Lecture-11: Behavior at stationarity

1 Limiting behavior

We are interested in characterizing the large system behavior at stationarity, i.e. we are interested in evaluating

$$\lim_{N\to\infty}\lim_{t\to\infty} A^N(t). \tag{1}$$

Recall that $A^N : \Omega \to \mathcal{M}_N(\mathcal{Z})^{\mathbb{R}_+}$ is a CTMC, and the limiting random variable $\lim_{t\to\infty} A^N(t)$ has a distribution, we denote it by $\pi^N \in \mathcal{M}(\mathcal{M}_N(\mathcal{Z}))$ when it exists. That is, for each $a \in \mathcal{M}_N(\mathcal{Z})$

$$\pi^N_a \triangleq \lim_{t\to\infty} P\left\{A^N(t) = a\right\}.$$

Definition 1.1 (rest points). Consider the mean-field model $\dot{a} = f(a)$ for a density dependent family of CTMCs $(A^N : \Omega \to \mathcal{M}(\mathcal{Z})^{\mathbb{R}_+} : N \in \mathbb{N})$. The set of rest points for the Mckean-Vlasov equations are denoted by

$$\mathcal{S} \triangleq \{a \in \mathcal{M}(\mathcal{Z}) : f(a) = 0\}.$$

Remark 1. We observe that if $a(0) \in \mathcal{S}$, then $a(t) = a(0)$ for all $t \in \mathbb{R}_+$.

The stationary limit of solution of Mckean-Vlasov equation is a rest point of the ordinary differential equation, and given by a solution of $f(a) = 0$ and denoted by $a^* \in \mathcal{S}$. That is,

$$\lim_{t\to\infty}\lim_{N\to\infty} A^N(t) = \lim_{t\to\infty} a(t) \in \mathcal{S}. \tag{2}$$

One of the key question is to find the conditions under which the two limits in (1) and (2) are equal.

Assumption 1.2 (Irreducibility). Assume that the CTMC $A^N : \Omega \to \mathcal{M}_N(\mathcal{Z})^{\mathbb{R}_+}$ is irreducible for each N.

Remark 2. We observe that $\mathcal{M}_N(\mathcal{Z}) \triangleq \left\{\alpha \in \left\{0, \frac{1}{N}, \dots, 1\right\}^{\mathcal{Z}} : \sum_{z\in\mathcal{Z}} \alpha_z = 1\right\}$ is a finite set. We observe that $\sum_{z\in\mathcal{Z}} N\alpha_z = 1$, where $N\alpha_z \in \{0, \dots, N\}$. That is, $(N\alpha_z : z \in \mathcal{Z})$ is a partition of N, and hence the cardinality of $\mathcal{M}_N(\mathcal{Z})$ is given by $|\mathcal{M}_N(\mathcal{Z})| = \frac{(N+|\mathcal{Z}|-1)!}{N!(|\mathcal{Z}|-1)!}$.

Proposition 1.3. *Under Assumption 1.2 for CTMC $A^N : \Omega \to \mathcal{M}_N(\mathcal{Z})^{\mathbb{R}_+}$, there exists a unique stationary distribution $\pi^N \in \mathcal{M}(\mathcal{M}_N(\mathcal{Z}))$.*

Proof. Since the CTMC A^N is irreducible and has a finite state space $\mathcal{M}_N(\mathcal{Z})$, it is positive recurrent with a unique stationary distribution $\pi^N \in \mathcal{M}(\mathcal{M}_N(\mathcal{Z}))$. □

