Lecture-06: Interacting particle Markov processes

1 Interacting particle systems

Consider a continuous time Markov process $X^N : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with countable state space $\mathcal{X} \subseteq \mathbb{R}_+$ defined over probability space $(\Omega, \mathcal{F}, P)$ that models an interacting particle system of N particles, where state of each particle at time t is denoted by $X_n^N(t) : \Omega \to \mathcal{Z}$. Then the aggregate state space $\mathcal{X} \triangleq \mathcal{Z}^N$ is exponentially growing in the number of particles N.

Proposition 1.1. *When the evolution of each particle is independent for process X, the generator matrix of the joint process is $Q^{X^N} = \bigotimes_{n\in[N]} Q^{X_n^N}$, and the joint distribution is given by $\pi^{X^N} = \bigotimes_{n\in[N]} \pi^{X_n^N}$.*

Remark 1. When the evolution of N particles is not independent, finding the invariant distribution $\pi \in \mathcal{M}(\mathcal{X})$ maybe too challenging.

Definition 1.2. For a countable set $\mathcal{Z}$ and finite $N \in \mathbb{N}$, we define the set of probability measures on $\mathcal{Z}$ as

$$\mathcal{M}_N(\mathcal{Z}) \triangleq \left\{ \alpha \in \left\{0, \frac{1}{N}, \dots, 1\right\}^{\mathcal{Z}} : \sum_{z\in\mathcal{Z}} \alpha_z = 1 \right\}.$$

Definition 1.3. For the N interacting particle process X^N, we define the associated empirical state distribution process $A^N \in \mathcal{M}_N(\mathcal{Z})^{\mathbb{R}_+}$ defined at time t as $A_z^N(t) \triangleq \frac{1}{N}\sum_{n=1}^N \mathbb{1}_{\{X_n^N(t)=z\}}$. When the state $X^N(t) = x$, we denote the associated empirical state distribution as $a \triangleq \frac{1}{N}\sum_{z\in\mathcal{Z}} e_z \sum_{n=1}^N \mathbb{1}_{\{x_n=z\}}$.

Proposition 1.4. *If the evolution X_n^N of particle n in the interacting particle process X^N depends on the state $X_n^N(t)$ and other particles $m \in [N]$ only through the empirical distribution $A^N(t)$ and this evolution is identical, then the empirical distribution process A^N is a Markov process.*

Proof. Since process X is a CTMC, there is only one possible transition in an infinitesimal time. Thus the only possible transitions for process X^N are $x \to y = x - x_n e_n + y_n e_n$ for some particle $n \in [N]$ and $x_n, y_n \in \mathcal{Z}$. It follows that the possible transitions for empirical distribution process A^N are of the form $a \to b = a - \frac{1}{N}e_z + \frac{1}{N}e_w$ for some $z, w \in \mathcal{Z}$. From the hypothesis, we have

$$Q_{xy}^{X^N} = \sum_{n=1}^N f(x_n, y_n, a)\mathbb{1}_{\{y_n \neq x_n\}}.$$

For any $a, b \in \mathcal{M}_N(\mathcal{Z})$ such that $a \to b$ is a possible transition, we can write $N(b-a) = -e_z + e_w$, and the corresponding transition rate as

$$Q_{ab}^{A^N} = \sum_{n=1}^N Q_{xy}^{X^N} \mathbb{1}_{\{(x_n,y_n)=(z,w)\}} = \sum_{n=1}^N \mathbb{1}_{\{(x_n,y_n)=(z,w)\}} f(z, w, a).$$

It follows that the transition rates $Q_{ab}^{A^N}$ depend only on a, b, and the result follows. □

Remark 2. Will show the Kurtz's theorem that implies that under some conditions

$$\pi_z^{(1)} = \lim_{t\to\infty} \lim_{N\to\infty} A_z^N(t) = \lim_{t\to\infty} P\{X_1(t) = z\}.$$

We will show the asymptotic independence of particles for any finite subset $F \subseteq [N]$, under certain conditions, i.e.

$$\lim_{t\to\infty} P\Big(\cap_{n\in F} \{X_n(t) = z_n\} \Big) = \prod_{n\in F} \pi_{z_n}^{(1)}.$$

1.1 Susceptible-infected-susceptible (SIS) epidemic model

Consider a population of N individuals, where $X_n^N(t)$ indicates that the individual n is infected at time t and susceptible otherwise. The fraction of susceptible individuals at time t is

$$A_0^N(t) \triangleq \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{\{X_n^N(t)=0\}}.$$

Assumption 1.5. We assume that each infected individual recovers independently after a random time distributed exponentially with mean 1. Further, each infected individual has an independent random infection time distributed exponentially with mean $1/\beta$ and infects a susceptible individual selected uniformly at random. In addition, we assume that each susceptible individual has a random immune time distributed exponentially with mean $1/\alpha$ after which it can get infected by an external source.

