On Kernelized Multi-armed Bandits

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ICML
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Overview

Problem Formulation

Algorithms

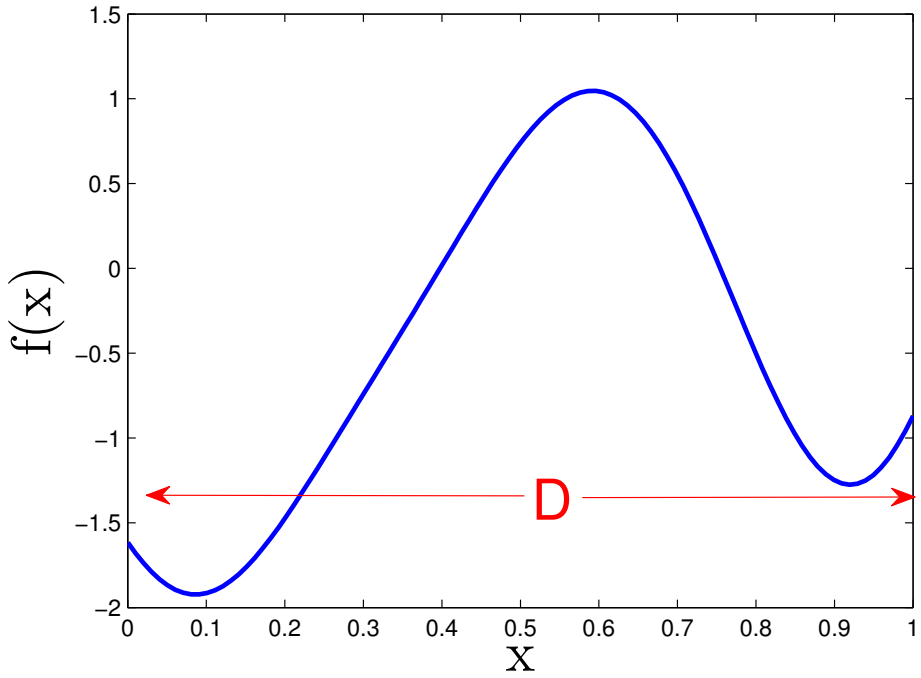
Regret Bounds

Numerical Results

Conclusion

Problem Statement

Sequentially Maximize $f : D \to \mathbb{R}$



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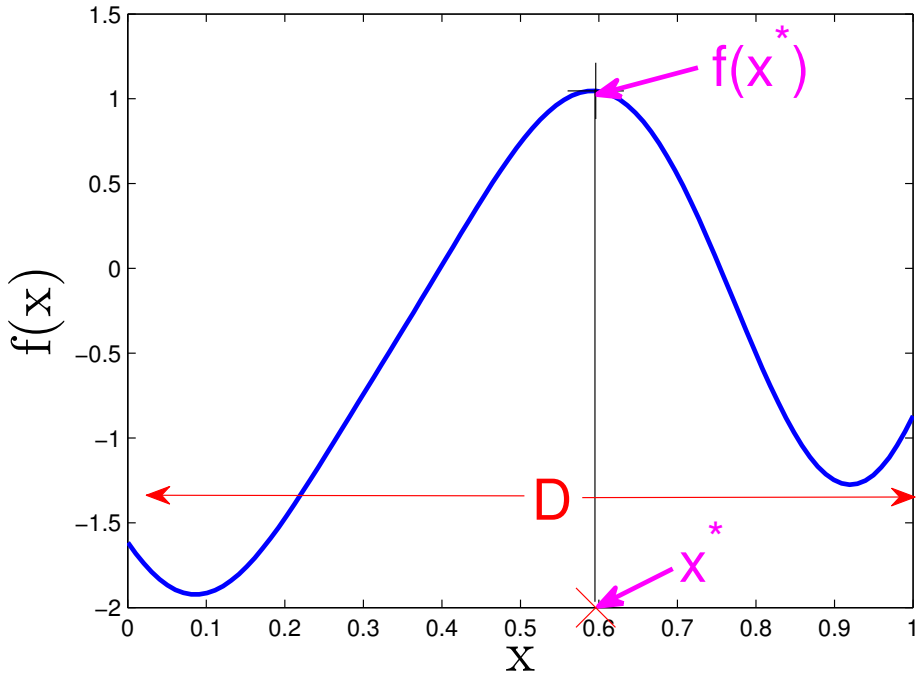

