Lecture 15

Agenda: * Universal portfolio formulation and constantly balanced portfolio

* Solution using probability assignment

* Lower bounds

[A] Universal Portfolio Formulation

Consider two stocks A and B.

Suppose over a duration of n days stock A is valued at Rs 100, and stock B is rather volatile and follows the price pattern

1000, 2000, 1000, 2000, ...

An investor with Rs 1 lakh at the outset splits his total wealth between the two stocks (in appropriate ratio).

If the investor puts all his money on any one stock, she will not make any money at the end.

On the other hand, with the crystal clarity of hindsight, an investor may choose to invest the 1 lakh amount in two stocks in equal proportion. Then, each day, the stock either increases by a ratio of $\left(\frac{1}{2}\cdot 1+\frac{1}{2}\cdot 2\right)=\frac{3}{2}$ or decreases by $\left(\frac{1}{2}\cdot 1+\frac{1}{2}\cdot\frac{1}{2}\right)=\frac{3}{4}$. Thus, on any two consecutive days the portfolio will grow by $9/8$ and double every 12 days!

Both strategies above are instances of a "constantly balanced (or rebalanced) portfolio" where every day we split our wealth in the same proportion b/w the two stocks. Note that the actual value of the stock does not enter our calculation since we allow the purchase of any fraction of the share. The performance of a given constantly balanced portfolio depends on the actual sequence of gains, which is $(1,2), (1,\frac{1}{2}), (1,2), (1,\frac{1}{2}), \ldots$ in the example above. Our goal is to compete with the best constantly balanced portfolio in hindsight.

Formally, consider the sequence $x_1, \ldots, x_n$ where each $x_i \in [0,1]^2$. At time t, we are allowed to see the performance x^{t-1} and decide on what fraction $\theta_t \in [0,1]$ of our wealth should we invest on stock A. Thus, the gain at time t is

$$r_t = \overset{\equiv \theta_{t1}}{\theta_t}\, x_{t1} + \overbrace{(1-\theta_t)}^{\equiv \theta_{t2}}\, x_{t2}.$$

The overall gain after n days is given by

$$T_n(\underline{x}, \hat{\theta}) \quad \prod_{t=1}^{n} r_t = \prod_{t=1}^{n} (\theta_{t1} x_{t1} + \theta_{t2} x_{t2})$$

For a constantly rebalanced portfolio (CRP),

$$\theta_t = \theta \quad \forall\, t.$$

Our goal is to remain multiplicatively competitive with ③
the best CRP in hindsight. To that end, for a
$\hat{\theta}_t : x^{t-1} \mapsto \theta_t = \hat{\theta}_t(x^{t-1})$,

denote

$$r_n(\underline{x}, \hat{\theta}) = \left(\max_{\theta} \log T_n(\underline{x}, \theta)\right) - T_n(\underline{x}, \hat{\theta})$$

$$= \max_{\theta} \log \frac{T_n(\underline{x}, \theta)}{T_n(\underline{x}, \hat{\theta})},$$

where $T_n(\underline{x}, \theta)$ denotes the performance of CRP and equals $\prod_{t=1}^{n} (x_{t1}\theta + (1-\theta)x_{t2})$.

Denote

$$r_n = \min_{\hat{\theta}} \max_{\underline{x} \in [0,1]^{2 \otimes n}} r_n(\underline{x}, \hat{\theta})$$

We will see that $r_n = \frac{1}{2}\log n + O(1)$ and is attained by a simple Bayesian strategy.

$\boxed{B}$ Probability assignment and optimal portfolio

Note that for $k = 2$,

$$T_n(\underline{x}, \underline{\theta}) = \prod_{t=1}^{n} (\theta_{t1} x_{t1} + \theta_{t2} x_{t2})$$

$$= \sum_{\underline{j} \in \{1,2\}^n} \prod_{t=1}^{n} \theta_{t j_t} x_{t j_t}$$

$$= \sum_{\underline{j} \in \{1,2\}^n} \theta(\underline{j})\, x(\underline{j})$$

$\theta(\underline{j}) \equiv \prod_{t=1}^{n} \theta_{t j_t}$

$x(\underline{j}) \equiv \prod_{t=1}^{n} x_{t j_t}$

Therefore,

$$\log \frac{T_n(\underline{x},\underline{\theta})}{T_n(\underline{x},\underline{\theta}')} = \log \frac{\sum_{\underline{j}\in\{1,2\}^n} \underline{\theta}(\underline{j})\,\underline{x}(\underline{j})}{\sum_{\underline{j}\in\{1,2\}^n} \underline{\theta}'(\underline{j})\,\underline{x}(\underline{j})}$$

$$\leq \max_{\underline{j}\in\{1,2\}^n} \log \frac{\underline{\theta}(\underline{j})}{\underline{\theta}'(\underline{j})}$$

