Uplink Power Control and Base Station Association in Multichannel Cellular Networks

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***Abstract*— A combined base station association and power control problem is studied for the uplink of multichannel multicell cellular networks, in which each channel is used by exactly one cell (i.e., base station). A distributed association and power update algorithm is proposed and shown to converge to a Nash equilibrium of a noncooperative game. We consider network models with discrete mobiles (yielding an atomic congestion game), as well as a continuum of mobiles (yielding a population game). We find that the equilibria need not be Pareto efficient, nor need they be system optimal. To address the lack of system optimality, we propose pricing mechanisms. It is shown that these mechanisms can be implemented in a distributed fashion.**

I. INTRODUCTION

Wireless communication systems have experienced tremendous growth over the last decade, and this growth continues unabated worldwide. The efficient management of resources is essential to the success of wireless cellular systems. In a mobile cellular system, users adapt to time varying radio channels by adjusting base station (BS) associations and by controlling transmitter powers. Doing so, they not only maintain their quality of service (QoS) but also enhance their transmitters' battery lives. In addition, such controls reduce the network interference, thus maximizing the spatial reuse of channels. Distributed control is of special interest, since the alternative of centrally orchestrated control involves added infrastructure, the need for distribution of measurements, and hence system complexity.

Distributed control algorithms for single channel multicell networks have been extensively studied (Foschini & Miljanic [1], Yates [2], Hanly [3]). The monograph by Chiang et al. [4] and references therein provide an excellent survey of the area. Noncooperative games have been a natural tool for analysis and design of distributed power control algorithms. Scutari et al. [5] and Heikkinen [6] model distributed power control problems as potential games, while Altman & Altman [7] show that many of the cellular power control algorithms can be modeled as submodular games. In contrast, uplink resource allocation for multichannel multicell networks poses several challenges as observed in Yates [2] and Jiang et al. [8].

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We address the resource allocation problem in the uplink of a multichannel multicell network with a single traffic class. Such a problem arises, if in order to reduce in-network interference a CDMA operator chooses to lease and utilize multiple frequency bands. The newer mobiles are typically radio agile, and thus have the option to choose from one of these distinct bands. We address a simplified version of this multichannel multicell problem where all BSs operate on different frequency bands.

A preview of our results is as follows. We propose a distributed algorithm for the combined base station association and power control problem, and subsequently model the problem as a player-specific congestion game. The equilibrium states of such algorithms, which are Nash equilibria of the corresponding games, may be far from the system optimal. We, thus, resort to pricing mechanisms to induce mobiles to behave in a way that optimizes system cost. We also show that such a mechanism can be employed in a distributed fashion. Towards this end, we model the network as having a continuum of (nonatomic) mobiles, each offering infinitesimal load, which leads to a population game formulation. We provide a marginal pricing mechanism that motivates a pricing strategy for the discrete mobiles case. Note that, unlike the case of transportation networks, mobiles are not really priced in cellular networks. The pricing is simply a part of the decision making routine built into each mobile in order bring about a *distributed* control mechanism that drives the system toward optimality.

The paper is organized as follows. In Section II we briefly discuss concepts of *finite noncooperative games* and *population games*. We study a network model with discrete mobiles in Section III. We propose a combined association and power control algorithm, model it as a noncooperative game, and analyze its performance. We extend this analysis to a network with a continuum of mobiles in Section IV. To address the inefficiency of the proposed algorithms, we design toll mechanisms in Section V. Finally, we conclude the paper and discuss future works in Section VI.

II. GAME PRELIMINARIES

A. Finite Noncooperative Games

A noncooperative strategic form game $(\mathcal{M}, (\mathcal{A}_i, i \in \mathcal{M}), (c_i, i \in \mathcal{M}))$ consists of a set of players $\mathcal{M} = \{1, \ldots, M\}$. Each player i is accompanied by an action set $\mathcal{A}_i$ and a cost function $c_i : \times_{i=1}^{M} \mathcal{A}_i \to \mathbb{R}$. In this work, we assume all action sets to be finite. An action profile $\mathbf{a} = (a_i, i = 1, \ldots, M)$ prescribes an action a_i for

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every player $i \in \mathcal{M}$. For $\mathbf{a} = (a_i, i = 1, \ldots, M)$, denote $\mathbf{a}_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_M)$ and $(b_i, \mathbf{a}_{-i}) := (a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_M)$.

Definition 2.1: Nash Equilibrium (NE): For an action profile $\mathbf{a}$, a mobile i's *best response*, $\mathcal{B}_i(\mathbf{a}) \subseteq \mathcal{A}_i$, is defined as $\mathcal{B}_i(\mathbf{a}) := \arg\min_{b_i \in \mathcal{A}_i} c_i(b_i, a_{-i})$. $\mathbf{a}$ is said to be a Nash Equilibrium for the game if $a_i \in \mathcal{B}_i(\mathbf{a})$ for all $i \in \mathcal{M}$.

Definition 2.2: Potential Game: A game $(\mathcal{M}, (\mathcal{A}_i, i \in \mathcal{M}), (c_i, i \in \mathcal{M}))$ is said to be an ordinal potential game if there exists a function $V : \times_{i=1}^{M} \mathcal{A}_i \to \mathbb{R}$, known as an ordinal potential function, that satisfies $c_i(b_i, \mathbf{a}_{-i}) < c_i(\mathbf{a}) \Leftrightarrow V(b_i, \mathbf{a}_{-i}) < V(\mathbf{a})$ for all $i \in \mathcal{M}, b_i \in \mathcal{A}_i, \mathbf{a} \in \times_{i=1}^{M} \mathcal{A}_i$.

Clearly all minimizers of an ordinal potential function V are Nash equilibria of the game. Thus all ordinal potential games $(\mathcal{M}, (\mathcal{A}_i, i \in \mathcal{M}), (c_i, i \in \mathcal{M}))$ admit at least one Nash equilibrium on account of their finiteness. They also have the finite improvement path property (FIP) (Monderer & Shapley [13, Lemma 2.3]). Thus, in a finite ordinal potential game when players update as per the *best response* (or even *better response*) strategy, *round-robin* or *random* update processes converge to a Nash equilibrium in a finite number of steps. With the same strategies, an *asynchronous* update process also converges (Neel [14, Chapter 5]).

Definition 2.3: Congestion Game: A game $(\mathcal{M}, (\mathcal{A}_i, i \in \mathcal{M}), (c_i, i \in \mathcal{M}))$ is said to be a player-specific singleton congestion game if

1) there exists a set $\mathcal{N}$ such that $\mathcal{A}_i = \mathcal{N}$ for all $i \in \mathcal{M}$, and
2) there exist a constant β and functions $f_{ij}, i \in \mathcal{M}, j \in \mathcal{N}$ such that $c_i(\mathbf{a}) = f_{ia_i}(\sum_{\substack{l \in \mathcal{M}: \\ a_l = a_i}} \beta)$ for all $\mathbf{a} \in \times_{i=1}^{M} \mathcal{A}_i, i \in \mathcal{M}$.

