Lecture-14: Perturbation theory

1 Perturbation theory

We focus on the perturbation of nonlinear autonomous systems, where only the initial condition is perturbed. Consider a finite state space $[d]$, and a compact subset $\mathcal{D} \subseteq [0,1]^d$.

Definition 1.1. Consider a nonlinear autonomous system $\Phi : \mathbb{R}_+ \times \mathcal{D} \to \mathcal{D}$ defined in terms of $f : \mathcal{D} \subseteq [0,1]^d \to \mathbb{R}^d$ for each $t \in \mathbb{R}_+$ and initial condition $\Phi_0 = a \in \mathcal{D}$ as

$$\Phi_t(a) \triangleq a + \int_0^t \Phi_s(a) ds. \tag{1}$$

We note that the time derivative of Φ is $\frac{d}{dt}\Phi_t(a) = f(\Phi_t(a))$.

Definition 1.2. We fix two distributions $a, b \in \mathcal{D}$ and define the error in the first order approximation of $\Phi_t(b)$ by $\Phi_t(a)$ at all times $t \in \mathbb{R}_+$ as

$$e(t) \triangleq \Phi_t(b) - \Phi_t(a) - (b-a)\nabla\Phi_t(a). \tag{2}$$

Remark 1. Let a^* be a rest point of Φ_t such that $f(\Phi_t(a^*)) = 0$, then we redefine $\Phi_t(a) \triangleq \Phi_t(a) - a^*$ for all $a \in \mathcal{D}$. The rest point for this redefined Φ is $a^* = 0$. Thus, without loss of any generality, we can assume $a^* = 0$.

Definition 1.3. We define $c \triangleq \frac{1}{\epsilon}(b-a)$ for some $\epsilon > 0$, and consider the case when $\|c\| = \frac{1}{\epsilon}\|b-a\| \leqslant \tilde{c}$.

Assumption 1.4 (Lipschitz partial derivative). For any $i \in [d]$, the function $f_i : [0,1] \to \mathbb{R}$ is twice continuously differentiable. Therefore, the Jacobian matrix $(\nabla f(a))_{j,i} = \frac{\partial f_i(a)}{\partial a_j}$ is Lipschitz, such that there exists $L > 0$ such that

$$\sup_{c \in \mathcal{D}} \|c(\nabla f(b) - \nabla f(a))\| \leqslant L\|c\|\|b-a\|.$$

Remark 2. Consider ϵ-balls $B(a,\epsilon) \triangleq \{b \in \mathcal{D} : \|b-a\| \leqslant \epsilon\}$, and $\{B(a,\epsilon) : a \in \mathcal{D}\}$ is an open cover for $\mathcal{D}$, From compactness of $\mathcal{D} \subseteq [0,1]^d$, it can be covered by a finite sub-cover such that $\{B(a,\epsilon) : a \in F\}$ covers $\mathcal{D}$ for some finite F. For any $b \in \mathcal{D}$ choose $b^* \in F$ such that $b \in B(b^*,\epsilon)$. Then, from triangle inequality, we can write for all $z \in \mathcal{Z}, b \in \mathcal{D}$

$$\|\nabla f_z(b)\| \leqslant \|\nabla f_z(b) - \nabla f_z(b^*)\| + \|\nabla f_z(b^*)\| \leqslant L\epsilon + \max\{\|\nabla f_z(a)\| : a \in F\} < \infty.$$