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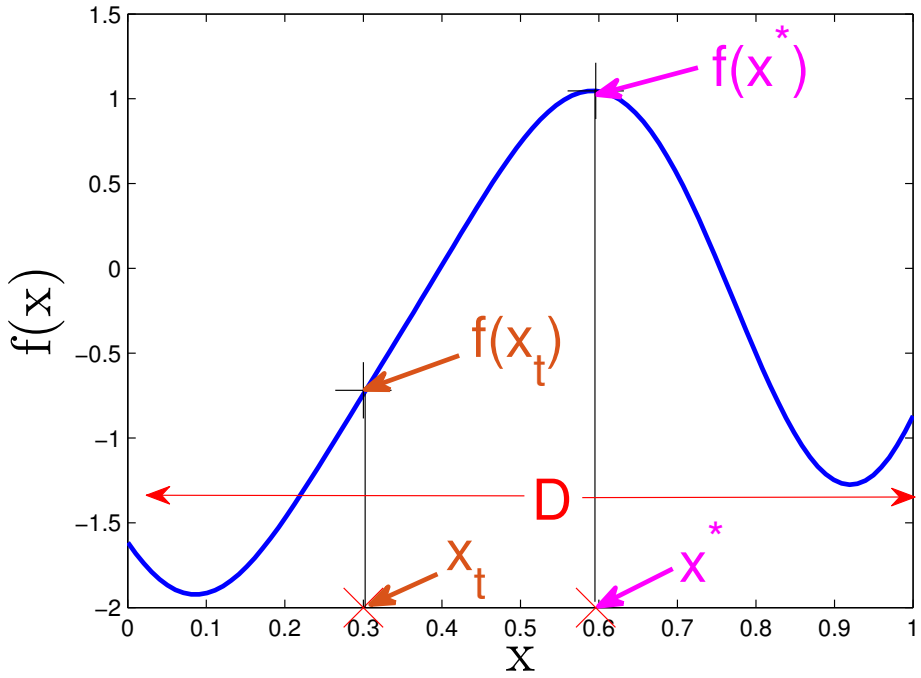

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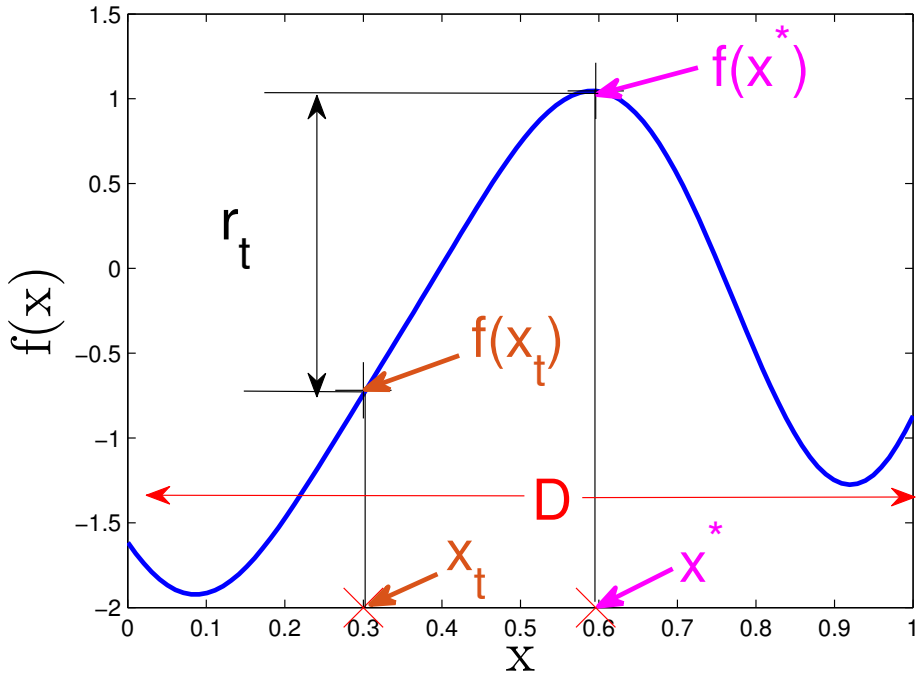

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Performance Metric

- **Regret** $r_t = f(x^\star) - f(x_t)$
- Goal: Minimize cumulative regret $\sum_{t=1}^{T} r_t$

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- Positive semi-definite kernel function $k : D \times D \to \mathbb{R}$ (known)
- Reproducing property: $f(x) = \langle f, k(x, \cdot) \rangle_k$
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- Bounded variance: $k(x, x) \leq 1$, for all $x \in D$

Example Kernels

- Squared Exponential kernel: $k(x, y) = \exp\left(\frac{-\|x-y\|_2^2}{2l^2}\right)$
- Matérn kernel: $k(x, y) = \frac{2^{1-\nu}}{\Gamma(\nu)}\left(\frac{\|x-y\|_2\sqrt{2\nu}}{l}\right)^{\nu} B_\nu\left(\frac{\|x-y\|_2\sqrt{2\nu}}{l}\right)$
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- Linear Kernel:
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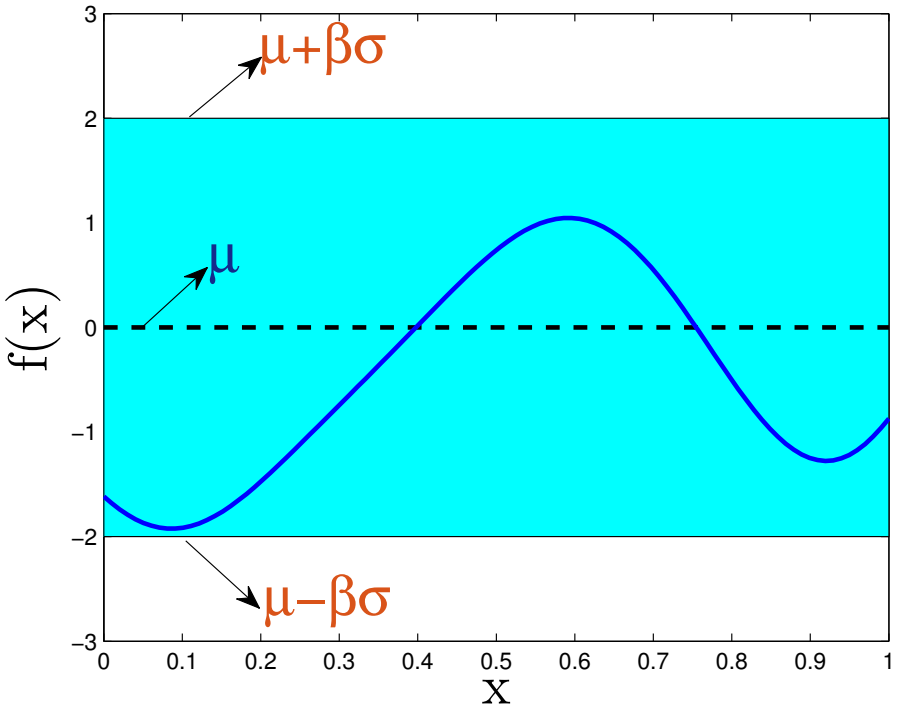
  - Reduces to parametric **linear bandit** problem (Dani et al., COLT 2008, Abbasi-Yadkori et al., NIPS 2011, ...)