In the above definition, we interpret $\mathcal{N}$ as a set of facilities and β as the load offered by each player. Then, $\sum_{\substack{l \in \mathcal{M}: \\ a_l = a_j}} \beta$ denotes the total load on facility j, under an action profile $\mathbf{a}$. The game is a *singleton* congestion game because each action picks exactly one facility. It is *player-specific* because the cost functions are player-specific.

Rosenthal [15] has defined congestion games with more general action sets while Milchtaich [9] studied weighted congestion games. Our interest however is only on singleton (unweighted) congestion games. We assume $f_{ij}(\cdot)$ to be a strictly increasing function of its argument for each $i \in \mathcal{M}$ and $j \in \mathcal{N}$.

B. Population Games

A population game (Sandholm [16]) $(\mathcal{M}, (\mathcal{A}_l, l \in \mathcal{L}), (c_{lj}, l \in \mathcal{L}, j \in \mathcal{A}_l))$ consists of $\mathcal{L} = \{1, \ldots, L\}$ classes of nonatomic populations of players. $\mathcal{M} = \cup_{l \in \mathcal{L}} \mathcal{M}_l$, and $M_l := |\mathcal{M}_l|$ denotes the total mass of the class l population. By a nonatomic population, we mean that the mass of each member of the population is infinitesimal. Players of class l are associated with an action set $\mathcal{A}_l$. Actions of these (class l) players lead to an action distribution $\mathbf{m}^l = (m_{lj}, j \in \mathcal{A}_l)$, where $\sum_{j \in \mathcal{A}_l} m_{lj} = M_l$. All the players within a class are alike. Thus the action distributions completely specify the play; we can characterize the states and dynamics of play solely in terms of action distributions. Let $\mathbf{m} = (\mathbf{m}^l, l \in \mathcal{L})$ denote the action distribution profile across the entire population, and $\mathcal{M}^*$ denote the set of all such profiles. A population l is also accompanied by continuous cost density functions $c_{lj} : \mathcal{M}^* \to \mathbb{R}$.

Definition 2.4: Nash Equilibrium (NE): An action distribution profile $\mathbf{m}$ is a pure strategy Nash equilibrium for the game $(\mathcal{M}, (\mathcal{A}_i, i \in \mathcal{M}), (c_{lj}, l \in \mathcal{L}, j \in (\mathcal{A}_l))$ if and only if for all $l \in \mathcal{L}$ and $j \in \mathcal{A}_l$, a positive mass $m_{lj} > 0$ implies $c_{lj}(\mathbf{m}) \leq c_{lk}(\mathbf{m})$ for all $k \in \mathcal{A}_l$.

Remark 2.1: At a Nash equilibrium $\mathbf{m}$, for a class l, if j and k are any two facilities in $\mathcal{A}_l$ such that $m_{lj} > 0, m_{lk} > 0$, then $c_{lj}(\mathbf{m}) = c_{lk}(\mathbf{m})$.

Definition 2.5: Potential Game: A game $(\mathcal{M}, (\mathcal{A}_l, l \in \mathcal{L}), (c_{lj}, l \in \mathcal{L}, j \in \mathcal{A}_l))$ is said to be a potential game if there exists a $\mathbf{C}^1$ function $V : \mathcal{M}^* \to \mathbb{R}$, known as a potential function, that satisfies

$$\frac{\partial V}{\partial m_{lj}}(\mathbf{m}) = c_{lj}(\mathbf{m}) \text{ for all } l \in \mathcal{L}, j \in \mathcal{A}_l, \mathbf{m} \in \mathcal{M}^*$$

It is well known that Nash equilibria are the profiles which satisfy the Kuhn-Tucker first order conditions for a minimizer of the potential function (Sandholm [16, Proposition 3.1]). For any dynamics with *positive correlation* and *noncomplacency* (in particular the best response dynamics), its all trajectories lead to Nash equilibria.

We are interested in *nonatomic congestion games* (Sandholm [16]), in which $\mathcal{A}_l = \mathcal{N}, \forall l \in \mathcal{L}$, for a given set $\mathcal{N}$. As before, we interpret $\mathcal{N}$ as a set of facilities. An action distribution profile $\mathbf{m}$ leads to a congestion profile $(m_j, j \in \mathcal{N})$, where $m_j = \sum_{l \in \mathcal{L}} m_{lj}$. The cost density functions c_{lj} depend on $\mathbf{m}$ only through m_j, and are increasing in m_j.

III. DISCRETE MOBILES

A. System Model

We now describe the model adopted in this work. We consider a cellular network consisting of several BSs and mobiles. *Each BS operates in a distinct frequency band.* Let $\mathcal{N} = \{1, \ldots, N\}$ and $\mathcal{M} = \{1, \ldots, M\}$ denote the set of BSs and the set of mobiles, respectively.

A mobile is free to choose the BS for connection, but can connect to only one BS at a time. Let h_{ij} denote the power gain from mobile i to base station j. Let the receiver noise at all BSs have common standard deviation σ. We consider only uplink performance in this work. Let p_i denote the power transmitted by mobile i, and a_i the BS to which it is associated. Let, under an association profile $\mathbf{a} = (a_i, i = 1, \ldots, M)$, $\mathcal{M}_j(\mathbf{a})$ be the set of mobiles associated with BS j. Under an association profile $\mathbf{a} = (a_i, i = 1, \ldots, M)$ and a power vector $\mathbf{p} = (p_i, i = 1, \ldots, M)$, the signal to interference ratio (SIR) of mobile i at BS a_i is

$$\frac{h_{ia_i} p_i}{\sum_{\substack{l \in \mathcal{M}_{a_i}(\mathbf{a}) \\ l \neq i}} h_{la_l} p_l + \sigma^2}$$

We assume single class traffic, i.e., all the mobiles have a common target signal to interference ratio (SIR) requirement γ.

B. The Proposed Algorithm

Yates [2] and Hanly [3] proposed an algorithm for distributed association and power control in single channel cellular networks. Convergence results for the algorithm are based on the concept of a *standard interference function*. The technique is based on a mobile reassociating itself with a BS with which it needs to use the least power; this fails to work in the case of a multichannel network and analogous convergence results for this algorithm may not hold (see Yates [2, Section VI]). Even in instances where the algorithm converges, it may get stuck at a power allocation that is not Pareto efficient.

We propose an alternative distributed algorithm for combined BS association and power control in multichannel multicell cellular networks. We also show its convergence. We make use of the following simple fact (see, for example Kumar et al. [17, Chapter 5]). Consider the subproblem of power control with a fixed association $\mathbf{a}$. Note that, under $\mathbf{a}$, $\mathcal{M}_j(\mathbf{a})$ is the set of mobiles associated with BS j. Define $\beta = \frac{\gamma}{1+\gamma}$, a measure of the "load" offered by a mobile to a BS.