Assumption 1.5 (Stability). The dynamical system Φ has a unique equilibrium point $a^* = 0$ and is exponentially stable. In other words, there exist positive constants α and κ such that starting from any initial condition $a \in \mathcal{D}$ and at any time $t \in \mathbb{R}_+$

$$\|\Phi_t(a)\| \leqslant \kappa\|a\|e^{-\alpha t}. \tag{3}$$

Proposition 1.6 (Khalil). *For any nonlinear autonomous system Φ defined in* (1) *under Assumption 1.5, there exist a Lyapunov function $V : \mathcal{D} \to \mathbb{R}_+$ and constants c_u, c_ℓ, c_d, c_p such that*

$$c_\ell\|a\|^2 \leqslant V(a) \leqslant c_u\|a\|^2, \tag{4}$$

$$\|\nabla V(a)\| \leqslant c_p\|a\|, \tag{5}$$

$$\dot{V}(a) \leqslant -c_d\|a\|^2. \tag{6}$$

Definition 1.7. For an autonomous nonlinear system Φ defined in (1), with initial condition a and perturbed initial condition $b = a + \epsilon c$, we define $\tilde{\Phi} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{D}$ as $\tilde{\Phi}(t, \epsilon) \triangleq \Phi_t(a + \epsilon c)$ for all $t, \epsilon \in \mathbb{R}_+$.

Remark 3. We observe that $\frac{d}{d\epsilon}\tilde{\Phi}(t, 0) = c\nabla\Phi_t(a)$, and can write the finite Taylor series for $\tilde{\Phi}(t, \epsilon)$ as

$$\tilde{\Phi}(t, \epsilon) = \tilde{\Phi}(t, 0) + \epsilon\frac{d}{d\epsilon}\tilde{\Phi}(t, 0) + e(t), \qquad \tilde{\Phi}(0, \epsilon) = a + \epsilon c.$$

We write the time derivative of autonomous nonlinear system with perturbed initial condition as

$$\dot{\tilde{\Phi}}(t, \epsilon) = \dot{\tilde{\Phi}}(t, 0) + \epsilon\frac{d}{dt}\frac{d}{d\epsilon}\tilde{\Phi}(t, 0) + \dot{e}(t) = f(\tilde{\Phi}(t, \epsilon)).$$

Definition 1.8. We define $\Psi : \mathbb{R}_+ \to \mathcal{D}^2$ as $\Psi(t) \triangleq (\Psi^0(t), \Psi^1(t))$, where $\Psi^0(t) \triangleq \tilde{\Phi}(t, 0)$ and $\Psi^1(t) \triangleq \frac{d}{d\epsilon}\tilde{\Phi}(t, 0)$ for all $t \in \mathbb{R}_+$.

Remark 4. We observe that $\Psi^0(t)$ is the unperturbed autonomous nonlinear system with initial condition a, such that $\dot{\Psi}^0(t) = f(\Psi^0(t))$ for all $t \in \mathbb{R}_+$, and $\Psi^0(0) = a$. Exchanging the two derivatives, we write the evolution of first order approximation in Taylor series of perturbed system, as

$$\dot{\Psi}^1(t) = \frac{d}{dt}\frac{d}{d\epsilon}\tilde{\Phi}(t, 0) = \frac{d}{d\epsilon}f(\tilde{\Phi}(t, 0)) = \Big[\frac{d}{d\epsilon}\tilde{\Phi}(t, 0)\Big]\nabla f(\tilde{\Phi}(t, 0)) = \Psi^1(t)\nabla f(\tilde{\Phi}(t, 0)), \qquad \Psi^1(0) = c.$$

Definition 1.9. We define two functions $\rho : \mathcal{D}^2 \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ and $\gamma : \mathcal{D}^2 \times \mathbb{R}_+ \to \mathbb{R}^d$ for any time $t \in \mathbb{R}_+$, as

$$\rho(\Psi(t), e(t), \epsilon) \triangleq f(e(t) + \Psi^0(t) + \epsilon\Psi^1(t)) - f(\Psi^0(t) + \epsilon\Psi^1(t)) - e(t)\nabla f(\Psi^0(t)), \qquad \rho(\Psi(t), 0, \epsilon) = 0,$$
$$\gamma(\Psi(t), \epsilon) \triangleq f(\Psi^0(t) + \epsilon\Psi^1(t)) - f(\Psi^0(t)) - \epsilon\Psi^1(t)\nabla f(\Psi^0(t)).$$