Algorithm Design Philosophy: Gaussian Processes



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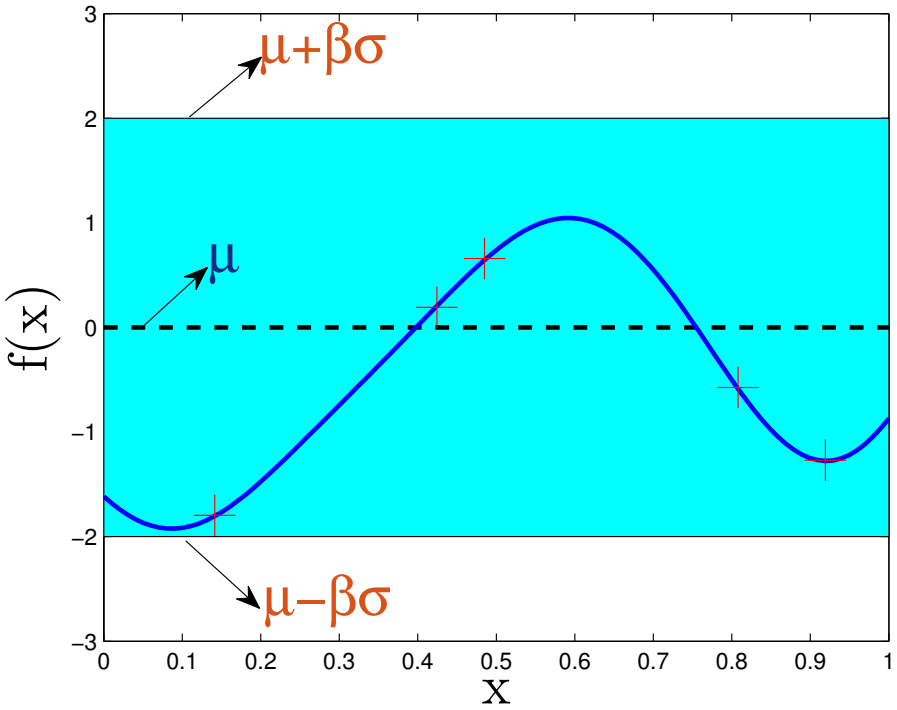

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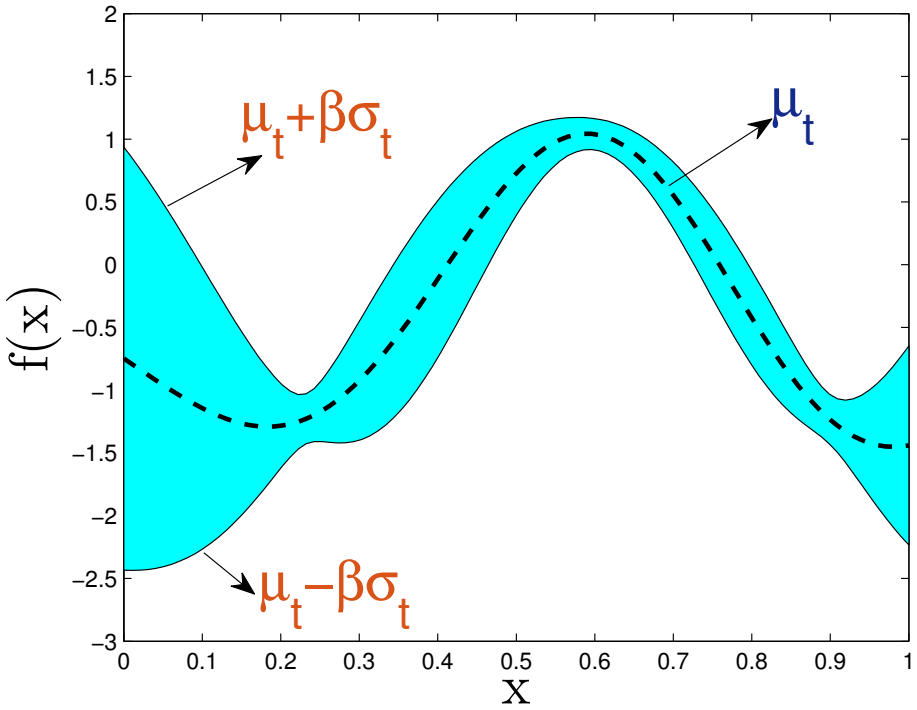

