

Approximate Mean Delay Analysis for a Signalized Intersection with Indisciplined Traffic

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Abstract—Mixed vehicular traffic comprising small cars and two-wheeled vehicles (called motorcycles in this paper) arrive at a lane of a signalized road intersection. The traffic do not follow lane-discipline, in that the arriving vehicles do not necessarily queue up one behind the other. The motorcycles are small enough to stand side-by-side with cars or other motorcycles, so as to fill up the width of the lane. With such queue joining behaviour, the waiting vehicles form batches, comprising motorcycles and at most one car. During the green signal period the vehicles in the head-of-the-line batch exit the intersection together. In this paper, assuming a Poisson point process model for vehicle arrivals, we have provided an approximate analysis of such a queueing system. Our approach is to use an assembly queue model for the batching process. The batches generated by the assembly queue enter an interrupted M/SemiMarkov/1 (or M/SM/1) queue. By analyzing the assembly queue we characterise the batch input process for the interrupted M/SM/1 queue. We then develop an extension of the Webster mean delay formula for obtaining the approximate mean delay in the interrupted M/SM/1 queue. Numerical results from the analysis are compared with simulation results. The analysis is shown to be accurate in predicting the increase in the system capacity due to the batching behaviour.

Index Terms—Mean traffic delay, signalized intersection, indisciplined traffic, ITS, interrupted queue.

I. INTRODUCTION

Due to their affordability, small two-wheeled vehicles (referred to as motorcycles, generically, in this paper) are predominant in road traffic in developing countries. As these vehicles are small in size, they often stand side-by-side in road lanes so as to occupy the whole width of the lane. When a bigger vehicle, such as a car or bus stands in the lane, incoming motorcycles occupy the empty spaces in-between and besides the larger vehicles. Indeed, small two-wheeled vehicles arriving into an intersection queue look for unoccupied places in the queue, all the way upto the head of the queue, and fill them up. This phenomenon is illustrated by Figure 1.

Such behaviour of motorcycles (which violates strict *lane discipline*) can cause drivers of outgoing vehicles to slow their exit speed, out of caution against side-swiping other vehicles that stand by their sides. However, such behaviour increases the service capacity of the intersection, which is measured as the number of vehicle exits per unit time.

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This work is an extended version of the material that appeared in [1]

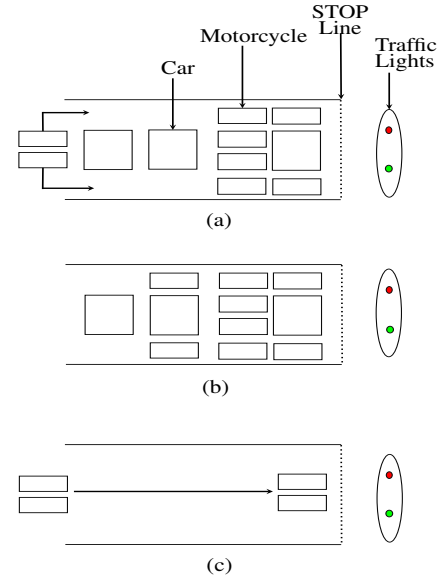


Fig. 1: Batch formation process: motorcycles arriving to a signalized intersection form batches by occupying vacant places between stationary vehicles and the boundaries of the lane (see (a) and (b) which show two motorcycles arriving, and taking up the empty spaces next to a standing car). In part (c), two motorcycles, arriving to an empty intersection, stand right at the STOP line (also called Head-Of-Line (HOL) in queueing theory jargon); subsequently if the signal turns green before any other motorcycles arrive, those two will depart as a batch of just two motorcycles.

modelling such a traffic system is of considerable interest. However, the batching phenomenon complicates the already difficult problem of modelling the intersection queue that is served by an intermittently available server. Classically, researchers like Clayton [2] and Webster [3] have looked into the relatively simplified problem of fixed-cycle traffic light problem, assuming Poisson arrival point process and deterministic service times. The strikingly simple empirical formula by Webster estimates the mean delay for this problem with great accuracy. Serfling [4] and Ohno [5] discuss the wide use of Poisson point process as a suitable and mathematically tractable model for vehicle arrival processes with light traffic density. McNeil [6] is one of the first to show that the problem of finding an exact expression for mean delay can be solved by finding an exact expression for mean overflow queue length (mean steady state queue length at the end of a green time). An exact but computationally expensive procedure to calculate the mean overflow queue length was proposed by Darroch [7]. Miller [8], and Newell [9] suggest

approximate formulas for calculating the mean overflow queue length. More recently, in 2006, Leeuwaarden [10] has taken an approach, similar to Darroch [7], to calculate queue length distribution, that leads to a closed form solution of the pgf (probability generating function) of the queue length of a fixed-cycle-traffic-length (FCTL) queue for a general arrival distribution, including Poisson. However, their approach needs to be supplemented by a numerical approach of finding the roots of the pgf, to calculate mean queue length. Oblakov et.al [11] have averted the computationally expensive issue of root-finding by expressing the mean queue length of a FCTL queue as a contour integral, and thereby using residue theorem to reduce computational burden. In the literature, the knowledge of distribution of Q , the number of passenger car units (PCU) in the queue at the beginning of a red time in a FCTL queue, is required to calculate the mean delay of a queue with lane-discipline, as suggested by Darroch [7], Mcneil [6], Newell [9] etc. Gertsbakh [12] has used a Markov chain analysis to obtain an approximate formula for Q .

An alternate approach is to consider a signalized intersection as a queueing system with server interruptions. In the late 1980s, Sengupta [13] and Federgruen and Green [14] used this approach to study the distribution of delay and mean delay in interrupted queues utilizing elegant queueing theory techniques. Sengupta [13], in particular, has obtained transform expressions. Since exact computation from these expressions, even of the mean delay, is difficult, reasonable approximations have been developed. Our own research, has also taken the interrupted queue approach.

The paper [15] has given an account of the recent spurt in interest for modelling traffic intersections without lane discipline in economically emergent nations with abundance in vehicular traffic congestion. Ali et al [16] have also addressed the difficulty associated with indisciplined traffic, in the context of measuring traffic parameters, in developing countries like India. Kiran and Verma [17] have given a review of the studies on various characteristics of lane indisciplined traffic with mixed vehicle types in developing countries. Their studies have reflected that the “gap-filling” (or “batching”, as referred to in our research) rather than “car-following” is the predominant behaviour unique to the indisciplined traffic, and requires further investigation to understand traffic congestion in indisciplined traffic.

Major contributions of this work: A shorter version of this work was presented in [1]. The major contributions of this paper are: **1)** We discuss the phenomenon of vehicle batching at a signalized single lane traffic intersection, and use the theory of interrupted queues in order to model such a signalized intersection with traffic indiscipline. **2)** We suggest a simplified approximate analysis of the system by decomposing the signalized queue into an assembly queue, that generates full batches, followed by a batch queue, where the batches so formed are queued as they wait for a green signal. **3)** We approximately model the resulting process of batch types departing from the assembly queue as a Markov process entering into the batch queue. **4)** We propose a generalization to the classical expected delay formula due to Webster [3] to calculate mean delay for the interrupted M/SM/1 queue.

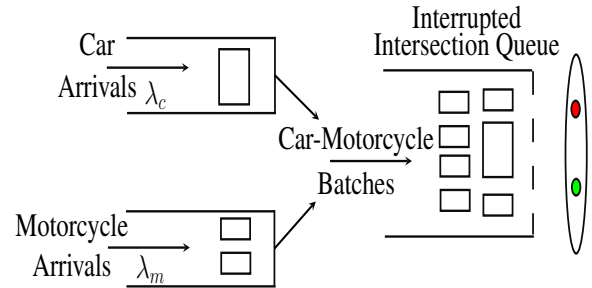


Fig. 2: System model capturing the batching of motorcycles with other vehicles, comprising an assembly queue followed by an interrupted signalized queue

This paper extends the material in [1] in the following ways: **1)** We have provided proofs of the theorems in the paper; these were omitted in [1]. **2)** We have provided an additional discussion on the structure of the mean delay in an interrupted queue, as provided in Sengupta [13], and the light it sheds on Webster’s formula, and the extension of this formula to the interrupted M/SM/1 queue. **3)** Moreover, we have explained how maximum arrival rate capacity is enhanced in indisciplined queues, and have provided an estimate for the factor of enhancement.

Organization of the paper: In Section II, the system model, comprising an assembly queue followed by an interrupted signalized queue, is proposed and described. Section III and Section IV describe the assembly queue model and the batch queue model respectively. Section V develops an approximate expression for the mean delay in the system, by approximately decomposing the delay into a sum of delays contributed by the assembly queue and the batch queue. In Section VI the efficacy of the proposed model is verified by numerically comparing simulation results to the approximate formula developed in the previous sections. Finally Section VII concludes the paper. The appendices A and B contain proofs of certain results in the main body of the paper.

II. SYSTEM DESCRIPTION AND APPROXIMATE ANALYSIS METHODOLOGY

In this section we first describe the system for which we aim to develop an approximate analysis. Then we will outline the methodology we use to develop the approximation.

A. System Description

We consider indisciplined traffic arriving to an isolated, signalized, single-lane traffic intersection. The system comprises: **1)** An arrival process consisting of mixed vehicle types. The arrival process into the system is decomposed into two distinct processes: car arrivals and motorcycle arrivals, each assumed to be Poisson point processes. **2)** A traffic light controller controlling the exit of the vehicles from the intersection using fixed-duration alternating green and red lights.

Cars arrive individually, with rate λ_c and motorcycles arrive

in pairs¹, with rate of arrival of the pair λ_m . Once vehicles enter the intersection, they can overtake each other. The road width is assumed to be such that only motorcycles can move towards the head-of-the-line (HOL) position through small lateral gaps available along the sides of vehicles ahead, thereby aiming to complete any incomplete batch at the front. This allows four possible batch types: one motorcycle pair batch, two motorcycle pair batch, a motorcycle and a car batch, and a single car batch, denoted by M_2 , M_4 , C_2 , and C_0 , respectively, as illustrated in Figure 1. When the signal turns green the HOL batch can begin to exit. It is assumed that an exiting batch comes to a halt if the light turns red, and completes its exit at the next green time.