Lemma 1.10. *In terms of ρ and γ defined in Definition 1.9, we can write the evolution of e as*

$$\dot{e}(t) = e(t)\nabla f(\Psi^0(t)) + \rho(\Psi(t), e(t), \epsilon) + \gamma(\Psi(t), \epsilon).$$

Proof. Using the definition of maps Ψ^0 and Ψ^1, we can write the error function as $e(t) = \tilde{\Phi}(t, \epsilon) - \Psi^0(t) - \epsilon\Psi^1(t)$ for all $t \in \mathbb{R}_+$. From the evolution of maps Ψ^0, Ψ^1, we can write the evolution of error function as

$$\dot{e}(t) = f(\tilde{\Phi}(t, \epsilon)) - f(\Psi^0(t)) - \epsilon\Psi^1(t)\nabla f(\Psi^0(t)), \qquad e(0) = 0. \tag{7}$$

From the definition of $\Psi^0, \Psi^1, \rho, \gamma$, we observe that

$$\rho(\Psi(t), e(t), \epsilon) + \gamma(\Psi(t), \epsilon) = f(\tilde{\Phi}(t, \epsilon)) - f(\Psi^0(t)) - e(t)\nabla f(\Psi^0(t)) - \epsilon\Psi^1(t)\nabla f(\Psi^0(t)).$$

The result follows from the expression for $\dot{e}$ in (7). □

Lemma 1.11. *Consider an autonomous nonlinear system Φ defined in* (1) *under Assumption 1.4, Ψ in Definition 1.8, and ρ, γ in Definition 1.9, we have*

$$\|\gamma(\Psi(t), \epsilon)\| \leqslant \epsilon^2 \|\Xi(t)\|, \qquad \|\rho(\Psi(t), e(t), \epsilon)\| \leqslant L\|e(t)\|(\epsilon\left\|\Psi^1(t)\right\| + \|e(t)\|),$$

where $\Xi_\ell(t) \triangleq \Psi^1(t)\nabla^2 f_\ell(\xi_\ell(t))(\Psi^1(t))^T$ for some $\xi_\ell(t) \in \Psi^0(t) + \epsilon[0, \Psi^1(t)]$ for all $\ell \in [d]$.

Proof. Recall that f is twice differentiable with partial derivatives being Lipschitz under Assumption 1.4.

1. Fix $\ell \in [d]$. From the mean value theorem applied to $f_\ell : [0, 1] \to [0, 1]$ for the vector duration $\Psi^0(t) + \epsilon[0, \Psi^1(t)]$, there exists $\alpha'_\ell \in [0, 1]$ such that

$$\gamma_\ell(\Psi(t), \epsilon) = \epsilon\left\langle\Psi^1(t), (\nabla f_\ell(\Psi^0(t) + \alpha'_\ell\epsilon\Psi^1(t)) - \nabla f_\ell(\Psi^0(t)))\right\rangle.$$

From the mean value theorem applied to $\nabla f_\ell : [0, 1]^d \to [0, 1]^d$ for the vector duration $\Psi^0(t) + \alpha'_\ell\epsilon[0, \Psi^1(t)]$, there exists $\alpha_\ell \in [0, 1]$ such that

$$\gamma_\ell(\Psi(t), \epsilon) = \alpha'_\ell\epsilon^2\Psi^1(t)\nabla^2 f_\ell(\Psi^0(t) + \alpha_\ell\alpha'_\ell\epsilon\Psi^1(t))(\Psi^1(t))^T.$$

For each $\ell \in [d]$, we define $\xi_\ell(t) \triangleq \Psi^0(t) + \alpha_\ell\alpha'_\ell\epsilon\Psi^1(t)$ to observe that $\xi_\ell(t) \in \Psi^0(t) + \epsilon[0, \Psi^1(t)]$ and $|\gamma_\ell(\Psi(t), \epsilon)| \leqslant \epsilon^2|\Xi_\ell(t)|$. The first result follows from taking the square root of the sum of the squares on both sides over all $\ell \in [d]$.