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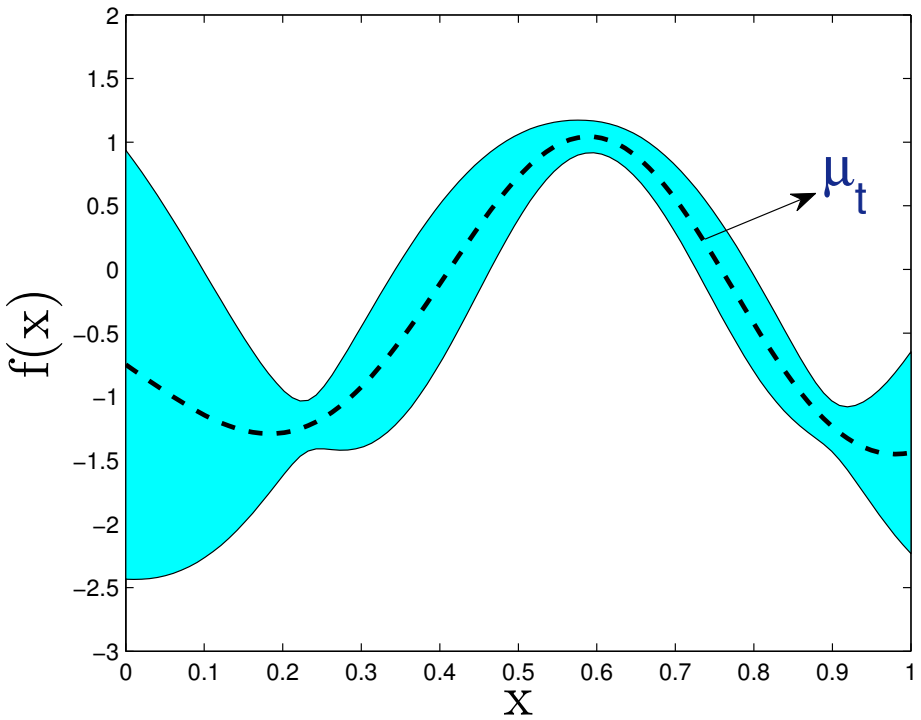
Posterior of f after t rounds: $GP(\mu_t(x), v^2 k_t(x,y))$

$$\begin{aligned} \mu_t(x) &= k_t(x)^T (K_t + \lambda I)^{-1} y_{1:t} \\ k_t(x,y) &= k(x,y) - k_t(x)^T (K_t + \lambda I)^{-1} k_t(y) \end{aligned}$$

Algorithm 1: Improved GP-UCB (IGP-UCB)

Key Idea: Play the arm with highest UCB



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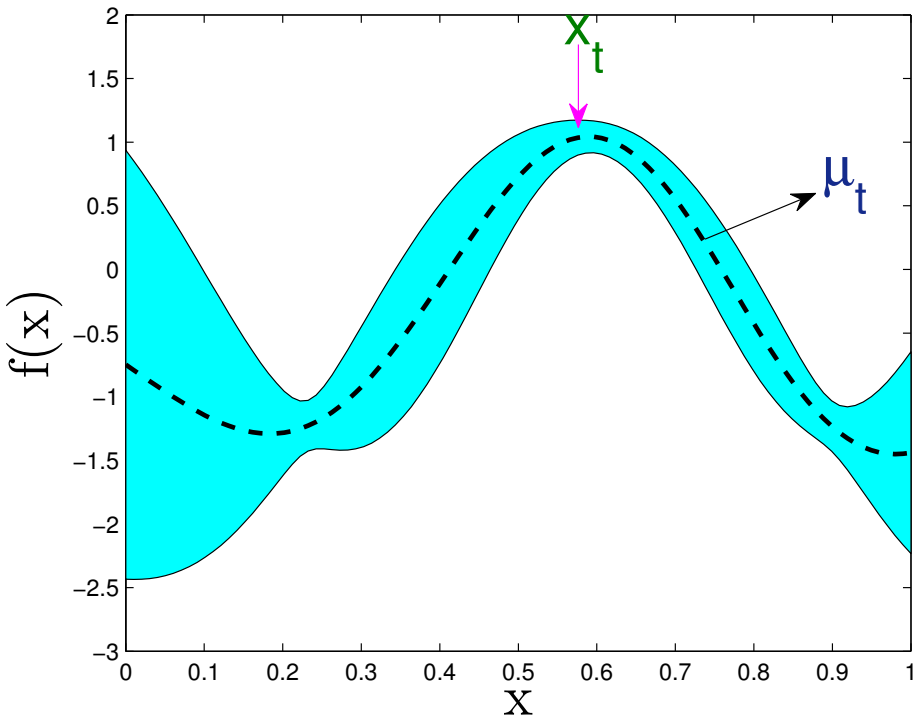