2 Limiting behavior of Mckean-Vlasov ODE

Definition 2.1. The solution to the Mckean-Vlasov ordinary differential equation for a mean-field model $\dot{a} = f(a)$ can be represented by a non-linear map $\Phi_t : \mathcal{M}(\mathcal{Z}) \to \mathcal{M}(\mathcal{Z})$ defined for any initial condition $a(0) \in \mathcal{M}(\mathcal{Z})$ and $t \in \mathbb{R}_+$ as

$$\Phi_t(a(0)) \triangleq a(t) = a(0) + \int_0^t f(a(s))ds.$$

Definition 2.2 (Invariant). Consider the map Φ associated with the solution to the Mckean-Vlasov equation for a mean-field model. A set $\mathcal{A} \subseteq \mathcal{M}(\mathcal{Z})$ is an **invariant set** with respect to map Φ if $\Phi_t(\mathcal{A}) = \mathcal{A}$ for all $t \in \mathbb{R}_+$.

Remark 3. If $a(0) \in \mathcal{A} \subseteq \mathcal{M}(\mathcal{Z})$ and $\mathcal{A}$ is an invariant set for map Φ associated with the Mckean-Vlasov equation, then $a(t) \in \mathcal{A}$ for all $t \in \mathbb{R}_+$.

Example 2.3 (Rest points). We recall that $\Phi_t(\mathcal{S}) = \mathcal{S}$ for all $t \in \mathbb{R}_+$ and hence the set of rest points is an invariant set for map Φ.

Definition 2.4 (Attractor). A compact invariant set $\mathcal{G} \subseteq \mathcal{M}(\mathcal{Z})$ is an **attractor set**, if there exists an open neighborhood $\mathcal{O}$ of $\mathcal{G}$ such that every trajectory initiated in $\mathcal{O}$ remains in $\mathcal{O}$ and converges to $\mathcal{G}$. That is, $\Phi_t(\mathcal{O}) = \mathcal{O}$ and $\lim_{t\to\infty} \Phi_t(\mathcal{O}) \subseteq \mathcal{G}$.

Remark 4. An attractor set is compact and invariant and has an open neighborhood that is an invariant set, such that for all initial conditions in this open neighborhood, the rest points are in the attractor.

Definition 2.5. For any subset $\mathcal{B} \subseteq \mathcal{M}(\mathcal{Z})$ and $\epsilon > 0$, we define

$$B_\epsilon(\mathcal{B}) \triangleq \left\{ a \in \mathcal{M}(\mathcal{Z}) : \inf_{b\in\mathcal{B}} \|a - b\|_{\text{TV}} \leqslant \epsilon \right\}.$$

Definition 2.6 (Stability). A compact invariant set $\mathcal{G} \subseteq \mathcal{M}(\mathcal{Z})$ is called **Lyapunov stable** if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\Phi_t(B_\delta(\mathcal{G})) \subseteq B_\epsilon(\mathcal{G})$ for all $t \in \mathbb{R}_+$. A compact invariant set $\mathcal{G} \subseteq \mathcal{M}(\mathcal{Z})$ is called **asymptotically stable** if it is both Lyapunov stable and an attractor. A compact invariant set $\mathcal{G} \subseteq \mathcal{M}(\mathcal{Z})$ is **globally asymptotically stable** if it is asymptotically stable and all trajectories converge to $\mathcal{G}$.

Example 2.7 (SIS epidemic model). Recall that the McKean-Vlasov equation for the SIS model for all $t \in \mathbb{R}_+$ is given by

$$\dot{a}_1(t) = (1 - a_1)\alpha + (1 - a_1)a_1\beta - a_1.$$

The rest points for the Mckean-Vlasov equation for the SIS model are solutions to the equation $-\frac{\alpha}{\beta} + a_1(\frac{1+\alpha}{\beta} - 1) + a_1^2 = 0$, and given by the following two rest points

$$r_1 = \frac{-(\frac{1+\alpha}{\beta} - 1) + \sqrt{(\frac{1+\alpha}{\beta} - 1)^2 + \frac{4\alpha}{\beta}}}{2} > 0, \qquad r_2 = \frac{-(\frac{1+\alpha}{\beta} - 1) - \sqrt{(\frac{1+\alpha}{\beta} - 1)^2 + \frac{4\alpha}{\beta}}}{2} < 0.$$