B. Approximate Analysis Methodology

Exact analysis of mean delay even at the simplest M/G/1 interrupted queue is only approximately available. For the above model, therefore, we seek to develop an approximate analysis.

In this section, we outline the methodology that we employ in the next sections to carry out an approximate analysis to find mean delay of a vehicle in this system.

1) *Separation into an Assembly Queue and an Interrupted Batch Service Queue*: Unlike disciplined queues, the mean delay for a queue with batching requires the full description of the batches. Our model adopts an approximate description by breaking the system into two subsystems, an *assembly queue* (see, [18]), succeeded by an *interrupted batch service queue* (in the sequel called as the *batch queue* for brevity) illustrated in Figure 2.

2) *The Assembly Queue* (See figure 2): The assembly queue aims to capture the complicated process of batching in a signalized intersection. An arriving car or a motorcycle enters the assembly queue via separate queues for motorcycles and cars. The assembly queue “assembles” (in zero time) two motorcycle pairs into a M_4 batch, and a motorcycle pair and a car into a C_2 batch to send them into the following batch queue. No batching is performed when only multiple cars are present in the assembly queue. This model for batching is adequate for a nonempty intersection. However, we need to address the case when a single car or a single motorcycle pair arrives at the assembly queue with the intersection being empty, thus allowing a lone car or a lone motorcycle pair to leave the assembly queue, resulting in a single car batch (C_0) or a single motorcycle pair batch (M_2) (See figure 1(c)). This possibility is captured approximately via certain probabilities that are derived from the downstream interrupted queue analysis; these probabilities are used in the transition structure of a Markov model of the assembly queue. See Section III.

3) *The Interrupted Batch Service Queue*: The output of the assembly queue are batches that enter the queue at the traffic light, thereby leading to an interrupted queue with batch arrivals. Furthermore, the analysis of the assembly queue

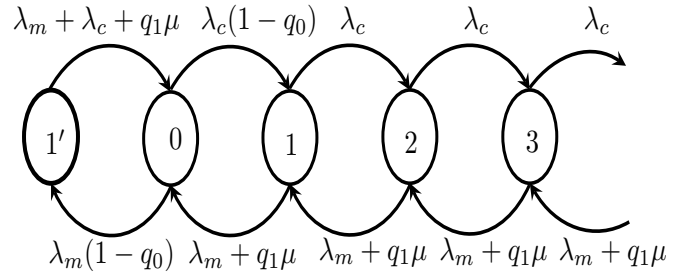


Fig. 3: State transition diagram for the assembly queue.

shows that the batch type arriving at the interrupted queue are not independent but can be modelled by a Markov chain. We approximate the instants of arrival of the batches into the interrupted queue by a Poisson process. This results in the queue at the traffic light becoming an interrupted M/SemiMarkov/1 queue (known as the M/SM/1 queue introduced by Neuts [19]). Inspired by the Webster delay formula for the interrupted M/D/1 queue, we propose and approximate formula for the mean delay in the interrupted M/SM/1 queue [20]. This analysis is detailed in Section V.

4) *Verification by Simulation*: We have proposed a model of a signalized single lane (described in Section II-A), and our main aim in this paper is to develop an accurate tractable analysis of this model, while providing insights into the queueing phenomenon. Since our analysis makes several approximations (as outlined earlier in this section), we verify its accuracy by performing a stochastic simulation of the *original* system described in Section II-A. Numerical results obtained from the simulation of the original system are compared with the numerical computations from our approximate analysis. The variation of mean delay is compared against the saturation level, which is taken as $\lambda\tau c/g$, where, λ is the mean arrival rate, τ is the mean service time, g is the green time duration, and c is the total cycle time. Several of these comparisons are done for separate sets of $(\alpha_c, \alpha_m, g, c)$, where α_c, α_m are the steady state probabilities of arriving cars and motorcycles. We have also included the mean delay plots for disciplined traffic, as well as the simulation results for both disciplined and undisciplined traffic for comparison purpose.

III. ASSEMBLY QUEUE MODEL ANALYSIS

Recalling the description of the assembly queue, we propose a transition rate diagram in Figure 3, which approximates the assembly queue process by a continuous time Markov chain (CTMC). The Markov description is approximate due to the coupling between it and the downstream interrupted queue, as explained in Section II-B.

- q_0 : steady state probability that the batch queue is empty
- q_1 : steady state probability that the batch queue has exactly one batch
- C : the random variable denoting the steady state *completion time* of a batch in the interrupted batch queue. The completion time of a batch is the time taken from the instant that the batch arrives at the HOL position until it exits from the intersection

¹This kind of side-by-side driving behaviour of motorcycles can commonly be observed in long high-speed lanes. However, with added complexity, the model can be extended to account for randomly sized grouped arrivals as well as singleton arrivals

- μ : steady state rate of completion of batches in the batch queue (i.e., $\mu = 1/\mathbb{E}(C)$)
- $\chi := \{1', 0, 1, 2, 3, \dots\}$, state space of the CTMC, which denotes the number of vehicles (cars or motorcycle pairs) present in the assembly queue in the steady state. State $1'$ denotes a pair of motorcycle, whereas state $0, 1, \dots$ represents as many unmatched cars.

A. Explanation of the CTMC in Figure. 3

In state $1'$: there is a lone motorcycle pair in the assembly queue with the batch queue nonempty (otherwise the motorcycle pair would have left). The only possible transition is $1' \rightarrow 0$ when either a car, or a motorcycle pair arrives, makes a C_2 or M_4 batch, respectively, and leaves the assembly queue; or the batch queue becomes empty. The corresponding rate is approximated as $\lambda_c + \lambda_m + q_1\mu$.

In state 0 : the assembly queue is empty (the batch queue might not be). If a car comes to this empty assembly queue, it joins the assembly queue only if the batch queue is nonempty, giving a transition $0 \rightarrow 1$ with approximate transition rate $\lambda_c(1 - q_0)$. If a motorcycle pair comes, it makes a $0 \rightarrow 1'$ transition (when the batch queue is nonempty), with approximate rate $\lambda_m(1 - q_0)$.

In state k , $k \geq 1$: a $k \rightarrow k + 1$ transition occurs with approximate rate λ_c if a car arrives, and a $k \rightarrow k - 1$ transition occurs when either a motorcycle pair arrives to form a C_2 batch (that leaves the assembly queue with approximate rate λ_m), or when the batch queue becomes empty (with approximate rate $q_1\mu$) giving an approximate transition rate $\lambda_m + q_1\mu$.

This CTMC is a birth-death chain, implying that the condition for stability of the assembly queue (positive recurrence of the chain) is $\lambda_c < \lambda_m + q_1\mu$. As a matter of fact, less than $\frac{1}{3}$ of the vehicles are cars ([21]); if the rest of the vehicles are motorcycles, we have $\lambda_m > \lambda_c$. Under steady state, the balance equations for the chain yields, as per Figure. 3:

$$\begin{aligned} (\lambda_c + \lambda_m + q_1\mu)\nu_{1'} &= \lambda_m(1 - q_0)\nu_0 \\ \lambda_c(1 - q_0)\nu_0 &= (\lambda_m + q_1\mu)\nu_1 \\ \lambda_c\nu_x &= (\lambda_m + q_1\mu)\nu_{x+1}, \quad x \geq 1 \end{aligned}$$

Solving these equations and then using $\sum_{x \in \chi} \nu_x = 1$, we get the following expressions for the stationary probabilities ν_x $x \in \chi$:

Lemma 3.1. *The stationary probability ν_x for $x \in \chi$ is given by*

$$\nu_x = \begin{cases} \left[1 + \left(\frac{\lambda_c + \lambda_m + q_1\mu}{\lambda_m - \lambda_c + q_1\mu} \right) \left(\frac{\lambda_m + q_1\mu - \lambda_c q_0}{\lambda_m(1 - q_0)} \right) \right]^{-1}, & x = 1' \\ \frac{\lambda_c + \lambda_m + q_1\mu}{\lambda_m(1 - q_0)} \nu_{1'}, & x = 0 \\ \frac{\lambda_c(1 - q_0)}{\lambda_m + q_1\mu} \nu_0, & x = 1 \\ \left(\frac{\lambda_c}{q_1\mu + \lambda_m} \right)^{x-1} \nu_1, & x \geq 1 \end{cases} \quad (1)$$

B. Analysis of the departure process from the assembly queue

The approximate CTMC modelling the assembly queue facilitates in finding approximate expression for batch departures from assembly queues as well. This, however, requires introducing further notation:

Denote $\mathcal{V} := \{M_2, M_4, C_0, C_2\}$. Then for $i \in \mathcal{V}$, and $x, y, \in \chi$:

- λ_i : departure rate of type i batches from the assembly queue
- f_{ij} : the conditional probability (called *following probability*) that an exiting batch of type i will be followed by a batch of type j
- a_x : rate of transition conditioned on the state of the assembly queue being at x
- $p_i(x)$: probability with which a transition from state x produces an exiting batch of type i
- $p_i(x, y)$: probability that a transition from state x ends up at state y by a departure of a type i batch
- $\phi_j(y)$: the conditional probability that the next exiting batch is of type j when the assembly queue is at state y

With a little effort, the quantities described in the notation can be found from the state transition diagram in Figure. 3, (see [22]). Using these quantities, it follows, for $i, j \in \mathcal{V}$,

$$\lambda_i = \sum_{x \in \chi} \nu_x a_x p_i(x) \quad (2)$$

$$f_{ij} = \frac{\sum_{x, y \in \chi} \nu_x a_x p_i(x, y) \phi_j(y)}{\lambda_i} \quad (3)$$

Equation (2) is justified by observing that a batch of type i departs state x of the assembly queue with rate $\nu_x a_x p_i(x)$. Equation (3) is deduced by noting that with rate $\nu_x a_x p_i(x, y)$, a type i batch departs from state x leaving the assembly queue at state y , and with probability $\phi_j(y)$, a type j batch exits succeeding the exit of a type i batch.