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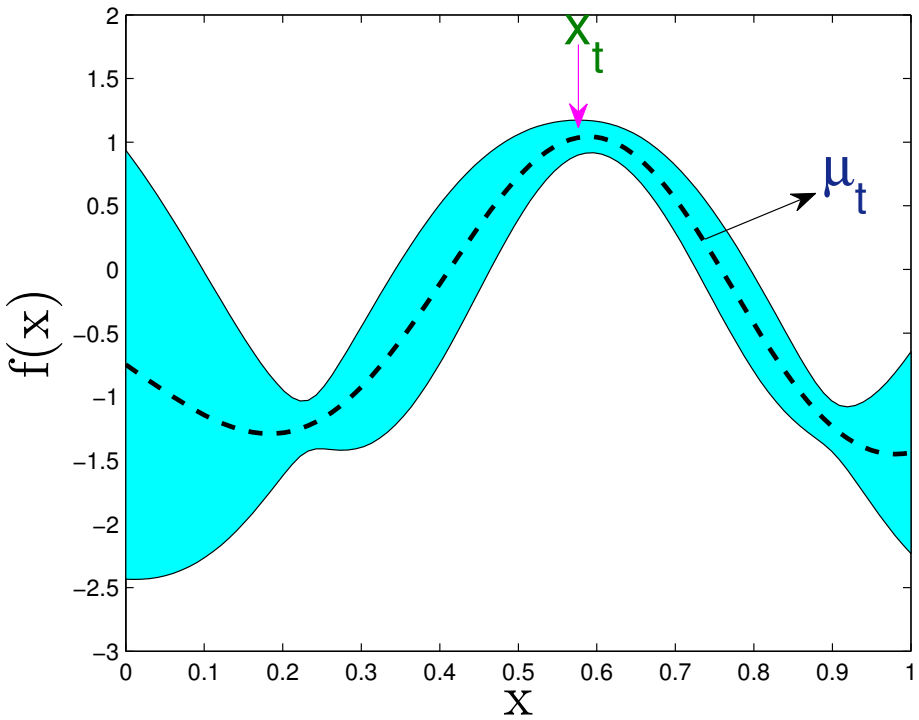

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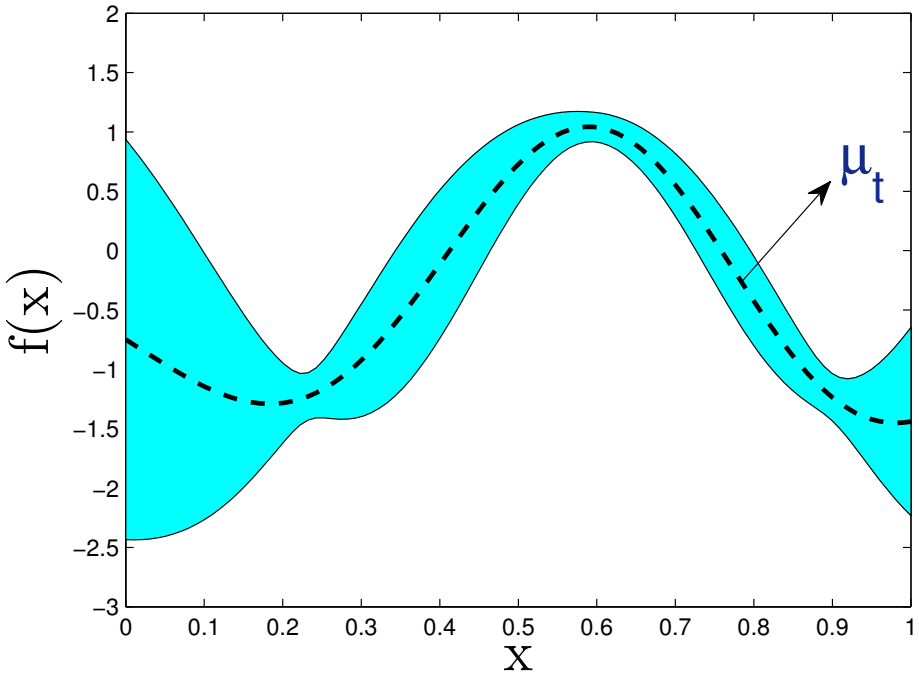
- β_t trades off b/w **exploration** and **exploitation**
- Reduced width (β_t) of confidence interval compared to GP-UCB (Srinivas et al., ICML 2010)

Algorithm 2: Gaussian Process Thompson Sampling (GP-TS)

Key Idea: Sample a random function and play its maximizer



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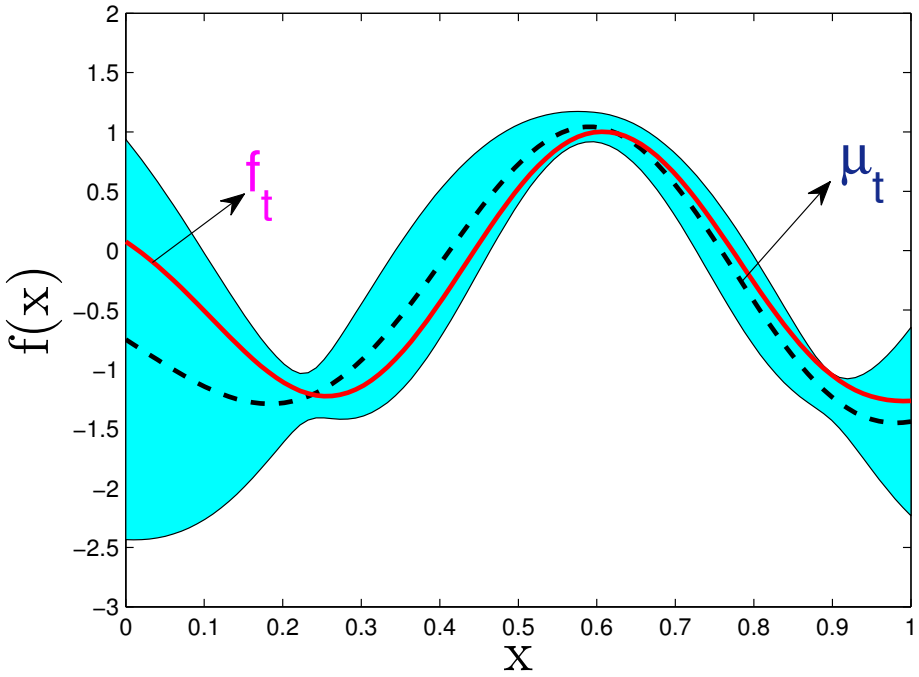