In particular,

$$\log \frac{T_n(\underline{x},\theta)}{T_n(\underline{x},\hat{\theta})} \leq \max_{\underline{j}\in\{1,2\}^n} \log \frac{\theta^{n(1|\underline{j})}(1-\theta)^{n-n(1|\underline{j})}}{\hat{\theta}(\underline{j}|\underline{x})},$$

where

$$\hat{\theta}(\underline{j}|\underline{x}) = \theta_1(j_1)\,\theta_2(j_2|x_1)\,\theta_3(j_3|x^2)\ldots$$
$$= \prod_{t=1}^{n} \theta_t(j_t|x^{t-1}),$$

where $x^0 = \phi$. Note that

$$\sum_{\underline{j}} \hat{\theta}(\underline{j}|\underline{x}) = 1.$$

Thus,

$$(1)\quad r_n \leq \min_{\hat{\theta}} \max_{\underline{x}} \max_{\underline{j}\in\{1,2\}^n} \max_{\theta} \log \frac{\theta^{n(1|\underline{j})}(1-\theta)^{n-n(1|\underline{j})}}{\hat{\theta}(\underline{j}|\underline{x})}.$$

We note that the cost function on the right-sides appears very similar to that for the probability assignment problem seen in the previous lecture. However, the observables are x^{t-1} and not j^{t-1}! Nevertheless, there is a very simple way to convert one problem to the other. Consider a distribution Q on $\{1,2\}^n$. Define $\hat{\theta} \equiv \hat{\theta}_Q$ by

$$\text{(2)}\quad \theta_t = \hat{\theta}(1|x^{t-1}) = \frac{\sum_{\underline{j}: j_t = 1} Q(\underline{j}) \prod_{i=1}^{t-1} x_{i j_i}}{\sum_{\underline{j}} Q(\underline{j}) \prod_{t=1}^{t-1} x_{i j_i}}$$

Then,
$$\hat{\theta}(\underline{j}|\underline{x}) = \prod_{t=1}^{n} \hat{\theta}(j_t | x^{t-1})$$

$$= \prod_{t=1}^{n} \left[\frac{\sum_{\underline{j}': j'_t = j_t} Q(\underline{j}') \prod_{i=1}^{t-1} x_{i j'_i}}{\sum_{\underline{j}'} Q(\underline{j}') \prod_{i=1}^{t-1} x_{i j'_i}} \right]$$

$$= \prod_{t=1}^{n} \left[\frac{\sum_{j'^{t-1}} Q(j_t | j'^{t-1}) \, Q(j'^{t-1}) \prod_{i=1}^{t-1} x_{i j'_i}}{\sum_{j'^{t-1}} Q(j'^{t-1}) \prod_{i=1}^{t-1} x_{i j'_i}} \right]$$

$$= Q(j_1) \cdot \left(\frac{Q(j_2|1)\, Q(1)\, x_{11} + Q(j_2|2)\, Q(2)\, x_{12}}{Q(1)\, x_{11} + Q(2)\, x_{12}} \right) \cdots$$

Note that

$$\left(\hat{\theta}(1)\, x_{11} + \hat{\theta}(2) x_{12}\right) \cdot \left(\hat{\theta}(1|x_1)\, x_{21} + \hat{\theta}(2|x_1)\, x_{22}\right)$$
$$= \sum_{\underline{j} \in \{1,2\}^2} Q(\underline{j}) \prod_{t=1}^{2} x_{t j_t},$$

and similarly for larger time horizons. Therefore, restricting the min. on the right-side of (1) to schemes $\hat{\theta}$ given by (2), we have

$$r_n \le \min_{Q \in \mathcal{P}_{[2]^n}} \max_{\underline{j} \in \{1,2\}^n} \max_{\theta \in [0,1]} \log \frac{\theta^{n(1|\underline{j})} (1-\theta)^{n(2|\underline{j})}}{Q(\underline{j})},$$

where the optimal Q for the right-side was obtained (using KT estimator) in the previous class.

$$\Rightarrow r_n \le r(2,n) = \frac{1}{2}\log n + O(1).$$

$\boxed{C}$ Lower bound

We have

$$\max_{\underline{x} \in [0,1]^{2n}} \log \frac{T_n(\underline{x}, \theta)}{T_n(\underline{x}, \hat{\theta})} \geq \max_{\underline{x} \in \{(1,0),(0,1)\}^n} \log \frac{T_n(\underline{x}, \theta)}{T_n(\underline{x}, \hat{\theta})}$$

For any $\underline{x} \in \{(1,0),(0,1)\}^n$ and $\underline{\theta}$, we have

$$T_n(\underline{x}, \underline{\theta}) = \prod_{t=1}^{n} \left(\theta_t \mathbb{1}_{\{x_{t1}=1\}} + (1-\theta_t)\mathbb{1}_{\{x_{t1}=0\}}\right)$$

$$= Q(\underline{x}_1)$$

where $Q(x_{1,t} = 1 \mid x_1^{t-1}) = \theta_t(1 \mid x^{t-1}) = \theta_t(1 \mid x_1^{t-1})$.

Therefore, when restricted to these inputs, the problem gets reduced to the probability assignment problem, and so,

$$r_n \geq r(2, n)$$

Overall, we have

$$r_n = r(2, n),$$

and the portfolio (2) corresponding to Q_{KT} is optimal.