IV. ANALYSIS OF THE INTERRUPTED BATCH QUEUE MODEL

Batches of vehicles leave the assembly queue to join the interrupted batch service queue. This queue has a first-in-first-out service discipline. Batches entering this interrupted queue each have a type i , $i \in \mathcal{V}$. In order to analyse the interrupted batch queue we require the ‘‘batch following’’ probabilities (i.e., the steady state fraction of batches of type j that follow a batch of type i). These are provided by the analysis of the assembly queue through equations Eq. (2) and Eq. (3). It is, however, clear from the transition diagram in Figure 3 that these quantities are in turn dependent on the stationary probabilities q_0 and q_1 , and also on the effective service rate μ of the batch queue. A natural approach to addressing this situation is to resort to setting up a fixed-point iteration, where we begin by assuming the unknown quantities at the interrupted batch queue, use these to characterise the output process of the assembly queue, thereby being able to analyse the interrupted batch queue so as to obtain the next iterate of the assumed stationary quantities for this queue.

A. Queueing model for interrupted batch queue

Arrival process model: The arrival process into the batch queue is modelled as a Poisson process of rate $\sum_{i \in \mathcal{V}} \lambda_i$. The arrival process is described by the following quantities

X_k : The type of the k^{th} arriving batch, $k \geq 0$, $X_k \in \mathcal{V}$
 f_{ij} : $= P(X_{k+1} = j | X_k = i)$, the transition probabilities of the chain $\{X_k\}_{k \geq 0}$ for vehicle types $i, j \in \mathcal{V}$

The batch type sequence, $\{X_k\}_{k \geq 0}$ is modelled to constitute a discrete time Markov Chain (DTMC) on \mathcal{V} with f_{ij} 's as the transition probabilities.

The batch arrival process into the batch queue, however, is not Poisson as the inter-batch departure times are not independent. At saturation, when the stationary probability of finding a lone car in the assembly queue is almost 0, the only batches are of type M_4 . In that case, the assembly queue CTMC represented in Figure 3 has only two states: 1' and 0. At that situation, once an M_4 batch departs, the time it takes for the next departure of an M_4 batch is T_{M_4} , which is equal to the time it takes for two successive pairs of motorcycles to come. Since the inter arrival time between motorcycles is distributed as $\exp(\lambda_m)$, the successive inter departure times of M_4 batches are i.i.d and are distributed as *second order Erlang*, i.e., the probability density function is given by $f_{T_{M_4}}(t) = \lambda_m^2 t e^{-\lambda_m t}$. If $\alpha_c > 0$, the distribution of inter-batch-departure times become more complicated. However, it turns out, the Poisson model provides a mathematically tractable model and yields a reasonable approximation, as demonstrated by the numerical results we present in section VI.

Interrupted queue service model of the intersection:

An incoming batch is said to enter service if the preceding batch leaves the lane when the lane has the green signal. The amount of time a batch takes to leave the intersection is its *completion time* (as defined in Section III). If the signal turns red in the middle of a batch crossing the intersection, the batch pauses at its position and resumes moving at the beginning of the following green time. This type of resuming method is called *preemptive resuming*.² When the i^{th} batch is followed by the j^{th} batch, only after the i^{th} batch starts moving and leaves a minimum distance from the tail of the last vehicle in batch i to the tail of first vehicle in batch j , the j^{th} batch begins moving. In our service model, we call this distance the minimum *lagging headway* [23] required for batch j when it is preceded by batch i , before batch j starts moving. For batches with parity in lengths of peripheral and central vehicles inside the batch, this quantity is equal for all the vehicles in the batch. The following notation compactly describes the quantities required for the vehicle batch service:

d_{ij} : Lagging headway of a batch of type j that is preceded by a batch of type i
 l_i : The length of a batch of type i
 v_s : Saturation speed (i.e., the uniform speed at which batches depart the intersection during a green time) of the vehicles in the lane

²In an alternative resuming method, called *non-preemptive resuming*, when interrupted by red light, the batch retreats some length so that the front end of the batch aligns with the STOP line, and resumes moving at the beginning of the following green time. To avoid confusion, the front ends of the two vehicles in the batch are assumed to be aligned.

The following gives a precise description of the distribution, assuming that the vehicle type is j :

$$s = \begin{cases} t_{ij} = \frac{d_{ij}}{v_s}, & \text{If the queue is nonempty and the} \\ & \text{vehicle is preceded by a vehicle of type } i \\ t_i = \frac{l_i}{v_s}, & \text{If the vehicle arrives in an empty queue} \end{cases}$$

The discussion in the last two paragraphs suggests an interrupted M/SM/1 model for the batch queue, with SM standing for Semi-Markov. This differs from the standard M/G/1 model due to the fact that the sequence of service times in an M/G/1 queue forms an independent and identically distributed (i.i.d.) sequence, contrasting an M/SM/1 queue, where the service time sequence forms a Markov chain.

Outline of the analysis approach: **1)** We find out the quantities q_0, q_1, μ of the batch queue in terms of f_{ij}, λ_i of the assembly queue in order to execute the fixed point iteration as described earlier. However, we find these quantities approximately using the analysis of an interrupted M/G/1 queue in Sengupta [13], as it is not clear if Sengupta's approach can be extended to find an analysis for an interrupted M/SM/1 queue. **2)** In Section V-B2, an approximation formula to calculate mean delay in an interrupted M/SM/1 queue is derived. This formula, however, requires us to first execute the fixed point iteration as discussed earlier.

In the following subsection we will concentrate on finding approximate expressions of the quantities q_0, q_1, μ of the batch service queue which will be approximated as an interrupted M/G/1 queue as discussed above. To facilitate the analysis, we use the following notation:

$\lambda = \sum_{i \in \mathcal{V}} \lambda_i$: batch arrival rate into the M/G/1 queue
 τ : mean service time of the M/G/1 queue
 g, r, c : deterministic on (green) and off (red) times of lengths g and r and the green-red cycle length $c (= g + r)$.

B. Approximate analysis of q_0, μ, q_1

Analysis of q_0 : Our approximate analysis for q_0 is carried out by adapting the work of Sengupta [13] to our case. This yields the following result

Lemma 4.1. *For an interrupted M/G/1 queue, the stationary probability of the queue being empty, is given by*

$$q_0 = \frac{g}{c} - \lambda\tau + \frac{u_0}{\lambda c} (1 - e^{-r\lambda}) \quad (4)$$

where u_0 is the stationary probability that the queue is empty at the beginning of an off-time.

Proof. The proof for this result requires introducing some of the definitions and notation used in Sengupta [13] and thus is postponed to the Appendix A. ■

It should be mentioned that the statement in Lemma. 4.1 is *exact* as far as an interrupted M/G/1 model is concerned. However, the lemma requires to know the variable u_0 in order to evaluate q_0 . It can be easily shown from the discussion in Sengupta [13] that $u_0 = (1 - \lambda\tau)w_0$ where w_0 denotes the steady state arrival point probability for the system size being 0 for a special G/G/1 queue, described in the proof of Lemma. 4.1 in Appendix A. Sengupta [13] has defined

this special G/G/1 queue with inter-arrival time distribution as the distribution of green time durations; in our case, this definition of inter-arrival time distribution reduces the G/G/1 queue to a D/G/1 queue. Unfortunately, obtaining w_0 for a D/G/1 queue is intractable, in general. We approximate this quantity by calculating the corresponding quantity \hat{w}_0 for a D/M/1 queue with mean service time as $\rho^{(1)} = \lambda\tau r/(1-\lambda\tau)$. The expression for \hat{w}_0 is well known [24] to be $\hat{w}_0 = 1 - r_0$ where r_0 is the unique solution of the equation $z = e^{-\frac{g(1-z)}{\rho^{(1)}}}$ in $(0, 1)$.

Analysis of μ (or, equivalently, $\mathbb{E}C$): Our analysis for determining μ is motivated by the technique of analysis of completion time introduced by Federgruen and Green [14]. The following notation will be required to perform the analysis:

- R' : the random variable denoting the residual red time seen by an arriving batch
- p_0 : steady state probability of a batch arriving within a red time and finding the system empty
- p_1 : probability of batch service interruption by a red time
- S : random variable denoting service time of a vehicle batch

Assuming that the service time of a batch (≈ 5 seconds to 10 seconds) is much smaller than the green time duration (≈ 20 seconds to 50 seconds), we can then see that the completion time of a batch can take three distinct values. If the batch arrives in a red time and finds the interrupted queue empty, then its completion time begins at its arrival time, and it gets served as the first customer in the next green time. The resulting completion time is $S + R'$, where we need to use the conditional distribution of R' given that the arriving batch finds the system in the red time and also empty. If either the batch arrives in a red time and finds the system nonempty, or it arrives in a green time, then it reaches the HOL position during a green time. Now there are two sub-cases. If the batch completes service within the green time in which it reaches the HOL position then its completion time is its service time. On the other hand, if the service of such a batch is interrupted by a red time, then it will complete service at the beginning of the next green time (where we use the assumption made at the beginning of this paragraph).

Thus the completion time C is characterized in the following way

$$C = \begin{cases} S + r, & \text{w.p. } p_1 \\ S + R', & \text{w.p. } p_0 \\ S, & \text{w.p. } (1 - p_1 - p_0) \end{cases} \quad (5)$$

Let r'_0 denote the expectation of R' conditioned on the batch arriving in a red time and finding the system empty. The quantities p_0, p_1, r'_0 can now be found from the following expressions.

Lemma 4.2. $p_0 = \frac{u_0(1 - e^{-r\lambda})}{\lambda c}$, $p_1 = \frac{1 - u_0}{\lambda c}$, $r'_0 = \frac{r}{1 - e^{-\lambda r}} - \frac{1}{\lambda}$ and $\mathbb{E}C = \tau + \frac{r}{\lambda c} \left(1 - \frac{u_0(1 - e^{-r\lambda})}{r\lambda}\right)$, where u_0 was defined in Lemma 4.1 and its determination was discussed in the paragraph following Lemma 4.1.