Proposition 3.1: (i) The power control subproblem of BS j is feasible iff $|\mathcal{M}_j(\mathbf{a})|\beta < 1$.
(ii) If the power control subproblem of BS j is feasible, there exists a unique Pareto efficient power vector $\mathbf{p}$ given by

$$p_i = \frac{\sigma^2}{h_{ij}} \frac{\beta}{1 - |\mathcal{M}_j(\mathbf{a})|\beta}.$$

This motivates the following algorithm:
Multichannel Multicell Distributed Power Control (MMDPC): Mobiles switch associations in a round-robin fashion by taking into account the steady state optimal power consumptions at the BSs with which these associate. As the load at a BS changes, it is immediately broadcast, and associated mobiles update their powers to the optimal required powers as per new loads. In other words, the algorithm proceeds as follows. Define

$$c_i(\mathbf{a}) = \frac{\sigma^2}{h_{ia_i}} \frac{\beta}{[1 - |\mathcal{M}_{a_i}(\mathbf{a})|\beta]^+}, \tag{1}$$

where $[x]^+ = \max(x, 0)$. For $t = 0, 1, 2, \ldots$, mobile i where $i = 1 + (t \mod M)$ updates its association and power at $t + 1$,

$$a_i(t+1) = \min_{j \in \mathcal{N}} c_i((j, \mathbf{a}(t)_{-i})), \tag{2a}$$

$$p_l(t+1) = c_l(\mathbf{a}(t+1)), \quad \forall l \in \mathcal{M}_{a_i(t)}(\mathbf{a}(t)) \cup \mathcal{M}_{a_i(t+1)}(\mathbf{a}(t+1)), \tag{2b}$$

where $\mathbf{a}(t+1) = (a_i(t+1), \mathbf{a}(t)_{-i})$.

Remark 3.1: A mobile i should not choose a BS if the device renders the corresponding power control subproblem infeasible. The situation is characterized by $|\mathcal{M}_{a_i}(\mathbf{a})|\beta \geq 1$, and Equation (1) justifiably yields infinite cost for the mobile.

Note that while only one mobile updates its association at a time, all mobiles that perceive a change in load at their BSs update their powers to optimal values based on the new loads. Simultaneous association updates are not allowed. In a framework with no synchronizing agent and with an arbitrarily fine time-scale, it is unlikely that two mobiles update simultaneously. If two or more BSs result in the same steady state power, one is chosen at random by the mobile.

This algorithm is also distributed in nature as the one proposed in Yates [2]. BS j broadcasts its total congestion $|\mathcal{M}_j(\mathbf{a})|$. In addition, each mobile i is told its scaled gains $\frac{h_{ij}}{\sigma^2}$ by each BS $j \in \mathcal{N}$.

Throughout we assume that there exists at least one feasible association and power vector. In our model, an infeasible association is reflected by infinite cost. So, even if the algorithm starts with an infeasible association, selfish moves of players eventually lead to a feasible one, and updates remain feasible thereafter.

C. A Congestion Game Formulation

To show the convergence properties of the proposed algorithm, we model the system as a strategic form game. Let the mobiles be the players and the action set for each player be the possible associations, i.e, $\mathcal{A}_i = \mathcal{N}$ for all $i \in \mathcal{M}$. Define the cost functions of the players to be $c_i(\mathbf{a})$ for all $i \in \mathcal{M}$. It can be seen that above is a player-specific singleton congestion game. In the following we refer to it as the strategic form game $(\mathcal{M}, \mathcal{N}, (c_i, i \in \mathcal{M}))$.

Proposition 3.2: The finite strategic form game $(\mathcal{M}, \mathcal{N}, (c_i, i \in \mathcal{M}))$ is an ordinal potential game and thus admits the FIP.

Proof: Note that the strategic form game $(\mathcal{M}, \mathcal{N}, (c_i, i \in \mathcal{M}))$ is better response equivalent (Neel [14, Chapter 5]) to $(\mathcal{M}, \mathcal{N}, (-\frac{1}{c_i}, i \in \mathcal{M}))$. Hence the former admits the FIP if and only if the latter does. Also note that

$$-\frac{1}{c_i(\mathbf{a})} = -\frac{h_{ia_i}}{\sigma^2} \frac{[1 - |\mathcal{M}_{a_i}(\mathbf{a})|\beta]^+}{\beta}.$$

For the game $(\mathcal{M}, \mathcal{N}, (-\frac{1}{c_i}, i \in \mathcal{M}))$, the function $V : \mathcal{N}^{\mathcal{M}} \to \mathbb{R}$ given by

$$V(\mathbf{a}) = -\frac{1}{\sigma^2 \beta} \prod_{i \in \mathcal{M}} h_{ia_i} \prod_{j \in \mathcal{N}} \left(\prod_{k=1}^{|\mathcal{M}_j(\mathbf{a})|} [1 - k\beta]^+ \right)$$

is an ordinal potential function. The game is thus an ordinal potential game; it is also finite which implies that the FIP property holds. ■

The FIP property ensures that **MMDPC** converges in a finite number of steps (see Section II-A). Consider the following variants of **MMDPC**.

1) At each t, one mobile is randomly chosen to update its association. All mobiles have strictly positive probabilities of being chosen.
2) At each t, each mobile i updates its association with probability $\epsilon_i \in (0, 1)$. There is thus a strictly positive probability that any subset of mobiles may update their associations simultaneously. As before, all mobiles update their powers based on the new loads. This

algorithm does not require any coordination among mobiles (to ensure one by one updates), and is thus fully distributed.

The FIP property of the game $(\mathcal{M}, \mathcal{N}, (c_i, i \in \mathcal{M}))$ implies that these two algorithms also converges to an NE with probability 1 (see Neel [14, Chapter 5]).

D. System Optimality

Unlike the case of single channel networks, joint association and power control problems in multichannel networks do not in general admit a unique Pareto efficient power allocation. A system optimal power allocation should effect the lowest interference environment. This motivates the following definition of system optimality.

Definition 3.1: For an association profile $\mathbf{a}$, define a system performance measure $C(\mathbf{a}) = \sum_{i=1}^{M} c_i(\mathbf{a})$ where $c_i(\mathbf{a})$ are as defined in (1). An association profile $\mathbf{a}^*$ is said to be system optimal if it minimizes $C(\mathbf{a})$ over all possible associations $\mathbf{a} \in \times_{i=1}^{M} \mathcal{A}_i$.

Remark 3.2:
1) Clearly any association profile that is system optimal is also Pareto efficient.
2) If there is a unique Pareto efficient association profile, it is also the unique system optimal one.

As the following example illustrates, **MMDPC** may settle at a Pareto inefficient association profile, and hence may not be system optimal.

Example 3.1: Consider a network with two BSs, two users, and a common SIR requirement γ. The two BSs operate in disjoint bands. Assume $h_{12} < h_{11} < \frac{h_{12}}{(1-\gamma)}$ and $h_{21} < h_{22} < \frac{h_{21}}{(1-\gamma)}$. Thus, the unique Pareto efficient power allocation is $(\frac{\sigma^2}{h_{11}}\gamma, \frac{\sigma^2}{h_{22}}\gamma)$. However, if we start with initial association $(a_1 = 2, a_2 = 1)$, **MMDPC** will not move forward, because a unilateral switch requires larger power to meet the target SIR. Neither user will switch to the BS with which it has a better channel. Hence, $(\frac{\sigma^2}{h_{12}}\gamma, \frac{\sigma^2}{h_{21}}\gamma)$ is a steady state power vector, at which the algorithm settles; it is Pareto inefficient.