In particular, we can write $\dot{a}_1 = -\beta(a_1 - r_1)(a_1 - r_2)$. The solution to this dynamical equation for all $t \in \mathbb{R}_+$ is

$$a_1(t) = \frac{r_1(a_1(0) - r_2)e^{(r_1-r_2)\beta t} - r_2(a_1(0) - r_1)}{(a_1(0) - r_2)e^{(r_1-r_2)\beta t} - (a_1(0) - r_1)}.$$

For external infection rate $\alpha = 0$, we have $r_1 = 0, r_2 = 1 - \frac{1}{\beta}$. Thus, we can write the evolution of the fraction of infected individuals as

$$a_1(t) = \frac{r_2\beta a_1(0)e^{\beta r_2 t}}{\beta a_1(0)e^{\beta r_2 t} - \beta(a_1(0) - r_2)} = \frac{a_1(0)(\beta - 1)e^{(\beta-1)t}}{\beta - 1 - \beta a_1(0)(1 - e^{(\beta-1)t})}.$$

Consider the case when $\beta > 1$. If $a_1(0) = r_1 = 0$, then $a_1(t) = r_1 = 0$ for every $t \geqslant 0$. If $a_1(0) > 0$, then $\lim_{t\to\infty} a_1(t) = r_2 = 1 - \frac{1}{\beta}$. It follows that r_2 is an attractor, but r_1 is not an attractor.

Remark 5. The set of rest points $\mathcal{S}$ need not be an attractor set, i.e. not all rest points are attractors. Sometimes even if the set of rest points is unique the solution to the McKean-Vlasov equation may not converge to the rest point at stationarity, i.e. even if $\mathcal{S} = \{\xi_0\}$, yet $\lim_{t\to\infty} a(t) \neq \xi_0$. This is an example of an unstable equilibrium. Another example of when all equilibria are unstable is when $\dot{a} = Ba$ where all eigenvalues of B are imaginary. It is also possible that a dynamical system has some stable and unstable equilibria.

3 Stationary limit of finite particle system

Lemma 3.1. *Assume the Lipschitz assumption on the transition rates of a single particle in empirical distribution and Assumption 1.2 on the irreducibility of CTMC A^N. Let $a^* \in \mathcal{S}$ be a globally asymptotically stable equilibrium. Then, for each $\epsilon > 0$, there exists $T > 0$, such that $\|\Phi_t(a(0)) - a^*\| < \epsilon$ for any initial condition $a(0) \in \mathcal{M}(\mathcal{Z})$ and time $t \geqslant T$.*

Proof. We will show this using the following steps.

Step 1. **Lyapunov stability.** Fix $\epsilon > 0$. By Lyapunov stability of compact invariant set $\{a^*\}$, there exists $\delta > 0$ such that if $\|a(0) - a^*\| < \delta$, then $\|a(t) - a^*\| < \epsilon$ for every $t \in \mathbb{R}_+$.

Step 2. **Global asymptotic tsability.** From the global asymptotic stability of compact invariant set $\{a^*\}$, we know that all trajectories converge to a^*. Thus, we can define a finite stopping time

$$\tau_{a(0)} \triangleq \inf \left\{ t \in \mathbb{R}_+ : \|\Phi_t(a(0)) - a^*\| < \frac{\delta}{2} \right\}. \tag{3}$$

Since the Lipschitz property of transition rates implies the Lipschitz property of rate function f, we can bound the difference of trajectories at time t for two different initial conditions $a(0), a'(0) \in \mathcal{M}(\mathcal{Z})$ as

$$\|\Phi_t(a(0)) - \Phi_t(a'(0))\| \leqslant \|a(0) - a'(0)\| + M \int_0^t \|\Phi_s(a(0)) - \Phi_s(a'(0))\| \, ds.$$