Parameters	Traffic model
\mathcal{V}	$\{C, M\}$
$\{l_C, l_M\}(m)$	$\{6, 2\}$
$\{d_{CC}, d_{CM}, d_{MC}, d_{MM}\}(m)$	$\{8, 3, 7, 2.5\}$
Exit Velocity, v_s	4.5m/s

TABLE I: Traffic model

Proof. See Appendix B. ■

Analysis of q_1 : Finding an exact expression for q_1 for an interrupted M/G/1 queue is in general intractable. However, if the identically distributed *dependent* sequence of completion times is approximated by a sequence of *i.i.d.* completion times, where the distribution of the completion times is retained, an approximate expression can be found by utilizing standard results for an uninterrupted M/G/1 queue:

Lemma 4.3. $q_1 \approx q_0 \frac{1 - C^*(\lambda)}{C^*(\lambda)}$ where $C^*(\lambda) = \frac{\tilde{B}(\lambda)}{\lambda c} [e^{-r\lambda}(1 + r\lambda u_0) + \lambda c - 1]$ is the Laplace-Stieltjes Transform (LST) of the distribution of C and $\tilde{B}(\cdot)$ is the LST of the steady state service distribution of a batch.

Proof. Assuming that all the moments of C exists, $C^*(s) = \sum_{k \geq 0} \frac{(-s)^k}{k!} \mathbb{E}(C^k)$. Using the characterization of C in Eq. (5), a straightforward calculation results in the expression for $C^*(\lambda)$. Assuming the independence of the completion time sequence, it is then a routine task to find the expression for q_1 as given in Lemma 4.1 (See [24]). ■

To obtain $\tilde{B}(\lambda)$, we compute the service distribution of the M/G/1 queue by first sampling a vehicle from the stationary distribution $\{\pi_i\}_{i \in \mathcal{V}}$ of the vehicle-type Markov chain, and then obtaining the distribution of service times from the *preceding probabilities*, p_{ij} , i.e. the probability that a vehicle of type i is preceded by a vehicle of type j (and can be computed from the following probabilities $\pi_i p_{ij} = \pi_j f_{ji}$). This yields the following service time distribution:

$$s = \sum_{i \in \mathcal{V}} \pi_i \sum_{j \in \mathcal{V}} p_{ij} 1_{\{s=t_{ji}\}} \quad (6)$$

It follows that the LST $\tilde{B}(s)$ is given by $\sum_{i \in \mathcal{V}} \pi_i \sum_{j \in \mathcal{V}} f_{ij} e^{-st_{ij}}$.

Discussion of numerical results: The plots in Figure 4 validate the analysis done in Section IV. The analysis is compared to the simulation results obtained by simulating a disciplined queue with traffic parameters in Table I. From Figure 4 we observe that the approximate analysis provides intermediate numerical results whose variation with the saturation level is qualitatively the same as yielded by the approximation. The values of q_0 and the mean completion time are captured very accurately, whereas there are errors in the values of q_1 yielded by the approximation, specially in the middle range of saturation values. This match of qualitative trends, and the reasonably good numerical match helps to verify our modelling process, as we can conclude that the good match of the final results is not just by coincidence; our approximations have captured internal system behaviour quite well.

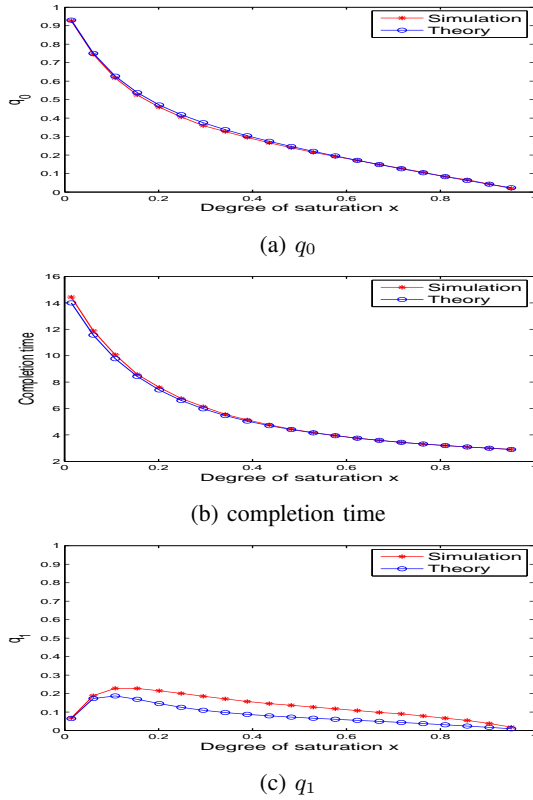


Fig. 4: $g = 20s, c = 60s$ with parameters in Table I

V. ANALYSIS OF MEAN DELAY IN BATCH QUEUE

Following our approach of decoupling the indiscriminate queue into an assembly queue followed by an intersection queue (which we call the batching queue), the mean delay of a vehicle (a car or a motorcycle batch) that enters the signalized indiscriminate queue, is approximately decomposed as $d_{\text{intersection}} \approx d_{\text{assembly}} + d_{\text{batch}}$ where d_{assembly} and d_{batch} are the expected delays of a vehicle in the assembly queue and the of the resulting batch in the batching queue, respectively.

A. Finding d_{assembly}

The following notation will be used to derive an approximate expression d_{assembly} :

- N_{C_0} : Random variable denoting the number of cars (alone) in the assembly queue
- N_{M_2} : Random variable denoting the number of motorcycle pairs in the assembly queue
- $\mathbb{E}N_{C_0}$: Mean number of lone cars in the assembly queue
- $\mathbb{E}N_{M_2}$: Mean number of motorcycle pairs in the assembly queue
- $\mathbb{E}W_{C_0}$: Mean delay experienced by a lone car in the assembly queue
- $\mathbb{E}W_{M_2}$: Mean delay experienced by a motorcycle pair in the assembly queue

Observe that each of a pair of motorcycles has mean arrival rate $2\lambda_m$. Hence, an incoming vehicle (a motorcycle or car) will see, w.p. $\frac{2\lambda_m}{\lambda_c + 2\lambda_m}$ a motorcycle pair in the assembly queue, or will see, w.p. $\frac{\lambda_c}{\lambda_c + 2\lambda_m}$ a car in the assembly queue. Since

the arrivals are Poisson, by PASTA [24] principle, the mean delay found by an vehicle arriving into the assembly queue is the mean delay in the assembly queue. Consequently, the expected delay of a vehicle in the assembly queue is found as

$$d_{\text{assembly}} = \frac{2\lambda_m}{\lambda_c + 2\lambda_m} \mathbb{E}W_{M_2} + \frac{\lambda_c}{\lambda_c + 2\lambda_m} \mathbb{E}W_{C_0} \quad (7)$$

Moreover, Little's law gives $\mathbb{E}W_{C_0} = \mathbb{E}N_{C_0}/\lambda_c$, $\mathbb{E}W_{M_2} = \mathbb{E}N_{M_2}/\lambda_m$. Furthermore, it follows from the CTMC in Section III, $\mathbb{E}N_{C_0} = \sum_{x \geq 1} x\nu_x$, and $\mathbb{E}N_{M_2} = \nu_1$. Thus evaluating d_{assembly} from Eq 7 requires evaluating ν_x $x \in \chi$ as defined in Section III. As explained in Section IV, due to the explicit dependence of ν_x , $x \in \chi$ on the batching queue occupancy probabilities q_0, q_1 , which in turn are implicitly dependent upon ν_x , $x \in \chi$, a fixed point iteration is called for that utilizes the analysis in Sections III and IV.

B. Finding d_{batch}

As explained in Section IV, since the arrival process into the batching queue is not, in general, a renewal process, d_{batch} can only be approximately analyzed. For an interrupted M/G/1 queue an elegant mean delay analysis is provide by Sengupta [13]. Since this analysis provides useful insight we first provide an outline of a heuristic derivation, in order to understand the difficulty in extending it to the interrupted M/SM/1 queue. Then we provide a simple adaptation of Webster's mean delay formula, and demonstrate its efficacy by comparison with simulation experiments.

1) *Outline of a heuristic derivation of a mean delay formula in Sengupta [13]*: We approximate the batch arrival process into the batching queue to be Poisson as discussed earlier in Section IV. Figure. 5 depicts the residual work-in-system in front of the k^{th} arrival into the batching queue, plus the work brought in by the arrival. Here A_k, U_k denote the arrival and departure times, respectively, of the k^{th} batch, and $W_k = U_k - A_k$ denotes its total sojourn time.

The following is an explanation of the diagram. When the k^{th} arrival joins the queue, the total amount of work to be done (until this arrival exits) becomes V_k (including the work brought in by the new arrival). The signal is in the red state as the work is not decreasing. When the signal turns green, the work begins to decrease as vehicles exit the intersection. Since this is a first-in-first-out system (at the batch level), the work seen by the k^{th} arrival cannot increase. Another red time is shown, yielding a total red time of R_k . Eventually, at U_k the value decreases to zero indicating that the k^{th} arrival departs. Due to first-in-first-out service, the sojourn time experienced by the k^{th} batch is the sum of the total work seen by it on arrival and the total red-time it encounters, i.e., $W_k = V_k + R_k$. It follows that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n W_k = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n V_k + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n R_k$, where the limits are assumed to exist with probability 1. Assuming that the batch average mean sojourn time exists, and the time average of the $V(t)$ process also exists, writing $\mathbb{E}W$ as the mean sojourn time and $\mathbb{E}V$ as the time average work in the system, almost

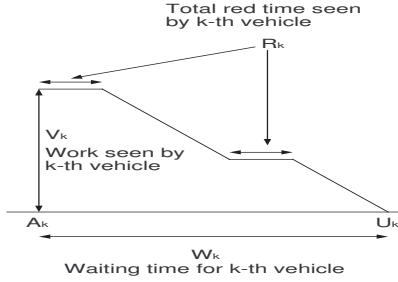


Fig. 5: The work-in-system process during the sojourn of vehicle k .