However, in the special case when all the mobiles are collocated and all the BSs are symmetrically placed with respect to the collocated mobiles, we have the following result.

Proposition 3.3: All the NEs in the game $(\mathcal{M}, \mathcal{N}, (c_i, i \in \mathcal{M}))$, with $h_{ij} = h$ for all $i \in \mathcal{M}, j \in \mathcal{N}$, are system optimal.

Proof: The mobiles as well as BSs are indistinguishable in this game. Let at an NE, m_j be the number of mobiles associated with BS j. We first prove that at any NE, the vector of costs to mobiles is unique up to a permutation. It is sufficient to prove that vectors $\mathbf{m} = (m_j, j \in \mathcal{N})$ are unique up to a permutation. Note that, since $\mathbf{m}$ yields an NE, the following must hold for all $j, k \in \mathcal{N}$:

$$\begin{aligned} \frac{\sigma^2}{h}\frac{\beta}{1-m_j\beta} &\leq \frac{\sigma^2}{h}\frac{\beta}{1-m_k\beta-\beta} \\ \text{or } m_j &\leq m_k + 1 \end{aligned} \quad (3)$$

Define $n = \lfloor \frac{M}{N} \rfloor$ and $l = M - nN$. From (3) we see that $\mathbf{m}$ given by $m_j = n+1, j = 1, \ldots, l, m_j = n, j = l+1, \ldots, N$ characterizes one of the NEs; other NEs are permutations of this vector. We now show that $\mathbf{m}$ is a system optimal congestion vector, and the system optimality of all other NEs follows. To do this observe that

$$C(\mathbf{a}) = \frac{\sigma^2}{h}\sum_{i\in\mathcal{M}}\frac{\beta}{1-m_{a_i}\beta} = \frac{\sigma^2}{h}\sum_{j\in\mathcal{N}}\frac{m_j\beta}{1-m_j\beta}$$

is a *Schur-convex* function in $(m_1, \ldots, m_N)$ because $\frac{x}{1-x}$ is a convex function. This implies that the minimum value is attained at a vector which is as close to uniform as possible, i.e., a vector that is *majorized* by any other vector (Marshall & Olkin [18]).[1] All such vectors are permutations of $\mathbf{m}$. Alternatively, if there exist BSs j and k such that $m_j \geq m_k + 2$, moving a mobile from BS j to BS k results in a strictly lower cost. This concludes the proof. ■

IV. CONTINUUM OF MOBILES

In this section, we consider a nonatomic version of the system in Section III-A. Such a model is of interest for two reasons. First, for many of the fixed QoS traffic classes (e.g., voice), the target SIR requirements in CDMA cellular systems are very small. In a typical IS 95 CDMA system with system bandwidth 1.25 MHz, chip rate 1.2288 Mcps, data rate 9.6 Kbps, and target $\frac{E_b}{N_0} = 6$ dB, the target SIR turns out to be -15 dB, i.e., $\frac{1}{32}$ (Kumar et al. [17, Chapter 5]). If we assume that at any time the number of mobiles associated with a BS is large, it is reasonable that an incoming mobile or an outgoing mobile affects the congestion in a negligible fashion. Secondly, we have seen that our proposed algorithm may end up with inefficient associations. There is extensive work on toll mechanisms that induce system optimality in networks with a continuum of users. The analysis of toll-mechanisms (or pricing) on a multichannel multicell network with a continuum of mobiles can be expected to shed light on the existence and properties of pricing mechanisms for networks with discrete mobiles.

A. System Model

Let $\mathcal{M} = \cup_{l=1}^{L}\mathcal{M}_l$ be an infinite set of $\mathcal{L} = \{1, \ldots, L\}$ classes of nonatomic mobiles. By nonatomic mobiles, we mean that the effect of a single mobile at a BS is infinitesimal. The population of class l mobiles has "mass" M_l. Assume $\mathcal{N}$ to be the finite set of BSs. As before, σ denotes the common standard deviation of receiver noise at all BSs. All the mobiles in a class are collocated and thus for all such mobiles their power gains to any of the BSs are same (gains from a mobile to different BSs can be different). Let h_{lj} give the power gain between a class l mobile and BS j. An association profile a is a measurable function $a : \mathcal{M} \to \mathcal{N}$. Any association a leads to a congestion profile $(\mathbf{m}(a) = m_{lj}(a), l \in \mathcal{L}, j \in \mathcal{N})$, $m_{lj}(a)$ being the mass of class l mobiles associated with BS j. Let $\mathcal{M}^*$ denote the set of all such profiles. Define $m_j(a) = \sum_{l=1}^{L} m_{lj}(a)$.

[1] The condition $h_{ij} = h_i$ for all $j \in \mathcal{N}$ is used to deduce that NE profiles are majorized by any non NE profile; the condition $h_{ij} = h_j$ for all $i \in \mathcal{M}$ is used to deduce Schur-convexity of $C(a)$.

Under an association profile a and a power density allocation $p : \mathcal{M} \to \mathbb{R}_+$, the SIR density for $x \in \mathcal{M}_l, l \in \mathcal{L}$ is

$$\frac{h_{la(x)}p(x)}{\sum_{l=1}^{L} \int_{\mathcal{M}_l} 1_a(x,z)h_{la(z)}p(z)dz + \sigma^2},$$

$$\text{where} \qquad 1_a(x,z) = \begin{cases} 1, \text{ if } a(x) = a(z) \\ 0, \text{ otherwise} \end{cases}$$

We assume that the minimum required SIR density (per unit mass) is uniform across the entire population (of all the classes); denote it by γ. This makes all the mobiles in a class alike, and so, congestion profiles are sufficient to characterize the system. In the sequel, we just use m_j for $m_j(a)$ for convenience. The dependence on a is understood.

Consider again the subproblem of power control with a fixed congestion profile $\mathbf{m}$. The following result is analogous to Proposition 3.1, and is shown in Appendix I.

Proposition 4.1: 1) The power control subproblem of BS j is feasible iff $m_j\gamma < 1$.

2) If the power control subproblem of BS j is feasible, there exists a unique Pareto efficient power density p given by

$$p(x) = \frac{\sigma^2}{h_{lj}} \frac{\gamma}{1 - m_j\gamma},$$

$\forall x \in \mathcal{M}_l$ such that $a(x) = j, l \in \mathcal{L}$, where a is the underlying association profile.

An evolutionary dynamics can be proposed to address the combined association and power control problem. To this end, we define functions $c_{lj} : \mathcal{M}^* \to \mathbb{R}_+$, where $c_{lj}(\mathbf{m})$ denotes the minimum power density for class l mobiles associated with BS j, under congestion profile $\mathbf{m}$.

$$c_{lj}(\mathbf{m}) = \frac{\gamma\sigma^2}{h_{lj}[1 - m_j\gamma]^+}$$

For notational convenience, define

$$g_{lj} = \frac{\gamma\sigma^2}{h_{lj}}$$

$$\text{and} \qquad c(z) = \begin{cases} \frac{1}{1-z\gamma}, \text{ if } z < \frac{1}{\gamma} \\ \infty, \text{ if } z \geq \frac{1}{\gamma} \end{cases}$$

Then, we have

$$c_{lj}(\mathbf{m}) = g_{lj}c(m_j) \tag{4}$$

Again we assume that the system is feasible, i.e., there exists a feasible assignment, as done in Section III-B. This boils down to assuming $|\mathcal{M}| < \frac{N}{\gamma}$ in the case of nonatomic mobiles.