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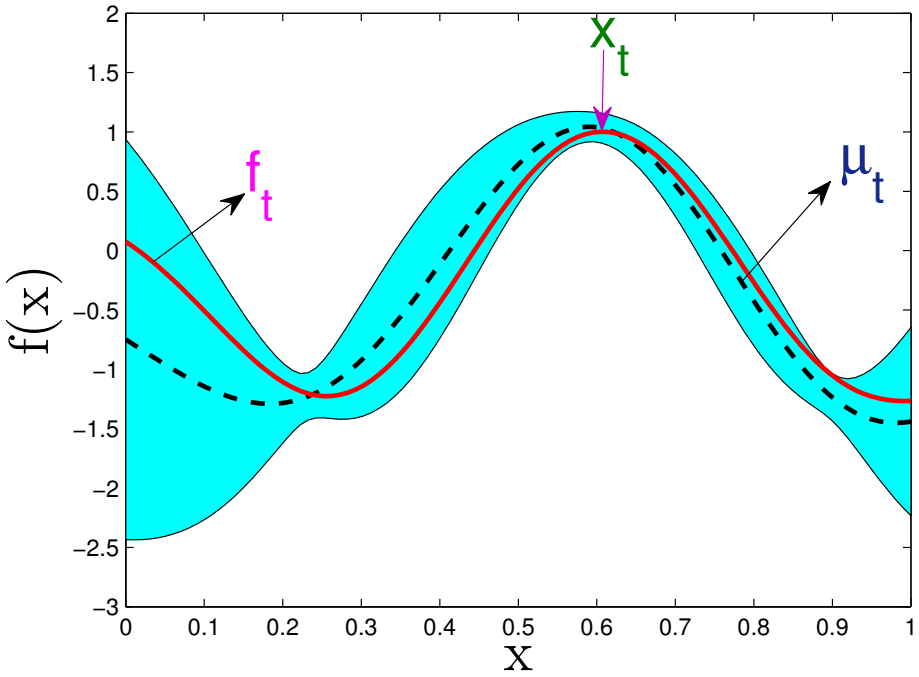

At each round t:

- Sample f_t from posterior of f
- Play $x_t = \operatorname*{argmax}_{x \in D_t} f_t(x)$

$D_t \subset D$: suitably chosen Discretization sets

Regret Bound for IGP-UCB

Result 1

Regret of IGP-UCB is $O\left(\sqrt{T}(B\sqrt{\gamma_T} + \gamma_T)\right)$ whp with the choice of confidence width $\beta_t \approx B + \sqrt{\gamma_t}$ for all t

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$$\gamma_T = \max_{A \subset D : |A| = T} I(y_A; f_A)$$

  - Mutual Information b/w function values and rewards at set A
  - Reduction in uncertainty about f after observing rewards
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- Regret of GP-UCB is $O\left(\sqrt{T}(B\sqrt{\gamma_T} + \gamma_T \ln^{3/2} T)\right)$ whp and so we improve by $O(\ln^{3/2} T)$!

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- Regret of GP-TS is $O\left(\sqrt{Td\ln(BdT)}(B\sqrt{\gamma_T}+\gamma_T)\right)$ whp
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Open Question: Can the logarithmic dependency be removed?

Recovering Regret Bounds for Linear Bandits

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- Lower Bound: $\Omega(d\sqrt{T})$ (Dani et al., COLT 2008)

Numerical Results

Algorithms Compared:

1. GP-Expected Improvement (Močkus, 1975)
2. GP-Probabilistic Improvement (Kushner, 1964)
3. GP-UCB (Srinivas et al., 2010)
4. IGP-UCB (this work)
5. GP-TS (this work)