surely, $\mathbb{E}W = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n W_k$, $\mathbb{E}V = \lim_{t \rightarrow \infty} \int_0^t V(u) du = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n V_k$. The second equality follows from ‘‘Poisson arrivals see time averages (PASTA)’’ (see, for example, [24]), since we have a Poisson arrival process. As a result, with $\mathbb{E}R = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n R_k$ almost surely, mean sojourn time of an arriving batch is obtained from $\mathbb{E}W = \mathbb{E}V + \mathbb{E}R$. This relation can be used to derive an approximation, whose heuristic derivation is shown in [20]. The term $\mathbb{E}R$ is split into the mean red time seen on arrival plus the mean delay due to interruptions by red periods. The first part is exactly $\frac{r}{c} \times \frac{r}{2}$, the probability an arrival finds the system in a red period, times the mean residual red time; here, since the red periods are deterministic the mean residual red time is just $\frac{r}{2}$. The second part is approximated as $\frac{r}{g} \times \mathbb{E}V$, since $\frac{\mathbb{E}V}{g}$ is roughly the number of times the work seen by the k^{th} arrival is interrupted by red periods.

$$\mathbb{E}W \approx \mathbb{E}V + \frac{r}{g} \mathbb{E}V + \frac{r^2}{2c}$$

This expression is the same as the one obtained from Sengupta’s exact analysis [13, Eq. (7)]. In order to use this formula, however, we require an expression for $\mathbb{E}V$. While such an expression is available for the interrupted M/G/1 queue from Sengupta [13], no such expression is available for the interrupted M/SM/1 queue. We, therefore, proceed as follows:

1) The above analysis suggests an interpretation of the well-known Webster mean delay formula as the sum of the mean delay due to interruptions and the mean delay in a queue with inflated service times. We have justified such an interpretation in [20]. **2)** Using this interpretation we propose an extension of the Webster formula to the M/SM/1 queue.

2) Webster’s approximate formula and its extension to interrupted M/SM/1 model:

Understanding Webster’s Formula: Webster [3] provided the following formula to approximate the mean delay in an interrupted M/D/1 queue with vehicle arrival rate λ , green time g , red time r , cycle time $c = r + g$, and mean vehicle service time τ ,

$$d \approx \frac{r^2}{2c(1 - \lambda\tau)} + \frac{c}{g} \frac{\lambda\tau^2}{2(g/c - \lambda\tau)} - 0.65 \left(\frac{c}{\lambda^2} \right)^{\frac{1}{3}} (\lambda c\tau/g)^{2+5g/c} \quad (8)$$

To find a version of this formula that can be used to approximately describe the mean delay in an interrupted M/SM/1 queue, it is essential to know how the underlying queue affects the three terms in which the mean delay is decomposed. A detailed numerical study in [20] verified this formula against the exact analysis of Sengupta [13], which finally gives a formula for mean delay, similar to the one found heuristically in Section V-B1. In light of this formula, we interpret the terms of Webster’s formula as follows:

1) We interpret the first term to approximate mean residual red time seen by a batch at arrival. Newell [9] interpreted the first term as a fluid arrival delay approximation term. **2)** We interpret the second term to approximate the mean delay of a batch excluding the residual red time seen at arrival. Newell [9] interpreted this term to approximate the mean overflow queue length. **3)** The third term is an empirical correction term accounting for the mismatch between the actual delay and the delay contributed by the first two terms. It was found by Webster by fitting a curve through Monte-Carlo simulations.

Moreover, observe that the second term is an expression for mean delay of an uninterrupted M/D/1 queue with mean service time enhanced by a factor of c/g . These observations motivate us to propose a heuristic extension of Webster’s formula to an interrupted M/SM/1 queue by *replacing the second term of Webster’s formula by the mean delay formula of an uninterrupted M/SM/1 queue with service time enhanced by a factor of c/g .*

Obtaining Mean Delay in an uninterrupted M/SM/1 model: Consider the M/SM/1 batch queue model elaborated in Section IV-A without interruptions. In order to find the mean delay for this queue, define the following:

D_k , $k \geq 0$: k^{th} departure instant.

Q_k , $k \geq 0$: Batch queue length embedded at D_{k+} , i.e. just after the k^{th} departure.

It then follows that the process $\{Q_k, X_k\}$ forms a two dimensional discrete time Markov chain on $\mathbb{N} \times \mathcal{V}$ with the transition probabilities given by

$$P(Q_{k+1} = y, X_{k+1} = j | Q_k = q, X_k = i) = f_{ij} a_{ij}^{(y-q+1)}, \quad q > 0 \\ = f_{ij} b_{ij}^{(y-q+1)}, \quad q = 0$$

where $a_{ij}^{(l)} = \frac{(\lambda t_{ij})^l e^{-\lambda t_{ij}}}{l!}$, $b_{ij}^{(l)} = \frac{(\lambda l^{(i)})^l e^{-\lambda l^{(i)}}}{l!}$, and $l^{(i)} = \frac{t_i}{v_s}$. The form of the transition probabilities for $\{Q_k, X_k\}$ can be appreciated recalling that $\{X_k\}$ is a DTMC on \mathcal{V} and the batch arrival process is (approximately) Poisson. The corresponding transition probability matrix is given by

$$P = \begin{bmatrix} B_0 & B_1 & B_2 & \dots & \dots \\ A_0 & A_1 & A_2 & \dots & \dots \\ 0 & A_0 & A_1 & \dots & \dots \\ 0 & 0 & A_0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad \text{where } \{B_l\}_{ij} = f_{ij} b_{ij}^{(l)},$$

and $\{A_l\}_{ij} = f_{ij} a_{ij}^{(l)}$. We now proceed in the following steps to find the mean delay: **1)** The ‘‘M/G/1 type’’ matrix Markov chain is solved for stationary distribution using standard techniques incorporated in Matlab toolset (see [25]). **2)** With the help of the stationary distribution of the process $\{X_k\}$, the stationary probability vector for the queue length right after

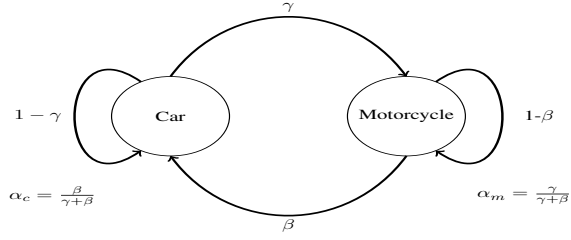


Fig. 6: Car-Motorcycle Markov Chain

departure instants is found and then used to find the mean queue length after departure epochs, q' . **3)** Since the departures occur in singletons, a standard level crossing argument is used, along with the PASTA property to conclude that the mean queue length is $\bar{q} = q'$. **4)** Finally, Little's law is applied to find the mean waiting time in the queue as $\bar{w} = \frac{\bar{q}}{\lambda} - \tau$ where λ is the batch arrival rate into the queue and τ is the mean service time of a batch. It is essential to mention that the mean service time is expanded by c/g to take care of service interruptions, as required by the earlier interpretation of Webster's formula.

Thus, for a given λ , the Webster's approximate formula, extended for the interrupted M/SM/1 model becomes,

$$d = \frac{c(1 - \frac{g}{c})^2}{2(1 - \frac{g}{c}x)} + \bar{w} - 0.65\left(\frac{c}{\lambda^2}\right)^{\frac{1}{3}}x^{2+5\frac{g}{c}} \quad (9)$$

where $x \triangleq \lambda\tau c/g$, which is called the *degree of saturation*.

Validation of the approximate delay formula for the interrupted M/SM/1 queue: In order to validate our extension of Webster's delay formula to the interrupted M/SM/1 queue, we simulate an M/SM/1 queue and compare the mean delay obtained from the simulation with that obtained from the approximate formula. The arrival process is Poisson; the arrivals are either motorcycles or cars, and these queue up in a disciplined manner. The arrival types form a Markov chain with transition structure shown in Figure 6.

Figure 7 compares the approximate formula for mean delay provided by our extension of Webster's formula for mean delay to the M/SM/1 case and the simulation results for disciplined M/SM/1 traffic with preemptive resuming (recall the interrupted queue model of the intersection in Section IV). The mean delay variation is plotted against the degree of saturation $x = \lambda\tau c/g$, where λ is the total arrival rate of motorcycles and cars, τ is the mean service time (obtained from the arrival type Markov chain in Fig 6, and traffic model in Table D), and c, g are the total cycle time and green time duration, respectively. Different plots are given for different sets of value of the motorcycle-to-car transition probability γ . The values of the headways used are given in Table. I. The percentages of the motorcycles and cars used are 80% and 20% respectively. The plots show that the approximation by the extended Webster's formula gives quite good estimate of the mean delay, with the approximation becoming tighter with increasing g . Also, Newell [9] showed that the original Webster's approximation approximation works better for lower values of arrival rates, which is why our extended Webster's model better approximates an M/SM/1 queue in this regime. Observe that, with g fixed, as the motorcycle-to-car transition

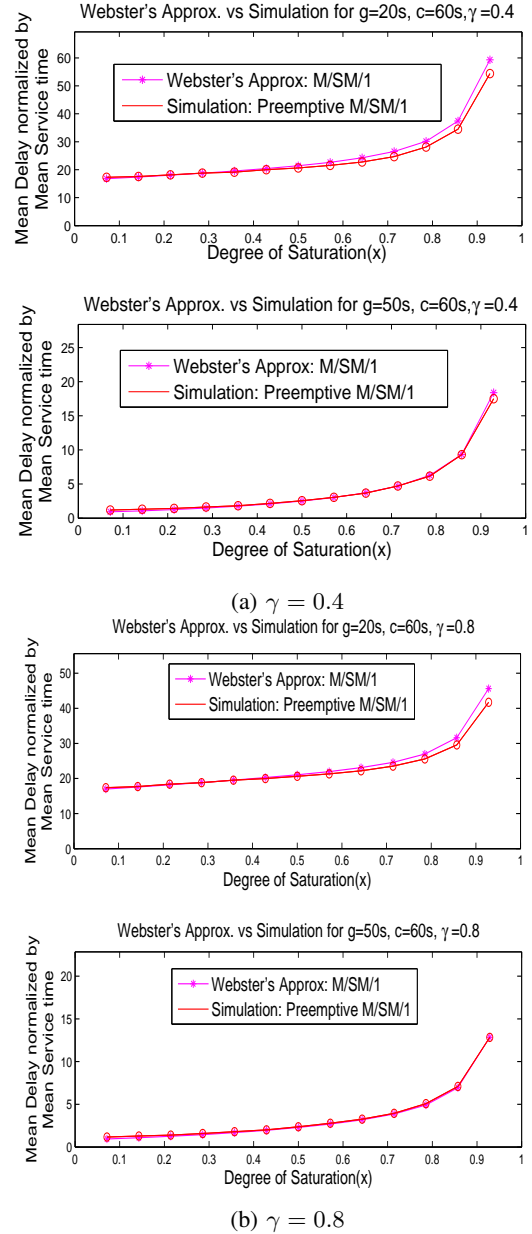


Fig. 7: Webster's extension for M/SM/1 vs Preemptive Simulation

probability γ increases, the mean delay decreases; this is expected since $1/\gamma$ is the mean size of a sequence of consecutive cars in the vehicle stream, so that an increase in γ populates the vehicle stream with longer sequence of motorcycles, which reduces mean delay because of smaller mean service time of motorcycles.