B. A Congestion Game Formulation

We model the problem as a nonatomic congestion game. The continuum of mobiles constitute the population, and $\mathcal{N}$ denotes the common action set for players of all the classes. Class l players are accompanied by cost functions $c_{lj}(\mathbf{m}), j \in \mathcal{N}$. In the following, we refer to it as the game $(\mathcal{M}, \mathcal{N}, (c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$.

Theorem 4.1: The nonatomic game $(\mathcal{M}, \mathcal{N}, (c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$ is a potential game. Furthermore, it admits at least one NE, and the set of NEs coincides with the set of local minimizers of the potential function.

Proof: Consider the following optimization problem:

$$\begin{aligned} \text{Minimize} \quad & \sum_{j\in\mathcal{N}} \sum_{l\in\mathcal{L}} g_{lj} \int_0^{m_j} c(x)dx && \text{(5a)} \\ \text{subject to} \quad & \sum_{j\in\mathcal{N}} m_{lj} = M_l, \; l \in \mathcal{L} && \text{(5b)} \\ & m_{lj} \geq 0, \; l \in \mathcal{L}, j \in \mathcal{N} && \text{(5c)} \end{aligned}$$

where $m_j = \sum_{l=1}^{L} m_{lj}, \; \forall j \in \mathcal{N}$. All the conditions are self-explanatory. The first claim follows from Sandholm [16, Section 2], which also shows that the objective function (5a) is a potential function for the population game $(\mathcal{M}, \mathcal{N}, (c_{lj}, 1 \leq l \leq L, j \in \mathcal{N}))$.

Structures of the cost functions along with feasibility assumption allow us to restrict attention to the region where $m_j < \frac{1}{\gamma}, \forall j \in \mathcal{N}$; there is at least one feasible point in this region, and objective function is infinite outside this. Then, $c(x)$ is strictly increasing in x, and $\int_0^{m_j} c(x)dx$ are strictly convex in m_j. Since we are minimizing a convex objective function subject to linear constraints, there exists at least one optimizer. Also, Kuhn-Tucker first order conditions are necessary and sufficient. Combining this with the fact that NEs are the profiles which satisfy the Kuhn-Tucker first order conditions for a minimizer of the potential function (see Section II-B), we see that the set of NEs coincides with the set of local minimizers of the potential function. ∎

Remark 4.1: At NEs, the congestions (at BSs) by class, m_{lj}, are not unique because the objective function (5a) is not strictly convex with respect to this set of variables. But the total congestion at BSs, m_j, are indeed unique.

Furthermore, NEs have the following property.

Proposition 4.2: The costs for a mobile is constant across all the NEs of the game $(\mathcal{M}, \mathcal{N}, (c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$.

Proof: Consider two NEs $\mathbf{m}$ and $\mathbf{m}'$. Remark 4.1 indicates that $m_j = m'_j \; \forall j \in \mathcal{N}$. Suppose that the statement of the proposition does not hold. Then there is class l and BSs j and k with $m_{lj} > 0, m'_{lk} > 0$, but $g_{lk}c(m'_k) < g_{lj}c(m_j)$. This leads to

$$g_{lk}c(m'_k) < g_{lj}c(m_j) \leq g_{lk}c(m_k)$$

where the second inequality follows because $\mathbf{m}$ is an NE and $m_{lj} > 0$. After cancellation of g_{lk} and after observing that c is a strictly increasing function, we get $m'_k < m_k$, a contradiction. ∎

C. System Optimality

Analogous to the one in Section III-D, we define a system performance measure

$$C(\mathbf{m}) := \sum_{j\in\mathcal{N}} \sum_{l=1}^{L} m_{lj} g_{lj} c(m_j) \tag{6}$$

A congestion profile $\mathbf{m}^*$ is said to be system optimal if it minimizes $C(\mathbf{m})$ over all possible profiles $\mathbf{m} \in \mathcal{M}^*$.

In contrast with the discrete mobiles case where equilibria need not be Pareto efficient (see Example 3.1), we have the following result for the nonatomic case.

Theorem 4.2: All NEs of the nonatomic game $(\mathcal{M}, \mathcal{N}, (c_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$ are Pareto efficient.

Proof: Let $\mathbf{m}$ be an NE congestion profile. Under an NE, the cost of all the mobiles of the same class remains same, irrespective of their associations (see Remark 2.1). Thus, it is sufficient to prove that there does not exist another congestion profile $\mathbf{m}'$ such that for every class l, and for all BSs j, k, with $m_{lj} > 0, m'_{lk} > 0$,

$$c_{lk}(\mathbf{m}') \leq c_{lj}(\mathbf{m}), \tag{7}$$

and strict inequality holds for some such l, j and k. Assume that such an $\mathbf{m}'$ exists. Then,

$$g_{lk} c(m'_k) < g_{lj} c(m_j) \leq g_{lk} c(m_k)$$

where the last inequality follows because $\mathbf{m}$ is an NE and $m_{lj} > 0$. This yields $m'_k < m_k$. This further implies that there is a BS s such that $m'_s > m_s$, and a class t such that $m'_{ts} > m_{ts}$. By the strictly increasing property of c, we have

$$g_{ts} c(m'_s) > g_{ts} c(m_s) \geq g_{tr} c(m_r)$$

for a BS r such that $m_{tr} > 0$. Such a BS exists and the latter inequality follows because $\mathbf{m}$ is an NE. The two inequalities imply $c_{ts}(\mathbf{m}') > c_{tr}(\mathbf{m})$, and so the tuple t, r, s violates (7). Thus the assumption that $\mathbf{m}'$ Pareto dominates $\mathbf{m}$ is incorrect. This completes the proof. ■

However, the equilibria need not be system optimal, as illustrated by the following example.