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f sampled from RKHS
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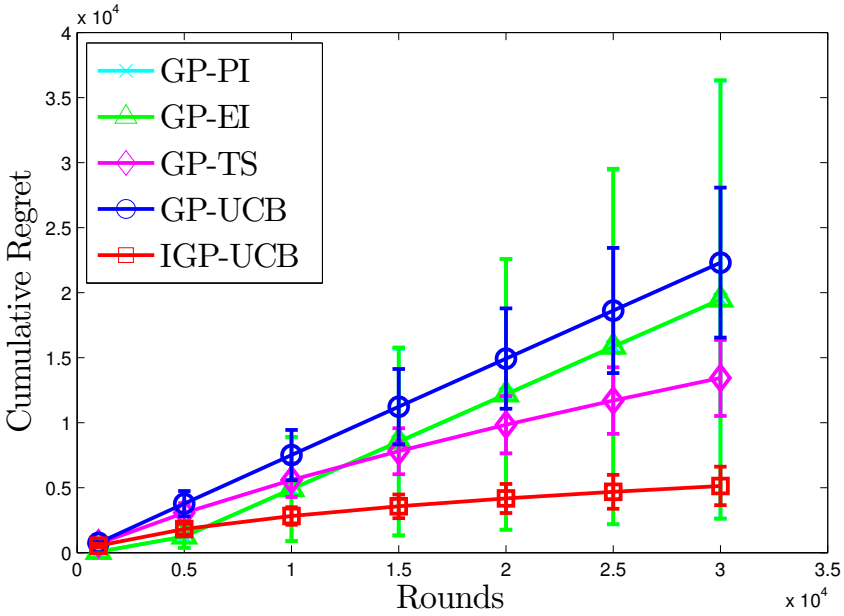

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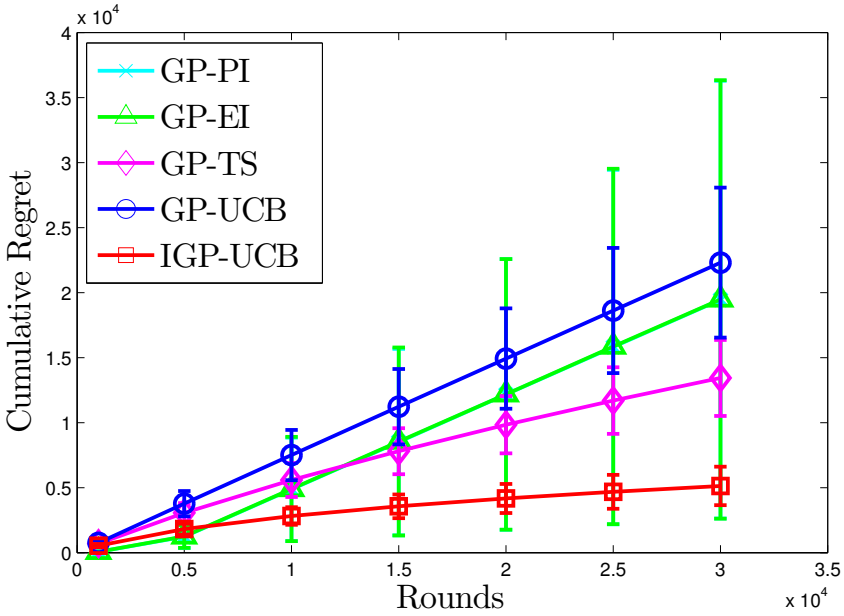

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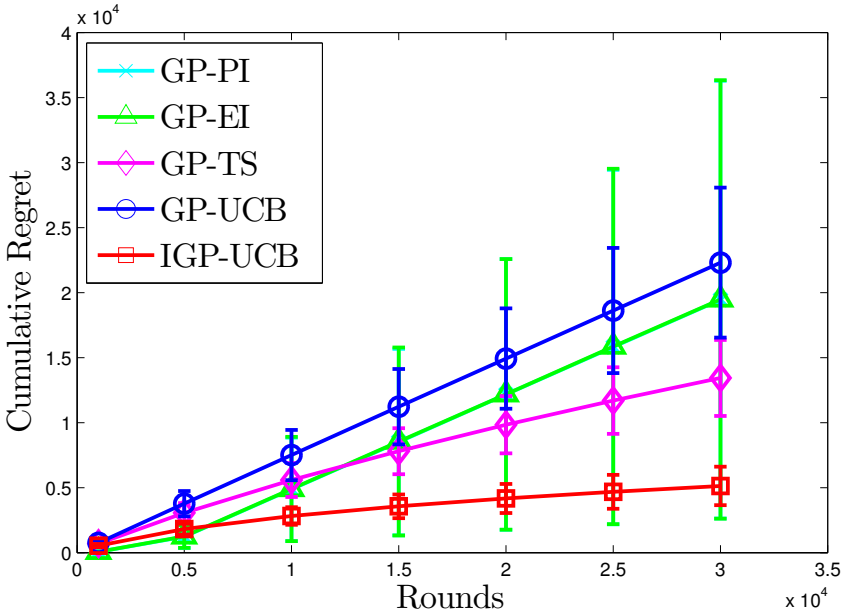
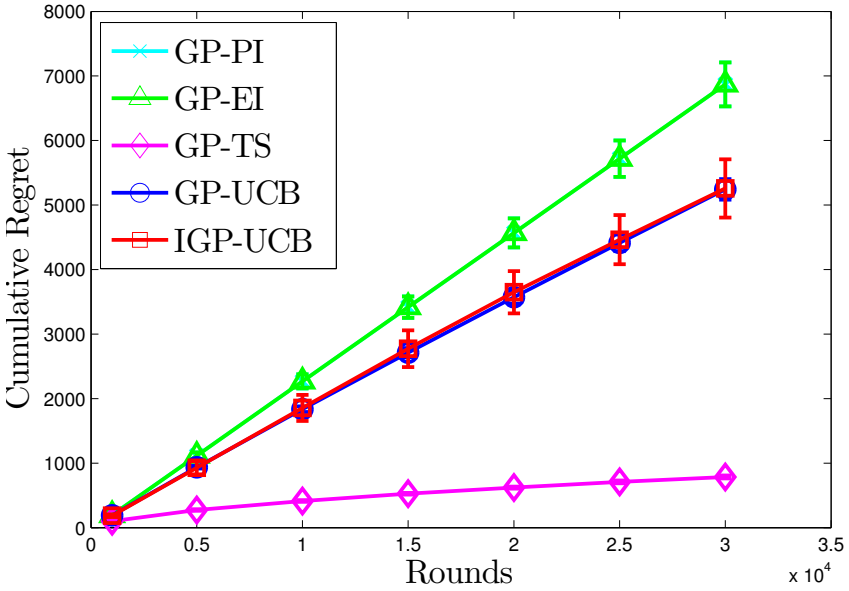