VI. NUMERICAL VALIDATION OF THE ANALYSIS APPROACH FOR INDISCIPLINED TRAFFIC

We simulate the model described in Section II, and compare the mean delay so obtained with the mean delay obtained from our analysis approach via splitting the system into an

i	j			
	C_0	M_2	C_2	M_4
C_0	8	3	9.5	3.5
M_2	7	2.5	8.5	3
C_2	8	3.5	9.5	3.5
M_4	7	2.5	8.5	3

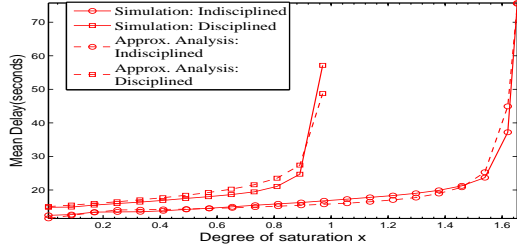
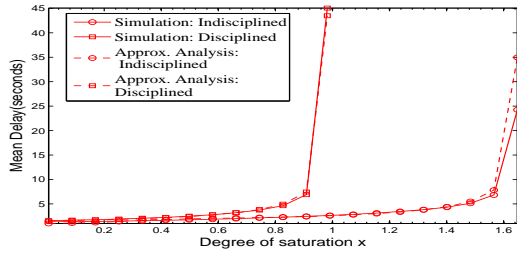
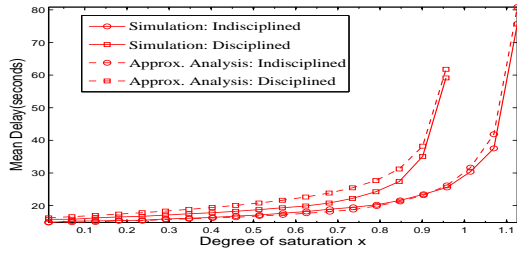
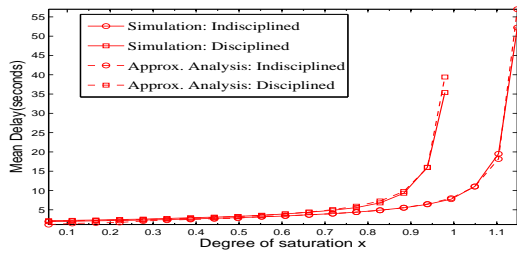
TABLE II: d_{ij} (in meters), indisdisciplined queue(a) $g = 20s, c = 60s, \alpha_c = 0.001, \alpha_m = 0.999$ (b) $g = 50s, c = 60s, \alpha_c = 0.001, \alpha_m = 0.999$ (c) $g = 20s, c = 60s, \alpha_c = 0.333, \alpha_m = 0.667$ (d) $g = 50s, c = 60s, \alpha_c = 0.333, \alpha_m = 0.667$

Fig. 8: Comparison of mean delays from simulation and approximate analysis for batch intersection queue for different fractions of cars and motorcycles

assembly queue and an interrupted M/SM/1 queue. We have simulated the arrivals of unbatched cars and motorcycle pairs by independent Poisson arrival processes, with rates $\alpha_m \lambda$, and $\alpha_c \lambda$, where λ is the total traffic arrival rate, and α_m, α_c are the fraction of motorcycle pairs and cars, respectively, in the arriving traffic. The variation of mean delay is plotted with respect to the degree of saturation (as defined in Section V),

for an indisciplined signalized intersection queue keeping g, c, α_c, α_m and the traffic parameters fixed. Additionally, plots of mean delays of a disciplined queue with traffic model in Table I, as well as the mean delay estimate produced by our proposed Webster's extension are plotted; these additional plots act as benchmarks. The following probabilities for the disciplined case are obtained from γ, β (from Fig 6), which are taken to be α_m, α_c respectively. For indisciplined queue with batches traffic parameters from Table II are used. As a practical choice, the lengths of the batches are taken as $l_{C_0} = l_{C_2} = l_C$ and $l_{M_2} = l_{M_4} = l_M$, where l_M and l_C denote the lengths of an unbatched car and an unbatched motorcycle pair respectively. The effect of reduction of exit speed is realized by increasing the headways of the batches, as in Table II, and the saturation speed v_s is taken to be 4.5 m/s, chosen to be the same³ for all the vehicles, both within and outside a batch. For the purpose of simulating heterogeneous traffic, we have used the two sets of vehicle probabilities (α_c, α_m) , (0.001, 0.999) and (0.333, 0.667).

We simulated an isolated signalized intersection with fixed cycle time length, with no arterial system interfering with its arrival process. For both the disciplined and indisciplined cases, given a fixed pair (α_c, α_m) , for each arrival rate λ , a sample process of the traffic intersection was generated and the process was allowed to run for a simulation time of 5×10^5 s. For fixed values of λ, c, g , and the lagging headway, d_{ij} values used in the experiments, (different values for disciplined and indisciplined cases) this simulation time was observed to be large enough to allow an average of 248500 arrivals, which was large enough for the queue to achieve steady state. We wrote the code for the simulation in C and ran the experiment on a laptop computer with Windows 8.1 operating system with 4 GB RAM, 2.4 GHz Intel Core-i5 processor.

To find the mean delay for the assembly queue-interrupted M/SM/1 batching queue using our approximate analysis, we first carried out a fixed point analysis to find out the arrival rates of the batches from the assembly queue into the batching queue, the following probabilities associated to the batch departure process from the assembly queue, the mean completion times of the batches, and the steady state probabilities q_0, q_1 of the batching queue. Using these estimates associated with the batch departure process from the assembly queue, the mean delay of the batching queue was estimated using Webster's extended formula in Eq. (9) for an interrupted M/SM/1 queue. Similarly, Webster's extended formula was used, with different traffic model parameters from Table I, to calculate mean delay estimate for disciplined queues. These calculations were done in on a laptop computer with Windows 8.1 operating system with 4 GB RAM, 2.4 GHz Intel Core-i5 processor.

The plots reveal that the phenomenon of batching in an indisciplined queue improves the capability of the system to handle higher arrival rate or discharge. Observe that with 99.9% motorcycles, the indisciplined behaviour elevates the capacity of the system, i.e., the maximum number of vehicles arrivals per second for which the queue remains stable, by over

³However, the model is flexible enough to incorporate unequal saturation speeds

60%, while with 33.3% motorcycles the capacity is elevated by over 10%. Although the batches exit with reduced speed, as reflected in their higher headways, the effect of multiple vehicles exiting together results in an overall decrease in mean delay. However, it remains to see to what extent the batches can reduce their exit speed until the “gain” in maximal arrival rate becomes 1. Our model also allows an analysis with parametric v_s , (again see [22]) which might be able to shed light into this question. The next paragraph shows how our model can heuristically explain the observed gain in maximum arrival rate capacity for indisciplined queues.

Heuristic explanation of enhancement of the capacity of an interrupted queue due to batching: We heuristically explain how the indisciplined behaviour of batching and batched service increases the arrival rate handling capacity of the system, and reduces the mean delay, by analysing the situation near arrival rate saturation. Near saturation, a valid assumption is $q_0 \approx q_1 \approx 0$. The assumption is justified by Figure 4, for our analysis approach is near exact in this regime. In this regime, there cannot be any isolated car or motorcycle pair, resulting in the only batches C_2, M_4 as defined in Section III. In this regime arrival rate of a car-motorcycle pair is $\lambda_{C_2} \approx \lambda_c$. It then follows that the rate of arrival of the rest of the motorcycle pairs that do not batch with cars is approximately $\lambda_m - \lambda_c$. Since two motorcycle pairs form an M_4 batch, the arrival rate of an M_4 batch becomes $\lambda_{M_4} \approx (\lambda_m - \lambda_c)/2$. Consequently, the total input rate to the batch queue can be approximated by $(\lambda_m + \lambda_c)/2 = \lambda_{in}/2$ where λ_{in} is the net arrival rate into the system. Invoking the assumption $q_0 \approx q_1 \approx 0$, from Equation (3) we can approximately get the 4 batch following probabilities as $f_{C_2C_2} \approx \alpha_c (\alpha_m + \frac{3}{2})$, $f_{C_2M_4} \approx \frac{(\alpha_m - \alpha_c)(1 + \alpha_m)}{2}$, $f_{M_4C_2} \approx \alpha_c(1 + \alpha_m)$, $f_{M_4M_4} \approx \alpha_m^2$. The corresponding stationary distribution of the batches are obtained as $\pi_{C_2} \approx \frac{\lambda_{C_2}}{\lambda_{C_2} + \lambda_{M_4}} = 2\alpha_c$, $\pi_{M_4} \approx 1 - \pi_{C_2} = \alpha_m - \alpha_c$, where $\alpha_m = \lambda_m/(\lambda_m + \lambda_c)$, $\alpha_c = \lambda_c/(\lambda_m + \lambda_c)$. The mean service time of the batches can be approximately calculated as

$$\begin{aligned} \tau_{\text{batch}} &= \sum_{i,j \in \{C_2, M_4\}} \pi_i f_{ij} t_{ij} \\ &\approx \alpha_c (\alpha_m - \alpha_c) (1 + \alpha_m) t_{C_2M_4} + \alpha_c^2 (3 + 2\alpha_m) t_{C_2C_2} \\ &\quad + (\alpha_m - \alpha_c) \alpha_c (1 + \alpha_m) t_{M_4C_2} + (\alpha_m - \alpha_c) \alpha_m^2 t_{M_4M_4} \end{aligned}$$