Example 4.1: Consider an infinite set $\mathcal{M}$ of nonatomic mobiles all belonging to same class. Assume common minimum SIR density requirement γ, and let $|\mathcal{M}|\gamma < 1$. Let there be two BSs with h_j the gain to BS j, $j = 1, 2$. An NE congestion profile $(\alpha^*|\mathcal{M}|, (1-\alpha^*)|\mathcal{M}|)$ is given as

1) if $\frac{h_1}{h_2} \leq (1 - |\mathcal{M}|\gamma)$, $\alpha^* = 0$,
2) if $\frac{h_2}{h_1} \leq (1 - |\mathcal{M}|\gamma)$, $\alpha^* = 1$,
3) otherwise, α^* satisfies

$$\begin{aligned} \frac{\gamma\sigma^2}{h_1(1-\alpha^*\gamma|\mathcal{M}|)} &= \frac{\gamma\sigma^2}{h_2(1-(1-\alpha^*)\gamma|\mathcal{M}|)} \\ \text{or, } \frac{1-\alpha^*\gamma|\mathcal{M}|}{1-(1-\alpha^*)\gamma|\mathcal{M}|} &= \frac{h_2}{h_1}. \end{aligned} \tag{8}$$

On the other hand, a congestion profile $(\alpha^o|\mathcal{M}|, (1-\alpha^o|\mathcal{M}|)$ will be system optimal if and only if α^o solves the following optimization problem:

$$\begin{array}{ll} \text{Minimize} & \dfrac{\alpha\gamma|\mathcal{M}|\sigma^2}{h_1(1-\alpha\gamma|\mathcal{M}|)} + \dfrac{(1-\alpha)\gamma|\mathcal{M}|\sigma^2}{h_2(1-(1-\alpha)\gamma|\mathcal{M}|)} \\ \text{subject to} & 0 \leq \alpha \leq 1. \end{array}$$

This is a convex optimization problem, and it is straightforward to show that

1) if $\sqrt{\frac{h_1}{h_2}} \leq (1 - |\mathcal{M}|\gamma)$, $\alpha^o = 0$,
2) if $\sqrt{\frac{h_2}{h_1}} \leq (1 - |\mathcal{M}|\gamma)$, $\alpha^o = 1$,
3) otherwise, α^o satisfies

$$\frac{1-\alpha^o\gamma|\mathcal{M}|}{1-(1-\alpha^o)\gamma|\mathcal{M}|} = \sqrt{\frac{h_2}{h_1}} \tag{9}$$

Hence, if $\min\{\frac{h_1}{h_2}, \frac{h_2}{h_1}\} > 1 - |\mathcal{M}|\gamma$, α^* and α^o must satisfy (8) and (9) respectively. In such a case, the NE will be system optimal if and only if $h_1 = h_2$.

Remark 4.2: Sandholm [16] shows that if the cost function for each mobile is a homogeneous function of a certain degree, then all NEs are system optimal. Note that in Example 4.1, NEs are not system optimal unless $h_1 = h_2$. We remark that the system optimality for the latter case does not follow from Sandholm [16] because the cost functions are not homogeneous functions.

V. PRICING FOR SYSTEM OPTIMALITY

A. Continuum of Mobiles

Levying of tolls is a conventional way to enforce system optimality in networks. Beckman [10] and Dafermos & Sparrow [11] studied optimal tolls in transportation networks with a single class of users. Later Dafermos [19] and Smith [20] extended the analysis to multiclass networks. Roughgarden & Tardos [12] applied these ideas in computer networks and analyzed tolls for optimal routing. In this section, we show that there is a toll mechanism that can induce system optimal associations and power allocations in a cellular network with multiple classes of mobiles. We also show that the mechanism can be employed in a distributed fashion.

Consider the BS j with congestion profile $m_{lj}, l \in \mathcal{L}$. Define

$$c'(z) := \begin{cases} \frac{d}{dz}c(z) = \frac{\gamma}{(1-z\gamma)^2}, & \text{if } z < \frac{1}{\gamma} \\ \infty, & \text{if } z \geq \frac{1}{\gamma} \end{cases}$$

The following theorem shows that if a mobile joining BS j is levied an additional toll

$$t_j(\mathbf{m}) = \sum_{l=1}^{L} m_{lj} g_{lj} c'(m_j), \tag{10}$$

the resulting NEs coincide with the system optimal association profiles.

Theorem 5.1: The nonatomic game $(\mathcal{M}, \mathcal{N}, (\bar{c}_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$ where $\bar{c}_{lj}(\cdot) = c_{lj}(\cdot) + t_j(\cdot), \forall l \in \mathcal{L}, j \in \mathcal{N}$, is a potential game. Furthermore, a congestion profile $\mathbf{m}$ is system optimal if and only if it is an NE for the game $(\mathcal{M}, \mathcal{N}, (\bar{c}_{lj}, l \in \mathcal{L}, j \in \mathcal{N}))$.

Proof: The problem of finding system optimal associations and powers is the following nonlinear optimization program:

$$\begin{array}{ll} \text{Minimize} & C(\mathbf{m}) \\ \text{subject to} & \text{Conditions (5b) - (5c).} \end{array}$$

The same reasoning as in the proof of Theorem 4.1 allows us to restrict attention to the region where $m_j < \frac{1}{\gamma}, \forall j \in \mathcal{N}$.

The Lagrangian of the optimization problem is

$$L(\mathbf{m},\mathbf{b},\mathbf{c}) = \sum_{l\in\mathcal{L}}\left(\sum_{j\in\mathcal{N}}(g_{lj}m_{lj}c(m_j)-b_{lj}m_{lj}) - c_l\left(\sum_{j\in\mathcal{N}}m_{lj}-M_l\right)\right)$$

The Kuhn-Tucker first order conditions are

$$g_{lj}c(m_j)+\sum_{i=1}^{L}m_{ij}g_{ij}c'(m_j)-c_l-b_{lj}=0,\ l\in\mathcal{L},j\in\mathcal{N}$$
$$b_{lj}\geq 0,\ l\in\mathcal{L},j\in\mathcal{N}$$
$$b_{lj}m_{lj}=0,\ l\in\mathcal{L},j\in\mathcal{N},$$

along with Conditions (5b) - (5c). Since the objective function is convex (though not strictly) in variables m_{lj}, and we are minimizing under linear constraint, there exists at least one NE. Also, the above conditions are necessary and sufficient. Thus, $\mathbf{m}^*$ is an optimizer if and only if it satisfies

$$g_{lj}c(m_j^*)+\sum_{i=1}^{L}m_{ij}^*g_{ij}c'(m_j^*)=c_l \qquad \text{if } m_{lj}^*>0$$
$$g_{lj}c(m_j^*)+\sum_{i=1}^{L}m_{ij}^*g_{ij}c'(m_j^*)\geq c_l \qquad \text{if } m_{lj}^*=0$$

along with (5b) - (5c). With t_j defined as in (10), $\mathbf{m}^*$ is an NE for the game $(\mathcal{M},\mathcal{N},(\bar{c}_{lj},l\in\mathcal{L},j\in\mathcal{N}))$.

As in Theorem 4.1, the objective function $C(\mathbf{m})$ serves as a potential function for this non atomic game, and we therefore have a potential game. ∎

Remark 5.1: 1) $\bar{c}_{lj}=c_{lj}+t_j$ can be interpreted as the marginal cost due to additional association of class l mobiles to BS j. The term c_{lj} is the power density incurred by these new mobiles, and t_j is the increase in power consumption densities of the mobiles already associated with BS j, integrated over all such mobiles. Economists call them "private cost" and "social cost", respectively. Selfish mobiles do not care for the social cost, while the social optimality criterion accounts for this marginal externality.

2) The cost functions for various classes have a certain structure in our case, which leads to uniform tolls across different mobile classes that consider a BS. Usually one does not see uniform tolls in the case of multiclass networks (see Dafermos [19], Smith [20]).

This toll mechanism can be implemented in a distributed fashion. All the BSs broadcast the tolls along with their aggregate congestions as before. All mobiles need to know their scaled gains $\frac{h_{lj}}{\sigma^2}$ to each BS $j\in\mathcal{N}$. A mobile then makes a choice taking both power density and toll into account.