From Gronwall's inequality applied to the above equation, we obtain $\|\Phi_t(a(0)) - \Phi_t(a'(0))\| \leqslant \|a(0) - a'(0)\| e^{Mt}$. In particular, there exists a neighborhood $\mathcal{O}_{a(0)}$ such that for every $a'(0) \in \mathcal{O}_{a(0)}$,

$$\left\|\Phi_{\tau_{a_0}}(a(0)) - \Phi_{\tau_{a_0}}(a'(0))\right\| < \frac{\delta}{2}. \tag{4}$$

From the definition of stopping time $\tau_{a(0)}$ in (3), the property of neighborhood $\mathcal{O}_{a(0)}$ in (4), and the triangle inequality for any $a'(0) \in \mathcal{O}_{a(0)}$, we obtain $\left\|\Phi_{\tau_{a(0)}}(a'(0)) - a^*\right\| < \delta$. From Lyapunov stability of compact invariant set $\{a^*\}$, it follows that $\|\Phi_t(a'(0)) - a^*\| < \epsilon$ for any $a'(0) \in \mathcal{O}_{a(0)}$ and $t \geqslant \tau_{a(0)}$.

Step 3. **Compactness of $\mathcal{M}(\mathcal{Z})$.** The compact set $\mathcal{M}(\mathcal{Z})$ can be covered with a finite open sub-cover $\{\mathcal{O}_b : b \in F\}$ for some finite subset $F \subset \mathcal{M}(\mathcal{Z})$. Defining a finite stopping time $T \triangleq \max_{b\in F} \tau_b$, we obtain that $\|\Phi_t(a) - a^*\| < \epsilon$ for any initial condition $a \in \mathcal{M}(\mathcal{Z})$ and all times $t \geqslant T$.

□

Theorem 3.2 (Asymptotic convergence and independence at stationarity). *Assume the Lipschitz assumption on the transition rates of a single particle in empirical distribution and Assumption 1.2 on the irreducibility of CTMC A^N. Let $a^* \in \mathcal{S}$ be a globally asymptotically stable equilibrium. Then,*

1. *For all $a \in \mathcal{M}(\mathcal{Z})$, we have $\lim_{N\to\infty} P\{A_N(\infty) = a\} = \mathbb{1}_{\{a=a^*\}}$.*
2. *For any finite $F \subseteq \mathbb{N}$, we have $\lim_{N\to\infty} P\Big(\cap_{n\in F} \{X_n^N(\infty) = z_n\} \Big) = \prod_{n\in F} a^*_{z_n}$.*

Proof. We denote the distribution of $A^N(t)$ by $\pi^{A^N(t)}$ such that for any $a \in \mathcal{M}_N(\mathcal{Z})$, we have