Now, the degree of saturation for the intersection queue with the C_2, M_4 batches as its input arriving vehicle batch types is $x_{\text{batch}} = \lambda_{\text{batch}} \tau_{\text{batch}} \frac{c}{g}$. But, at saturation flow, $x_{\text{batch}} = 1 \Rightarrow \lambda_{\text{in}} = 2g/(c\tau_{\text{batch}})$. For the unbatched queue, the degree of saturation is $x = \lambda_{\text{in}} \tau c/g$ which becomes 1 at saturation, i.e., the queue becomes saturated when $\lambda_{\text{in}} = g/(c\tau)$ where τ is the mean service time of the unbatched vehicles i.e.

$$\tau = \alpha_c (f_{CC} t_{CC} + f_{CM} t_{CM}) + \alpha_m (f_{MC} t_{MC} + f_{MM} t_{MM})$$

This gives us an approximate expression for the “gain” of vehicle arrival rate for an indisciplined queue around saturation,

$$\frac{\lambda_{\text{in,sat}}^{(\text{batch})}}{\lambda_{\text{in,sat}}^{(\text{nobatch})}} \approx \frac{2\tau}{\tau_{\text{batch}}} \quad (10)$$

this serves as an estimate of the maximum “degree of saturation” that can be handled by the batching queue. Using the

traffic headways in Tables I, II, and the vehicle-type Markov chain in Fig 6, the calculated value of the above estimate are found to be ≈ 1.166 for $(\alpha_c = 0.333, \alpha_m = 0.667)$, and ≈ 1.67 for $(\alpha_c = 0.001, \alpha_m = 0.999)$, respectively, which can be verified to be pretty exact from the plots in Fig 8. We can also observe from Eq (10), that for the “gain” in capacity to become 1, at saturation, $\tau_{\text{batch}} \approx 2\tau$, from which an estimate of the maximum reduction of exit speed of the batches can be found as a function of $\alpha_m, \alpha_c, \{f_{ij}, t_{ij}\}_{i,j \in \{C,M\}}$.

VII. CONCLUSION

In this paper we have considered heterogeneous traffic consisting of smaller motorcycles and larger cars arriving into a single lane, isolated signalized intersection queue with fixed duration green and red cycles. The smaller motorcycles fill the side-ways gaps next to waiting cars or other motorcycles, resulting in the formation of “batches”. We model the batch formation process via an assembly queue and model the service of the batches by an interrupted queue. We decompose the mean delay of the system, approximately, as the sum of mean delays in the two proposed queues. A by-product of our analysis is a generalization of the approximate mean delay formula of Webster to the interrupted M/SemiMarkov/1 queue.

By comparison with a detailed simulation of the original system, we find that our analysis approach, which involves several approximations, provides an accurate estimate of the mean delay. One obvious consequence of batching behaviour is that the intersection capacity increases, provided the saturation speed remains unchanged due to batch formation. Our model permits us to estimate the increase in capacity. Further, since the saturation speed is a parameter in the analysis, our analysis permits sensitivity analysis with respect to this parameter. Thus, our analytical approach can be applied to the traffic engineering of intersections with indisciplined traffic, in a manner similar to Webster’s mean delay model.

APPENDIX A

PROOF OF LEMMA 4.1

We begin with a review of the queueing model in Sengupta’s work [13]. The model considers a queue whose arrivals and service processes are defined by an alternating renewal process. The states of the alternating renewal process are 1 (ON) and 2 (OFF). The distribution of time spent in state i ($i = 1, 2$) is $F_i(t)$. Arrivals into state i constitutes a Poisson point process with rate λ_i . The service time distribution is given by $B_i(t)$. The service times of successive customers are assumed to be independent. Sengupta splits the process $\{X(t), t \geq 0\}$, the amount of work in the system at time t , into two stationary processes $\{Y(t), t \geq 0\}$, $\{Z(t), t \geq 0\}$. $Y(t)$ is constructed from process $X(t)$ by deleting all times when environment is in state 2, and $Z(t)$ by deleting all times when environment is in state 1.

We require the following notation from Sengupta [13] for our proof.

$G(t)$ distribution of work brought in at renewal epochs of $Y(t)$. (i.e work accumulated during the previous OFF period)

$\tilde{\rho}(s)$	LST of busy period of a special M/G/1 queue with arrival rate λ_1 , and service time distribution $B_1(t)$ whose amount of work at time 0 has distribution $G(t)$.
$f_i^{(k)}(t)(b_i^{(k)}(t))$	k^{th} moment of $F_i(t)$ ($B_i(t)$)
$\tilde{B}_1(s)$	stationary LST of $B_1(t)$
$\tilde{R}_1(s)$	stationary LST of the process $Y(t)$
$\tilde{R}_2(s)$	stationary LST of the process $Z(t)$
$\tilde{R}(s)$	stationary LST of the process $X(t)$

A special GI/GI/1 queue is defined, with inter-arrival duration $F_1(t)$ and LST of service time distribution is $\tilde{\rho}(s)$.

$\tilde{V}(s)$	LST of work in the special GI/G/1 queue.
$\tilde{W}(s)$	LST of customer arrival stationary distribution of the special GI/G/1 queue.
$\tilde{U}(s)$	LST of steady-state distribution of, $Z(t)$ observed at instants just after the renewal epochs.

We also recall the following results derived in Sengupta [13]

$$\tilde{R}_1(s) = \frac{1 - \lambda_1 b_1^{(1)}}{1 - \lambda_1 [(1 - \tilde{B}_1(s))/s]} \tilde{V}(s - \lambda_1(1 - \tilde{B}_1(s))) \quad (11)$$

$$\tilde{R}_2(s) = \frac{\tilde{U}(s)(1 - \tilde{F}_2(\lambda_2(1 - \tilde{B}_2(s))))}{f_2^{(1)} \lambda_2 (1 - \tilde{B}_2(s))} \quad (12)$$

$$\tilde{R}(s) = c_1 \tilde{R}_1(s) + c_2 \tilde{R}_2(s) \quad (13)$$

where, $c_1 = \frac{f_1^{(1)}}{f_1^{(1)} + f_2^{(1)}}$; $c_2 = 1 - c_1$.

With the notation all set, we find that the steady state probability that the interrupted queue is empty is $q_0 = \mathbb{P}(X(t) = 0) = \lim_{s \rightarrow \infty} \tilde{R}(s)$. Using Equation (11), Equation (12) and Equation (13), and noting that $\lambda_1 = \lambda_2 = \lambda$, $f_1^{(1)} = g$, $f_2^{(1)} = r$ and $b_1^{(1)} = b_2^{(1)} = \tau$ it follows that

$$q_0 = \frac{g}{c} (1 - \lambda\tau)v_0 + \frac{r}{c} \cdot \frac{1 - e^{-\lambda r}}{r\lambda} u_0 \quad (14)$$

where $v_0 := \lim_{s \rightarrow \infty} \tilde{V}(s)$, $u_0 := \lim_{s \rightarrow \infty} \tilde{U}(s)$. To find v_0 , we look at the special G/G/1 queue considered by Sengupta [13]. Let the work be denoted by W_r in steady state, which allows one to write $W_r = \sum_{j=0}^{N_r} S_j$ where N_r is the number of Poisson arrivals of rate λ_2 in a red time and S_j is the work brought by the j^{th} arrival. Observe that S_j 's are i.i.d $\sim B_2(\cdot)$ and are independent of N_r . Consequently, $G(t) = \int_0^\infty \sum_{k=0}^\infty e^{-\lambda_2 u} \frac{(\lambda_2 u)^k}{k!} B_2^{(n)}(t) dF_2(u)$, where $B_2^{(n)}(\cdot)$ denotes the n -fold convolution of $B_2(\cdot)$ with itself. Denote by $B_G(t)$ the service distribution of the special G/G/1 queue. It follows that $v_0 = p_0$ where p_0 is the steady state probability that the system size of the special G/G/1 queue is 0. Using, Little's law [24], $v_0 = p_0 = 1 - \frac{\lambda_1}{\mu_G}$, where $1/\mu_G$ is the mean service time of the special G/G/1 queue. Finding μ_G requires finding the mean busy period of an M/G/1 queue with non-zero amount of work at time 0. Let \hat{b} denotes this mean busy period of this special M/G/1 queue. Let \bar{G} denotes the mean amount of work brought at time 0 to the queue. Then, evidently, the busy period of the queue starts when the server starts serving this work. In the meantime, arrivals come during the removal of this work in the queue and

they bring additional work into the queue. Once the server removes the initial work it has to finish off the additional work brought by the arrivals during the removal of the initial work. Had there been no initial work in the system, this later work would constitute the busy period of the queue which, by a branching process argument, can be shown to be $b = b_1^{(1)}/(1 - \lambda_1 b_1^{(1)})$ leading to an expression of mean busy period: $\hat{b} = \bar{G} + \lambda_1 \bar{G} b = \bar{G}/(1 - \lambda_1 b_1^{(1)})$. Furthermore, the description of $G(t)$ implies $\bar{G} = \lambda_2 b_2^{(1)} f_2^{(1)}$. Thus,

$$v_0 = 1 - \frac{1}{f_1^{(1)}} \frac{\lambda_2 b_2^{(1)} f_2^{(1)}}{1 - \lambda_1 b_1^{(1)}} = 1 - \frac{1}{g} \frac{\lambda\tau r}{1 - \lambda\tau} \quad (15)$$

Using Equation (15) in Equation (14) and using the relation $c = r + g$, the final result follows.