Proposition 1.6. *Defining $\mathcal{Z} \triangleq \{0,1\}$ and $\mathcal{X} \triangleq \mathcal{Z}^N$, we observe that $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ is a continuous time Markov chain with generator matrix $Q : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ defined for all $x, y \in \mathcal{X}$ as*

$$Q^{X^N}_{x,y} = \begin{cases} \bar{x}_n(\alpha+\beta a_1), & y = x + \bar{x}_n e_n,\\ x_n, & y = x - x_n e_n,\\ -Na_0(\alpha+\beta a_1) - Na_1, & y = x. \end{cases}$$

Proof. Consider a single infected individual and its *i.i.d.* exponentially distributed inter-infection time sequence $T : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with rate β and *i.i.d.* indicator sequence $\xi : \Omega \to \{0,1\}^{\mathbb{N}}$ for infecting a susceptible individual from the population. Let $\tau \triangleq \{n \in \mathbb{N} : \xi_n = 1\}$ be the first time a susceptible person is infected by this individual and we assume that no-other transition takes place until this time. Then, we observe that $\sum_{n=1}^{\tau} T_n$ is exponentially distributed with rate $\beta\mathbb{E}\xi$. To see this, we observe that for $\theta > -\beta$ and $p = \mathbb{E}\xi$

$$\mathbb{E}e^{-\theta\sum_{n=1}^{\tau} T_n} = \mathbb{E}\Big(\frac{\beta}{\theta+\beta}\Big)^{\tau} = \frac{p\frac{\beta}{\beta+\theta}}{1-\frac{\beta\bar{p}}{\beta+\theta}} = \frac{\beta p}{\theta+\beta p}.$$

At any time t, the remaining recovery time and infection time to a susceptible individual for an infected individual n are denoted by $Y_n(t)$ and $Z_n(t)$ respectively, and the remaining immune time for a susceptible individual is denoted by $W_n(t)$. The transition time in a state x at time t is given by

$$\min\Big\{\min_{X_n^N(t)=1}\{Y_n(t), Z_n(t)\}, \min_{X_n^N(t)=0} W_n(t)\Big\}.$$

For $A^N(t) = a$, we observe that there are Na_0 susceptible and Na_1 infected individuals and $Y_n(t), Z_n(t), W_n(t)$ are independent exponential random variables with rates $1, \beta a_0, \alpha$ respectively. It follows that the transition times are exponentially distributed with rate $Na_1(1+\beta a_0) + Na_0\alpha$.

The probability of nth individual to get infected from being susceptible is

$$P\Big\{\min\Big\{W_n(t), \min_{X_m^N(t)=1}\{Z_{m,n}(t)\}\Big\} < \min\Big\{\min_{X_m^N(t)=1} Y_m(t), \min_{n'\neq n}\Big\{W_{n'}(t), \min_{X_m^N(t)=1}\{Z_{m,n'}(t)\}\Big\}\Big\}\Big\},$$

where $\min\{W_n(t), \min_{X_m^N(t)=1}\{Z_{m,n}(t)\}\}$ is exponentially distributed with mean $\alpha + a_1\beta$ independent of $\min\{\min_{X_m^N(t)=1} Y_m(t), \min_{n'\neq n}\{W_{n'}(t), \min_{X_m^N(t)=1}\{Z_{m,n'}(t)\}\}\}$ exponentially distributed with mean $Na_1 + (Na_0-1)(\alpha+\beta a_1)$. It follows that this probability is $\frac{\alpha+a_1\beta}{Na_1+Na_0(\alpha+\beta a_1)}$. Similarly, we can find the probability of nth individual to become susceptible from being infected as $\frac{1}{Na_1+Na_0(\alpha+\beta a_1)}$. □

Remark 3. That is, for state $X^N(t) = x$ and associated empirical distribution $A^N(t) = a \triangleq (\frac{1}{N}\sum_{n=1}^{N}\bar{x}_n, \frac{1}{N}\sum_{n=1}^{N} x_n)$, we can isolate the generator matrix for particle n as

$$Q^{X_n^N}_{x_n,y_n} = \begin{cases} (\alpha+\beta a_1), & (x_n, y_n) = (0,1)\\ 1, & (x_n, y_n) = (1,0). \end{cases}$$

Corollary 1.7. *For the Markov process X^N, the associated empirical distribution process $A^N : \Omega \to \mathcal{M}_N(\mathcal{Z})^{\mathbb{R}_+}$ is a continuous time Markov chain with the generator matrix $Q^{A^N} : \mathcal{M}_N(\mathcal{Z}) \times \mathcal{M}_N(\mathcal{Z}) \to \mathbb{R}$ is defined for all $a, b \in \mathcal{M}_N(\mathcal{Z})$ as*

$$Q^{A^N}_{a,b} \triangleq \begin{cases} Na_0\alpha + a_0 Na_1\beta, & b = a - \frac{1}{N}e_0 + \frac{1}{N}e_1, \\ Na_1, & b = a + \frac{1}{N}e_0 - \frac{1}{N}e_1, \\ -Na_0\alpha - a_0Na_1\beta - Na_1, & b = a. \end{cases}$$