B. Discrete Mobiles

Pricing mechanisms for networks with discrete mobiles are relatively difficult to design and analyze (Fotakis & Spirakis [21]). In atomic congestion games, players may incur different costs in different NEs as is clear from Example 3.1. This is unlike the case in nonatomic games, see Proposition 4.2. Therefore, when considering atomic games, one has to distinguish between the following two cases. A toll mechanism is said to *weakly enforce* the optimal solution if there is some NE for the game with tolls that is system optimal. It is said to *strongly enforce* the optimal solution if all the NEs of the game with tolls are system optimal. In this section, we propose a toll mechanism that weakly enforces the optimal solution in all cases and strongly enforces it in a special setting. The mechanism is motivated by the one for the nonatomic case (Theorem 5.1).

Consider the network model of Section III-A and an association profile $\mathbf{a}'$. Let mobile i evaluate BS j for association. Define $\mathbf{a}=(j,\mathbf{a}'_{-i})$. Analogous to the nonatomic case, define "private" and "social" costs as

$$c_i(\mathbf{a}) = \frac{\sigma^2}{h_{ij}}\frac{\beta}{[1-|\mathcal{M}_j(\mathbf{a})|\beta]^+},$$
$$\text{and } t_i(\mathbf{a}) = \sum_{\substack{l\in\mathcal{M}:\\ l\neq i,a_l=j}}\frac{\sigma^2}{h_{lj}}\left(\frac{\beta}{[1-|\mathcal{M}_j(\mathbf{a})|\beta]^+} - \frac{\beta}{[1-(|\mathcal{M}_j(\mathbf{a})|-1)\beta]^+}\right), \qquad (11)$$

respectively.[2] Clearly, $c_i(\mathbf{a})$ is the required power of mobile i if it joins BS j, while $t_i(\mathbf{a})$ is the aggregate increase in power consumption of all other mobiles associated with BS j. We propose a toll mechanism with tolls $t_i:\mathcal{N}^M\to\mathbb{R}$ given by (11). This yields a new game $(\mathcal{M},\mathcal{N},(\bar{c}_i,i\in\mathcal{M}))$ with cost functions for an association profile $\mathbf{a}$ given by

$$\bar{c}_i(\mathbf{a}) = c_i(\mathbf{a})+t_i(\mathbf{a}) = \sum_{\substack{l\in\mathcal{M}:\\ a_l=a_i}}\frac{\sigma^2}{h_{la_i}}\frac{\beta}{[1-|\mathcal{M}_{a_i}(\mathbf{a})|\beta]^+} - \sum_{\substack{l\in\mathcal{M}:\\ l\neq i,a_l=a_i}}\frac{\sigma^2}{h_{la_i}}\frac{\beta}{[1-(|\mathcal{M}_{a_i}(\mathbf{a})|-1)\beta]^+} \qquad (12)$$

Proposition 5.1: The finite strategic form game $(\mathcal{M},\mathcal{N},(\bar{c}_i,i\in\mathcal{M}))$ is an ordinal potential game and thus admits FIP.

Proof: For the game $(\mathcal{M},\mathcal{N},(\bar{c}_i,i\in\mathcal{M}))$, the function $V:\mathcal{N}^{|\mathcal{M}|}\to\mathbb{R}$ given by

$$V(\mathbf{a})=\sum_{i\in\mathcal{M}}\frac{\sigma^2}{h_{ia_i}}\frac{\beta}{[1-|\mathcal{M}_{a_i}(\mathbf{a})|\beta]^+}$$

is an ordinal potential function. Thus $(\mathcal{M},\mathcal{N},(\bar{c}_i,i\in\mathcal{M}))$ is an ordinal potential game. Since it is also a finite game, the FIP property holds. ∎

Note that the potential function $V(\mathbf{a})$ equals the system performance measure $C(\mathbf{a})$ defined in Section III-D. Hence an association profile $\mathbf{a}^o$ that optimizes system performance

[2] In (11), when both terms within parentheses are ∞, the expression is taken to be ∞; we may think of driving β to the true value from below, and the first term always dominates the second. Same remark holds for other such expressions also.

is also a (global) minimizer of $V(\mathbf{a})$, and therefore an NE of the potential game with tolls. *So, we see that tolls $t_i(\mathbf{a})$ weakly enforce a system optimal association profile.*

1) Collocated Mobiles: Let us consider the special case when all the mobiles are collocated, i.e., $h_{ij} = h_j$ for all $i \in \mathcal{M}, j \in \mathcal{N}$. The potential function for this special case can be written as

$$V(\mathbf{a}) = \sum_{j\in\mathcal{N}} \frac{\sigma^2}{h_j} \frac{|\mathcal{M}_j(\mathbf{a})|\beta}{[1-|\mathcal{M}_j(\mathbf{a})|\beta]^+}$$

Define $g_j = \frac{\sigma^2}{h_j}, f(m) = \frac{m\beta}{[1-m\beta]^+}$ and $m_j(\mathbf{a}) = |\mathcal{M}_j(\mathbf{a})|$ for all $j \in \mathcal{N}$. $\mathbf{m}(\mathbf{a}) = (m_j(\mathbf{a}), j \in \mathcal{N})$ denotes the congestion profile under $\mathbf{a}$. Since mobiles are indistinguishable, any two association profiles that lead to identical congestion profiles are essentially indifferent from the point of view of analysis. Thus we talk solely in terms of congestion profiles. Abusing notation (the argument of $V(\cdot)$ was earlier defined to be the association profile $\mathbf{a}$), we write

$$V(\mathbf{m}) = \sum_{j\in\mathcal{N}} g_j f(m_j).$$

Since $(\mathcal{M}, \mathcal{N}, (\bar{c}_i, i \in \mathcal{M}))$ is a finite potential game, an association profile $\mathbf{m}^*$ will be an NE if and only if

$$\begin{aligned} g_j f(m_j^*) + g_k f(m_k^*) \quad &\le \quad g_j f(m_j^* - 1) + g_k f(m_k^* + 1) \\ &\qquad \forall k \ne j,\ j,k \in \mathcal{N} \end{aligned}$$

The following proposition shows that tolls $t_j(\mathbf{a})$ strongly enforce a system optimal association profile in case of collocated mobiles.

Proposition 5.2: All the NEs in the game $(\mathcal{M}, \mathcal{N}, (\bar{c}_i, i \in \mathcal{M}))$, with $h_{ij} = h_j$ for all $i \in \mathcal{M}, j \in \mathcal{N}$, are system optimal. In other words, the tolls strongly enforce system optimality.