2. Fix $i, \ell \in [d]$. We observe that $\frac{\partial \rho_\ell}{\partial e_i}(\Psi(t), e(t), \epsilon) = \nabla_i f_\ell(e(t) + \Psi^0(t) + \epsilon\Psi^1(t)) - \nabla_i f_\ell(\Psi^0(t))$. From the mean value theorem applied to $f_\ell : [0,1] \to [0,1]$ for the vector duration $\Psi^0(t) + \epsilon\Psi^1(1) + [0, e(t)]$, there exists $a \in [0,1]^d$ such that

$$\rho(\Psi(t), e(t), \epsilon) = e(t)\Big(\nabla f(\Psi^0(t) + \epsilon\Psi^1(t) + a \circ e(t)) - \nabla f(\Psi^0(t))\Big).$$

The second result follows from the Lipschitz condition on partial derivatives of f_ℓ under Assumption 1.4 and Cauchy-Schwartz inequality.

□

Proposition 1.12. *Consider an autonomous nonlinear system Φ defined in* (1) *under Assumption 1.4 and Assumption 1.5. There exists a Lyapunov function $V : \mathbb{R}^d \to \mathbb{R}_+$ such that*

$$V(e(t)) \leqslant \frac{c_p^2}{4c_\ell}\epsilon^4 \Big(\int_0^t \phi(t,\tau) \left\|\Xi(\tau)\right\| d\tau\Big)^2,$$

where $\phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is defined as $\phi(t,\tau) \triangleq \exp\Big(-\frac{c_d}{2c_u}(t-\tau) + L\frac{c_p}{2c_\ell}\int_\tau^t (2\left\|e(s)\right\| + \left\|\Psi^0(s)\right\| + \epsilon\left\|\Psi^1(s)\right\|)ds\Big)$ for all $t, \tau \in \mathbb{R}_+$ and constants c_ℓ, c_u, c_p, c_d are defined in Proposition 1.6.

Proof. Under Assumption 1.5, there exists a Lyapunov function $V : \mathbb{R}^d \to \mathbb{R}_+$ that satisfies conditions (4), (5), (6) from Proposition 1.6. Further,

$$\dot{V}(e) = \langle \dot{e} - f(e), \nabla V(e)\rangle + \langle f(e), \nabla V(e)\rangle.$$

Recall that $\dot{e} = e\nabla f(\Psi^0(t)) + \rho(\Psi, e, \epsilon) + \gamma(\Psi, \epsilon)$ from Lemma 1.10. From the Cauchy-Schwartz inequality, we obtain

$$\langle \dot{e} - f(e), \nabla V(e)\rangle \leqslant \left\|\nabla V(e)\right\| \Big(\left\|e(\nabla f(\Psi^0) - \nabla f(0))\right\| + \left\|e\nabla f(0) - f(e)\right\| + \left\|\rho(\Psi, e, \epsilon)\right\| + \left\|\gamma(\Psi, \epsilon)\right\|\Big).$$

From the Lipschitz property of partial derivatives of f from Assumption 1.4, we obtain

$$\left\|e(\nabla f(\Psi^0) - \nabla f(0))\right\| \leqslant L\left\|e\right\|\left\|\Psi^0\right\|.$$

From the mean value theorem applied to f, there exists $\beta \in [0,1]^d$ such that $f(e) = f(0) + e\nabla f(\beta \circ e)$, where $a^* = 0$ is a rest point of Φ and hence $f(0) = 0$. Together with Lipschitz property of partial derivatives of f from Assumption 1.4, we obtain

$$\left\|e\nabla f(0) - f(e)\right\| = \left\|e(\nabla f(0) - \nabla f(\beta \circ e))\right\| \leqslant L\left\|e\right\|^2.$$