Temperature Sensor Data
(Intel Berkeley Research lab)



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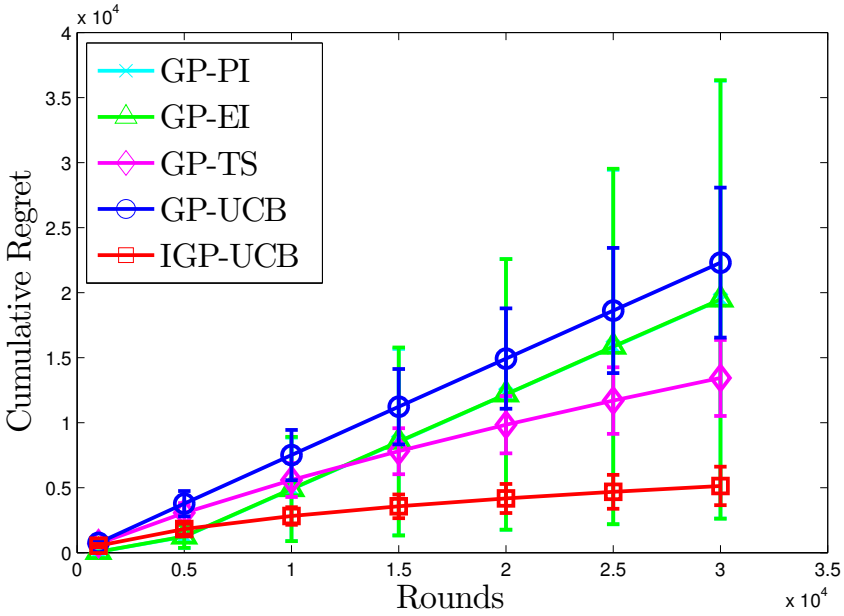

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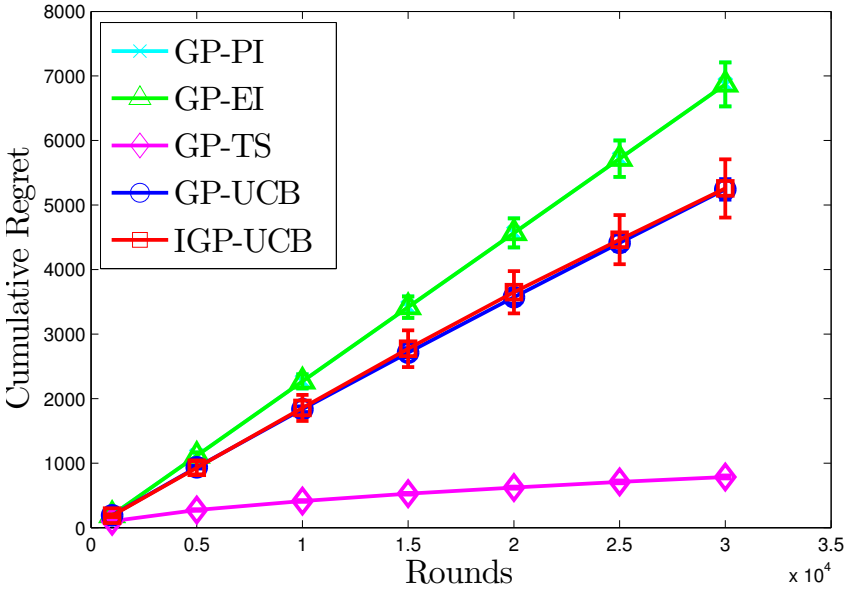

- IGP-UCB performs similar to GP-UCB ✓
- GP-TS performs the best ☺

Key Tool: New Concentration Inequality

Setup:

- Feature map $\varphi : D \to \text{RKHS}$
- $S_t = \sum_{s=1}^{t} \varepsilon_s \varphi(x_s) \leftarrow$ RKHS-valued Martingale
- $V_t = I + \sum_{s=1}^{t} \varphi(x_s)\varphi(x_s)^T \leftarrow$ possibly of infinite dimension

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Result 3: Self-Normalized CI for RKHS-valued Martingales

- For all t: $\|S_t\|^2_{V_t^{-1}} \leq 2R^2 \ln(\frac{\sqrt{\det(K_t+I)}}{\delta})$ with probability at least $1-\delta$ if K_t is positive-definite
- **Generalizes** finite-dimensional Inequality for vector-valued Martingales (Abbasi-Yadkori et al., NIPS 2011)
- Curse of Dimensionality $\to$ Mixing over Gaussian Processes

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Future Work:

- Kernel function not known to the learner
- Time varying functions from RKHS

Thank You

Poster Tonight