Proof. We observe that the state evolution of particle n depends on its current state $x_n \in \mathcal{Z}$ and is coupled with other particles via the empirical distribution $A^N_1(t) = \frac{1}{N}\sum_{n=1}^N x_n$. We observe that possible transitions for x_n are from 0 and 1 and from 1 to 0. This results in transitions for empirical distribution from $a \to a - \frac{1}{N}e_0 + \frac{1}{N}e_1$ and from $a \to a + \frac{1}{N}e_0 - \frac{1}{N}e_1$. The former transition takes place when any x_n transitions from $0 \to 1$ and the latter for transition $1 \to 0$. It follows that

$$Q^{A^N}_{a_0} = \sum_{n \in [N]: x_n = 0} Q_{x, x+e_n}, \qquad Q^{A^N}_{a_1} = \sum_{n \in [N]: x_n = 1} Q_{x, x-e_n}.$$

□

1.2 Random multiple access

Consider N devices trying to access a wireless channel in a non-coordinated fashion. For each device n, we denote by $X^N_n(t)$ the number of transmission attempts at time t. Let r be the maximum number of attempts after which the head of line packet is discarded. We denote the state space for each device as $\mathcal{Z} \triangleq \{0, \dots, r\}$, the number of attempts they have made.

Assumption 1.8. We assume that each device makes a transmission attempt after waiting for an independent and random amount of time distributed exponentially. If a device has made $z \in \mathcal{Z}$ transmission attempts, then the mean waiting time is $1/c_z$. A transmission attempt is successful, if no other device transmits during its transmission attempt waiting time, and the number of transmission attempts return to 0, otherwise they increment by unity. If the transmission attempt fails rth time, the head of the line packet is discarded and the number of attempts returns to zero.

Definition 1.9. We define inner product of vectors $c \in \mathbb{R}^{\mathcal{Z}}_+$ and $a \in \mathcal{M}_N(\mathcal{Z})$ as $\langle c, a \rangle \triangleq \sum_{z \in \mathcal{Z}} c_z a_z$.

Proposition 1.10. *Defining $\mathcal{X} \triangleq \mathcal{Z}^N$ and assuming an infinite backlog of packets, we observe that $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ is a continuous time Markov chain with the associated generator matrix is defined for all $x, y \in \mathcal{X}$ as*

$$Q^{X^N}_{xy} = \begin{cases} c_{x_n}, & y = x - x_n e_n, \\ N\langle c, a \rangle - c_{x_n}, & y = x + e_n. \end{cases}$$

Proof. At a time t, we denote the excess transmission attempt time for particle n by $Y_n(t)$. Then, the probability of successful transmission attempt by nth device as

$$P\Big(\cap_{m \neq n} \{Y_n(t) < Y_m(t)\} \Big) = \mathbb{E} \prod_{m \neq n} e^{-c_{X^N_m(t)} Y_n(t)} = \mathbb{E} e^{-(\sum_{z \neq x_n} NA^N_z(t) c_z + (NA^N_{x_n}(t) - 1)c_{x_n}) Y_n(t)}.$$

From the definition of $\langle c, a \rangle$ and the fact that $Y_n(t)$ is exponential with rate c_{x_n}, we get

$$P\Big(\cap_{m \neq n} \{Y_n(t) < Y_m(t)\} \Big) = \frac{c_{x_n}}{N \langle c, A^N(t) \rangle}.$$

Given that $A^N(t) = a$ at time t, then we observe that Na_z devices have made z attempts, and the rate of attempts from them is $Na_z c_z$. It follows that the holding rate for any state $x \in \mathcal{X}$ with empirical distribution $a \in \mathcal{M}_N(\mathcal{Z})$ is given by $N\langle c, a \rangle$, and the result follows. □

Corollary 1.11. *For the continuous time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$, the associate empirical distribution process $A^N : \Omega \to \mathcal{M}_N(\mathcal{Z})^{\mathbb{R}_+}$ is a continuous time Markov chain with the associated generator matrix is defined for all $a, b \in \mathcal{M}_N(\mathcal{Z})$ as*

$$Q^{A^N}_{ab} = \begin{cases} N\langle c, a \rangle - Nc_z a_z, & b = a - \frac{1}{N}e_z + \frac{1}{N}e_{z+1}, \\ Nc_z a_z, & b = a - \frac{1}{N}e_z + \frac{1}{N}e_0. \end{cases}$$