From Lemma 1.11, we have $\left\|\gamma\right\| \leqslant \epsilon^2\left\|\Xi\right\|$ and $\left\|\rho\right\| \leqslant L\left\|e\right\|(\epsilon\left\|\Psi^1\right\| + \left\|e\right\|)$. Aggregating these results, we obtain

$$\langle \dot{e} - f(e), \nabla V(e)\rangle \leqslant L\left\|\nabla V(e)\right\|\left\|e\right\|\Big(2\left\|e\right\| + \left\|\Psi^0\right\| + \epsilon\left\|\Psi^1\right\|\Big) + \epsilon^2\left\|\Xi\right\|\left\|\nabla V(e)\right\|. \tag{8}$$

For autonomous non-linear system Φ defined in (1) with initial condition e, we have $\dot{V}(\Phi_t(e)) = \langle f(\Phi_t(e)), \nabla V(\Phi_t(e))\rangle$. Since $\Phi_0(e) = e$, we observe that

$$\dot{V}(\Phi_t(e))\Big|_{t=0} = \langle f(\Phi_0(e)), \nabla V(\Phi_0(e))\rangle = \langle f(e), \nabla V(e)\rangle.$$

We have $\dot{V}(a) \leqslant -c_d\left\|a\right\|^2$ from (6) and $-\left\|a\right\|^2 \leqslant -\frac{V(a)}{c_u}$ from (4). Substituting these results in the above equation, we obtain $\langle f(e), \nabla V(e)\rangle = \dot{V}(\Phi_t(e))\Big|_{t=0} \leqslant -\frac{c_d}{c_u}V(e)$. We have $\left\|\nabla V(a)\right\| \leqslant c_p\left\|a\right\|$ from (5) and $\left\|a\right\|^2 \leqslant \frac{V(a)}{c_\ell}$ from (4). Substituting these upper bounds in (8), we obtain

$$\dot{V}(e) \leqslant -\frac{c_d}{c_u}V(e) + L\frac{c_p}{c_\ell}V(e)\Big(2\left\|e\right\| + \left\|\Psi^0\right\| + \epsilon\left\|\Psi^1\right\|\Big) + \frac{c_p}{\sqrt{c_\ell}}\epsilon^2\left\|\Xi\right\|\sqrt{V(e)}.$$

Defining $W : \mathbb{R}_+ \to \mathbb{R}_+$ as $W(t) \triangleq \sqrt{V(e(t))}$, we observe $\dot{W} = \frac{1}{2}\frac{\dot{V}(e)}{\sqrt{V(e)}}$. We further define $h : \mathbb{R}_+ \to \mathbb{R}$ as $h(t) \triangleq -\frac{c_d}{2c_u} + L\frac{c_p}{2c_\ell}\left(2\left\|e(t)\right\| + \left\|\Psi^0(t)\right\| + \epsilon\left\|\Psi^1(t)\right\|\right)$ for all $t \in \mathbb{R}_+$. In terms of W and h, we write

$$\dot{W} \leqslant -\frac{c_d}{2c_u}W + L\frac{c_p}{2c_\ell}W\left(2\left\|e\right\| + \left\|\Psi^0\right\| + \epsilon\left\|\Psi^1\right\|\right) + \frac{c_p}{2\sqrt{c_\ell}}\epsilon^2\left\|\Xi\right\| = hW + \frac{c_p}{2\sqrt{c_\ell}}\epsilon^2\left\|\Xi\right\|.$$

We observe that $\phi(t,\tau) = e^{-\int_\tau^t h(s)ds}$ for each $t, \tau \in \mathbb{R}_+$, and hence we obtain from Gronwall's inequality

$$W(t) \leqslant \phi(t,0)W(0) + \frac{c_p}{2\sqrt{c_\ell}}\epsilon^2\int_0^t \phi(t,\tau)\left\|\Xi(\tau)\right\| d\tau.$$

Result follows from the fact that $e(0) = 0$ and $V(0) = 0$. $\square$