$$\pi_a^{A^N(t)} \triangleq P\left\{A^N(t) = a\right\}.$$

1. If $\pi^{A_N(0)} = \pi^{A^N(\infty)}$, then $\pi^{A^N(t)} = \pi^{A^N(\infty)}$ for all $t \in \mathbb{R}_+$. Since $\mathcal{M}(\mathcal{M}(\mathcal{Z}))$ is a compact set, for any sequence $(\pi^{A^N(\infty)} : N \in \mathbb{N}) \subseteq \mathcal{M}(\mathcal{M}(\mathcal{Z}))$, there exists a converging sub-sequence, $(\pi^{A^{N_\ell}(\infty)} : \ell \in \mathbb{N})$ such that $\lim_{\ell\to\infty} \pi_a^{A^{N_\ell}(\infty)} \to \pi_a^*$ for all $a \in \mathcal{M}(\mathcal{Z})$. We observe that if $\pi^{A^N(0)} = \pi^{A^N(\infty)}$, then $\pi^{A^N(t)} = \pi^{A^N(\infty)}$ for all $t \in \mathbb{R}_+$. Since we have $\lim_{\ell\to\infty} \pi_a^{A^{N_\ell}(0)} = \pi_a^*$ for all $a \in \mathcal{M}(\mathcal{Z})$, it follows that $\lim_{\ell\to\infty} \pi_a^{A^{N_\ell}(t)} = \pi_a^*$ for all $t \in \mathbb{R}_+$ and $a \in \mathcal{M}(\mathcal{Z})$. From the continuity of map $\Phi_t : \mathcal{M}(\mathcal{Z}) \to \mathcal{M}(\mathcal{Z})$, we have $\lim_{N\to\infty} A^N(t) = \Phi_t(\lim_{N\to\infty} A^N(0))$. Thus, we have $\pi^* = \pi^* \circ \Phi_t^{-1}$, i.e. π^* is stationary under the map Φ_t. From global asymptotic stability of a^*, we observe that for every $\epsilon > 0$ there exists $T_\epsilon > 0$ such that support$(\pi^* \circ \Phi_t^{-1}) =$ support$(\pi^*) \subseteq B_\epsilon(a^*)$ for all $t \geqslant T_\epsilon$. Since the choice of $\epsilon > 0$ was arbitrary, it follows that support$(\pi^*) = \{a^*\}$, i.e. $\pi_a^* = \mathbb{1}_{\{a=a^*\}}$ for all $a \in \mathcal{M}(\mathcal{Z})$.

2. Let $\Phi_n \in C_b(\mathcal{Z})$ be bounded continuous functions for all $n \in \mathbb{N}$, then it suffices to show that

$$\lim_{N\to\infty} \mathbb{E} \prod_{n\in F} \Phi_n(X_n^N(\infty)) = \prod_{n\in F} \langle \Phi_n, a^* \rangle.$$

We will show the result for $|F| = 1$ and $|F| = 2$, and the result for general finite F follows by induction. We note that if the distribution of $(X_n^N(0) : n \in [N])$ is exchangeable, then so is the distribution of $(X_n^N(t) : n \in [N])$ at all $t \in \mathbb{R}_+$.

$|F| = 1$. Without any loss of generality, we assume $F = \{1\}$. From exchangeability, we can write

$$\mathbb{E}\Phi_1(X_1^N(\infty)) = \mathbb{E}\Big[\frac{1}{N}\sum_{n=1}^{N}\Phi_1(X_n^N(\infty))\Big] = \mathbb{E}\left\langle \Phi_1, A^N(\infty)\right\rangle.$$

Since $\lim_{N\to\infty}\pi_a^{A^N(\infty)} = \mathbb{1}_{\{a^*=a\}}$ for all $a \in \mathcal{M}(\mathcal{Z})$, the result follows.

$|F| = 2$. Without any loss of generality, we assume $F = \{1, 2\}$. From exchangeability, we can write

$$\mathbb{E}\Big[\Phi_1(X_1^N(\infty))\Phi_2(X_2^N(\infty))\Big] = \mathbb{E}\Big[\frac{1}{N(N-1)}\sum_{m\neq n=1}^{N}\Phi_1(X_m^N(\infty))\Phi_2(X_n^N(\infty))\Big].$$

Further, we can write

$$\mathbb{E}\Big[\left\langle \Phi_1, A^N(\infty)\right\rangle\left\langle \Phi_2, A^N(\infty)\right\rangle\Big] = \frac{1}{N^2}\mathbb{E}\Big[\sum_{n=1}^{N}\Phi_1(X_n^N(\infty))\Phi_2(X_n^N(\infty)) + \sum_{m\neq n=1}^{N}\Phi_1(X_m^N(\infty))\Phi_2(X_n^N(\infty))\Big].$$

Since $\lim_{N\to\infty}\pi_a^{A^N(\infty)} = \mathbb{1}_{\{a^*=a\}}$ for all $a \in \mathcal{M}(\mathcal{Z})$, the result follows.

$\square$