is positive recurrent.

Corollary 1.11. *Consider an irreducible recurrent CTMC $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with sojourn time rates $\nu \in \mathbb{R}_+^{\mathcal{X}}$ and the transition matrix p for the embedded Markov chain. Let u be any strictly positive solution to $u = up$. Then, CTMC X is positive recurrent iff $\sum_{x\in\mathcal{X}} \frac{u_x}{\nu_x} < \infty$. In particular, the CTMC is positive recurrent iff $\sum_{x\in\mathcal{X}} \frac{u_x}{\nu_x} = 1$.*

2 Invariant Distribution

Remark 4. Let $\nu(0) \in \mathcal{M}(\mathcal{X})$ denote the marginal distribution of initial state X_0, then by definition of probability transition kernel for Markov process X, we can write the marginal distribution of X_t as

$$\nu(t) = \nu(0)P(t), \quad t \in \mathbb{R}_+.$$

In general, we can write $\nu(s + t) = \nu(s)P(t)$, and hence if there exists a stationary distribution $\pi \triangleq \lim_{s\to\infty} \nu(s)$ for this process X, then we would have $\pi = \pi P(t)$ for all times $t \in \mathbb{R}_+$.

Definition 2.1. A distribution $\pi \in \mathcal{M}(\mathcal{X})$ is an **invariant distribution** of a homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with probability transition kernel $P : \mathbb{R}_+ \to \mathcal{M}(\mathcal{X})^{\mathcal{X}}$ if $\pi P(t) = \pi$ for all $t \in \mathbb{R}_+$.

Remark 5. Recall that an irreducible DTMC is positive recurrent iff it has a strictly positive stationary distribution.

Corollary 2.2. *For a homogeneous continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with generator matrix Q, a distribution $\pi : \mathcal{X} \to [0, 1]$ is an equilibrium distribution iff $\pi Q = 0$.*

Proof. Recall that we can write the transition probability matrix $P(t)$ at any time $t \in \mathbb{R}_+$ in terms of generator matrix Q as $P(t) = e^{tQ}$. Using the exponentiation of a matrix, we can write

$$\pi P(t) = \pi e^{tQ} = \pi + \sum_{n\in\mathbb{N}} \frac{t^n}{n!} \pi Q^n, \quad t \in \mathbb{R}_+.$$

Therefore, $\pi Q = 0$ iff π is an equilibrium distribution of the Markov process X. □

Theorem 2.3. *Let $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ be an irreducible recurrent homogeneous CTMC with probability transition kernel $P : \mathbb{R}_+ \to [0,1]^{\mathcal{X}\times\mathcal{X}}$, the transition rate sequence $\nu \in \mathbb{R}_+^{\mathcal{X}}$, and the transition matrix for embedded jump chain $p \in [0,1]^{\mathcal{X}}$. Then for all states $x, y \in \mathcal{X}$ the $\lim_{t\to\infty} P_{xy}(t)$ exists, this limit is independent of the initial state $x \in \mathcal{X}$ and denoted by π_y. Let u be any strictly positive invariant measure such that $u = up$. If $\sum_{x\in\mathcal{X}} \frac{u_x}{\nu_x} = \infty$, then $\pi_x = 0$ for all $x \in \mathcal{X}$. If $\sum_{x\in\mathcal{X}} \frac{u_x}{\nu_x} < \infty$ then for all $y \in \mathcal{X}$,*

$$\pi_y = \frac{\frac{u_y}{\nu_y}}{\sum_{x\in\mathcal{X}} \frac{u_x}{\nu_x}} = \frac{\nu_y^{-1}}{\mathbb{E}_y \tau_y^+}.$$

Proof. Fix a state $y \in \mathcal{X}$, and define a process $W : \Omega \to \{0,1\}^{\mathbb{R}_+}$ such that $W_t = \mathbb{1}_{\{X_t = y\}}$. Then, from the regenerative property of the homogeneous CTMC and renewal reward theorem, we have

$$\lim_{t\to\infty} P_x \{X_t = y\} = \frac{\nu_y^{-1}}{\mathbb{E}_y \tau_y^+}.$$

□