APPENDIX B PROOF OF LEMMA 4.2

Consider the alternating process described by the alternating red and green times. The beginning of the k^{th} red time is denoted by T_k , for $k = 0, 1, 2, \dots$, and marks the beginning of the k^{th} red-green cycle. The cycle time is deterministic and is fixed to $c (= r + g)$. $I_k \in \{0, 1\}$ is an indicator random variable taking value 1 if a batch arrives in $(T_k, T_k + r]$ and finds an empty system; evidently, in a cycle, the number of batches that can arrive in a red time to find the system empty can either be 0, or 1. Let $V(t)$ denote the residual work in the interrupted queue at time t , and let $V_k := V(T_k)$. Let $A(t)$ be the total number of arrivals in $[0, t)$, and $A_{r,0}(t)$ be the number of arrivals in $[0, t)$ that arrive in an empty system in red time. Then, the fraction of batches that arrive at an empty system in a red time is $p_0 = \lim_{t \rightarrow \infty} \frac{A_{r,0}(t)}{A(t)} = \lim_{t \rightarrow \infty} \frac{\sum_{k=1}^{A(t)} I_k}{A(t)} = \frac{\lim_{t \rightarrow \infty} \sum_{k=1}^{A(t)} I_k}{\lim_{t \rightarrow \infty} \frac{A(t)}{t}}$. Since the batch arrivals constitute a Poisson point process with arrival rate λ , using elementary renewal theorem [26] it follows that $\lim_{t \rightarrow \infty} A(t)/t \stackrel{a.s.}{=} \lambda$. Furthermore, $V(t)$ is a Markov regenerative process with stopping times $\{T_k, k = 0, 1, 2, \dots\}$ and hence $\{(V_k, T_k), k = 0, 1, 2, \dots\}$ is a Markov renewal sequence [27]. Now, note that an arrival in a red time finds the queue empty only if the red time begins with zero residual work in the system. In such a case, the first arrival in such a red time will find the queue empty. Thus, a Markov regenerative analysis [27] argument allows us to write $\lim_{t \rightarrow \infty} \frac{\sum_{k=1}^{A(t)} I_k}{\sum_{k=1}^{A(t)} I_k} \stackrel{a.s.}{=} \frac{\mathbb{E}_{V_k}(I_k)}{\mathbb{E}X} = \frac{u_0 \mathbb{P}(\text{a customer arrives in } (0, r))}{c} = \frac{u_0 (1 - e^{-r\lambda})}{c}$. Here u_0 is the stationary probability that the system is empty at the beginning of a red time. Hence, combining terms in the previous limit expression, $p_0 = \lim_{t \rightarrow \infty} \frac{A_{r,0}(t)}{A(t)} \stackrel{a.s.}{=} \frac{u_0 (1 - e^{-r\lambda})}{\lambda c}$. Let ζ denote the arrival instant of the first batch to arrive in a red-green cycle with ζ measured from the beginning of the red time to the instant of arrival of the batch. The residual red time seen by the arrival will then be just $(r - \zeta)^+$. Then the residual red time seen by the first batch arriving in the red time, given that a batch does arrive is given by $r'_0 = \mathbb{E}(r - \zeta | r > \zeta) = \frac{\int_0^r (r-u)\lambda e^{-\lambda u} du}{1 - e^{-r\lambda}} = \frac{r}{1 - e^{-r\lambda}} - \frac{1}{\lambda}$.

To find p_1 , let $D(t)$ be the number of departures in $(0, t]$, and $D_i(t)$ be the number of departures in $(0, t]$ that are inter-

rupted. Then $p_1 = \lim_{t \rightarrow \infty} \frac{D_i(t)}{D(t)} = \frac{\lim_{t \rightarrow \infty} \frac{D_i(t)}{t}}{\lim_{t \rightarrow \infty} \frac{D(t)}{t}}$. Now, for a stable queueing system $\lim_{t \rightarrow \infty} A(t)/t = \lim_{t \rightarrow \infty} D(t)/t \stackrel{a.s.}{=} \lambda$. For the limit in the numerator we again use a Markov regenerative analysis. Observe that a departure is interrupted at T_k if $V(T_k-) \neq 0$. Again considering the red-green cycles, if the red time at the beginning of the cycle starts with non-zero work in the system, it implies that the service of a batch has been interrupted. Due to the assumption of service time being smaller than the green time, this batch will depart in the green time in that cycle, thus yielding a “reward” of 1. It follows, using Markov regenerative analysis, that $p_1 = \frac{1}{\lambda} \lim_{t \rightarrow \infty} \frac{D_i(t)}{t} \stackrel{a.s.}{=} \frac{1}{\lambda} \cdot \frac{(1-u_0) \cdot 1}{c}$. Finally, using the description of the completion time in Eq (5), the desired expression for $\mathbb{E}C$ is obtained.

REFERENCES

- [1] S. Mukhopadhyay, P. MJ, and A. Kumar, “An approach for analysis of mean delay at a signalized intersection with indiscriminate traffic,” in *Communication Systems and Networks (COMSNETS), 2015 7th International Conference on*, Jan 2015, pp. 1–6.
- [2] A. Clayton, “Road traffic calculations.” *Journal of the ICE*, vol. 16, pp. 247–264, Jun., 1941.
- [3] F. Webster, *Traffic Signal Settings*. Dept. of Sci. and Ind. Research, Road Research Technical Paper No. 39, Her Majesty’s Stationary Office, London, 1958.
- [4] R. J. Serfling, “A class of stochastic point process models for traffic flow,” Department of Statistics, Florida State University, Tallahassee, Florida, Tech. Rep., January, 1970.
- [5] K. Ohno, “Computational algorithm for a fixed cycle traffic signal and new approximate expressions for average delay,” *Transportation Science*, vol. 12, no. 1, pp. 29–47, Feb., 1978.
- [6] D. R. McNeil, “A solution to the fixed-cycle traffic light problem for compound poisson arrivals,” *J. of Applied Probability*, vol. 5, no. 3, pp. 624–635, Dec., 1968.
- [7] J. Darroch, “On the traffic-light queue,” *Ann. Math. Statist.*, vol. 35, no. 1, pp. 380–388, 1964.
- [8] A. J. Miller, “Settings for fixed-cycle traffic signals,” *Operational Research Quarterly*, vol. 14, no. 4, pp. 373–386, Dec., 1963.
- [9] G. Newell, “Approximation methods for queues with application to the fixed-cycle traffic light,” *SIAM Review*, vol. 7, no. 2, pp. 223–240, Apr., 1965.
- [10] van Leeuwen, “Delay analysis for the fixed-cycle traffic-light queue,” *Transportation Science*, vol. 40, no. 2, pp. 189–199, May, 2006.
- [11] A. Oblakova, A. Al Hanbali, R. Boucherie, J. Ommeren, and W. Zijm, “Exact expected delay and distribution for the fixed-cycle traffic-light model and similar systems in explicit form,” *Memorandum Faculty of Mathematical Sciences University of Twente*, no. 2056, 2016.
- [12] I. B. Gertsbakh, “Mixed vehicle flow at signalized intersection: Markov chain analysis,” *Transport and Telecommunication Journal*, vol. 16, no. 3, pp. 190–196, 2015.
- [13] B. Sengupta, “A queue with service interruptions in an alternating random environment,” *Operations Research*, vol. 38, pp. 308–318, Mar-April, 1990.
- [14] A. Federgruen and L. Green, “Queueing systems with service interruptions,” *Operations Research*, vol. 34, no. 5, pp. 752–768, Sep. - Oct., 1986.
- [15] G. Nagaraja and A. Verma, “Traffic signal timing optimization for Indian traffic conditions using modified webster’s delay model,” 2014, submitted to *Transportation Science*.
- [16] S. S. M. Ali, B. George, and L. Vanajakshi, “An efficient multiple-loop sensor configuration applicable for undisciplined traffic,” *IEEE Transactions on Intelligent Transportation Systems*, vol. 14, no. 3, pp. 1151–1161, Sept 2013.
- [17] M. S. Kiran and A. Verma, “Review of studies on mixed traffic flow: Perspective of developing economies,” *Transportation in Developing Economies*, vol. 2, no. 1, p. 5, 2016. [Online]. Available: <http://dx.doi.org/10.1007/s40890-016-0010-0>
- [18] E. Lipper and B. Sengupta, “Assembly-like queues with finite capacity: bounds, asymptotics and approximations,” *Queueing Systems*, vol. 1, pp. 67–83, 1986.

- [19] M. Neuts, “The single server queue with poisson inputs and semi-markov service times,” *J. of Applied Probability*, vol. 3, pp. 202–230, June, 1966.
- [20] M. Pramod, “Topics in the modeling, analysis, inference, and control of congestion in road networks,” http://ece.iisc.ernet.in/network_labs/nel/files/pramodmj13me-thesis.pdf, 2013.
- [21] P. S. Praveen and V. T. Arasan, “Influence of traffic mix on PCU value of vehicles under heterogeneous traffic conditions,” *International Journal of Traffic and Transport Engineering*, vol. 3, no. 3, pp. 302–330, 2013.
- [22] S. Mukhopadhyay, “Topics in the performance modeling and control of signalized road intersections,” http://ece.iisc.ernet.in/network_labs/nel/files/mukhopadhyay14me-thesis.pdf, 2014.
- [23] E. L. Seguin, K. W. Crowley, and W. D. Zweig, “Passenger car equivalents on urban freeways,” U.S. Federal Highway Administration, Tech. Rep., 1982.
- [24] D. Gross, J. F. Shortle, J. M. Thompson, and C. M. Harris, *Fundamentals of queueing theory*. John Wiley & Sons, 2013.
- [25] G. Latouche and V. Ramaswami, *Introduction to Matrix Analytic Methods in Stochastic Modeling*. ASA/SIAM Series on Statistics and Applied Probability, 1999.
- [26] S. M. Ross, *Stochastic processes*. John Wiley & Sons New York, 1996, vol. 2.
- [27] V. G. Kulkarni, *Modeling and analysis of stochastic systems*. CRC Press, 2016.



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