Proof: Let $\mathbf{m}^o$ be a system optimal congestion profile, and $\mathbf{m}^*$ any other profile such that $V(\mathbf{m}^*) > V(\mathbf{m}^o)$. We can partition the set $\mathcal{N}$ as $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_+ \cup \mathcal{N}_-$ such that

$$\begin{aligned} j \in \mathcal{N}_0 \quad &\iff \quad m_j^* = m_j^o \\ j \in \mathcal{N}_+ \quad &\iff \quad m_j^* \ge m_j^o + 1 \\ j \in \mathcal{N}_- \quad &\iff \quad m_j^* \le m_j^o - 1 \end{aligned}$$

We start with the congestion profile $\mathbf{m}^*$, and move mobiles from BSs $\mathcal{N}_+$ to BSs $\mathcal{N}_-$ one mobile at a time, so that we end up with the congestion profile $\mathbf{m}^o$. In this process we get a succession of congestion profiles, each of which satisfies

$$\begin{aligned} m_j = m_j^* \quad &\forall \quad j \in \mathcal{N}_0 \\ m_j \le m_j^* \quad &\forall \quad j \in \mathcal{N}_+ \\ m_j \ge m_j^* \quad &\forall \quad j \in \mathcal{N}_- \end{aligned}$$

Additionally, there exists a pair of successive congestion profiles $\mathbf{m}'$ and $\mathbf{m}''$ such that $V(\mathbf{m}') > V(\mathbf{m}'')$ ($\mathbf{m}''$ can be the ultimate congestion profile $\mathbf{m}^o$). Let $\mathbf{m}''$ be obtained from $\mathbf{m}'$, when we move a mobile from BS $j \in \mathcal{N}_+$ to another BS $k \in \mathcal{N}_-$. This, then implies that

$$g_j f(m_j') + g_k f(m_k') > g_j f(m_j' - 1) + g_k f(m_k' + 1)$$

i.e.,

$$g_j(f(m_j') - f(m_j' - 1)) > g_k(f(m_k' + 1) - f(m_k')) \quad (13)$$

Recall that c is a convex function and $m_j' \le m_j^*, m_k' \ge m_k^*$. Using these in (13), we get

$$g_j(f(m_j^*) - f(m_j^* - 1)) > g_k(f(m_k^* + 1) - f(m_k^*))$$

i.e.,

$$g_j f(m_j^*) + g_k f(m_k^*) > g_j f(m_j^* - 1) + g_k f(m_k^* + 1)$$

which implies that $\mathbf{m}^*$ is not an NE. This completes the proof. ■

Remark 5.2:

1) On the other hand, tolls $t_j(\mathbf{a})$ may fail to strongly enforce a system optimal association profile, if mobiles are not collocated. For instance reconsider Example 3.1. The association profile $(a_1 = 2, a_2 = 1)$ is inefficient, but an NE for the game $(\mathcal{M}, \mathcal{N}, (\bar{c}_i, i \in \mathcal{M}))$.
2) While tolls at a BS are equal for all the mobiles not associated with it, they are mobile dependent for all associated ones (see (11)). This is unlike in nonatomic case where we saw uniform tolls.
3) The modified algorithm (the one accounting for tolls) can be implemented in distributed fashion. All the BSs broadcast quantities $t_j^o(\mathbf{a})$ given by

$$t_j^o(\mathbf{a}) = \sum_{l:a_l=j} \frac{\sigma^2}{h_{lj}} \frac{\beta}{[1-|\mathcal{M}_j(\mathbf{a})|\beta]^+}$$

along with their aggregate congestions $|\mathcal{M}_j(\mathbf{a})|$. All the mobiles need to know the scaled gains $\frac{h_{ij}}{\sigma^2}$ of their own channels to all the BSs $j \in \mathcal{N}$. Mobiles use these broadcast information to calculate their powers and tolls, and choose a BS taking both into account.

VI. CONCLUSION

A. Conclusions

We studied the combined association and power control problem in multichannel multicell cellular networks. We studied the cases of discrete mobiles and a continuum of mobiles. We proposed several distributed mechanisms motivated by the techniques of game theory. To mitigate the inefficiency of the distributed algorithms, we proposed toll mechanisms in both the settings. Several open questions remain.

1) Can the inefficiency of the distributed algorithms motivated by game theory be quantified? This is the so-called price of anarchy.
2) The specific setting was one where all the mobiles had a common SINR requirement. Can it be extended to multiple classes of traffic?
3) We assumed only one BS per channel. It would be of interest to extend the work to multiple BSs per channel.

Appendix I
Nonatomic Power Control

Assume $\mathcal{M}$ to be an infinite set of mobiles, and a single BS. The required SIRs of mobiles are given by the function $\gamma : \mathcal{M} \to \mathbb{R}_{++}$. Let the function $h : \mathcal{M} \to \mathbb{R}_{++}$ gives the gains of mobiles while a power allocation is another function $p : \mathcal{M} \to \mathbb{R}_{++}$. $p(x)$ and $\gamma(x)$ are interpreted as power density and target SINR density, respectively, per unit mass. The feasibility condition for p can be written as

$$\begin{aligned}
\frac{p(x)h(x)}{\int_{\mathcal{M}} p(y)h(y)dy + \sigma^2} &\geq \gamma(x),\ \forall x \in \mathcal{M} \\
\text{or, } p(x)h(x) &\geq \gamma(x) \int_{\mathcal{M}} p(y)h(y)dy + \gamma(x)\sigma^2, \\
&\quad \forall x \in \mathcal{M}
\end{aligned}$$

Integrating the inequalities over the set of all mobiles, we get the following necessary condition for feasibility.

$$\begin{aligned}
\int_{\mathcal{M}} p(x)h(x)dx &\geq \int_{\mathcal{M}} \gamma(x)dx \int_{\mathcal{M}} p(y)h(y)dy \\
&\quad + \sigma^2 \int_{\mathcal{M}} \gamma(x)dx \qquad (14) \\
&> \int_{\mathcal{M}} \gamma(x)dx \int_{\mathcal{M}} p(y)h(y)dy
\end{aligned}$$

where the strict inequality arises because $\sigma^2 \int_{\mathcal{M}} \gamma(x)dx > 0$. Thus, we find that a necessary condition is

$$\int_{\mathcal{M}} \gamma(x)dx < 1 \qquad (15)$$

Assuming that this necessary condition holds, we see that the following is a feasible power allocation.

$$p(x) = \frac{\gamma(x)\sigma^2}{h(x)(1 - \int_{\mathcal{M}} \gamma(x)dx)} \qquad (16)$$

Hence (15) is necessary as well as sufficient condition for feasibility of power control problem.

The power allocation given by Equation (16) is Pareto efficient. Suppose $q : \mathcal{M} \to \mathbb{R}_{++}$ is another feasible power allocation such that $q(x) \leq p(x)$ and strict inequality holds for a set of mobiles having a positive measure. Note the with power allocation p, we get an equality in (14). Since

$$\int_{\mathcal{M}} q(x)h(x)dx < \int_{\mathcal{M}} p(x)h(x)dx,$$

q will violate the necessary condition

$$\int_{\mathcal{M}} q(x)h(x)dx \geq \frac{\sigma^2 \int_{\mathcal{M}} \gamma(x)dx}{1 - \int_{\mathcal{M}} \gamma(x)dx},$$

a rearrangement of (14), and hence cannot be feasible.

In fact p can be shown to be the unique Pareto efficient and hence the system optimal (system optimality criteria being minimizing the sum of power consumptions over all the mobiles) power allocation. Indeed if q is another feasible power vector which is also Pareto efficient, pointwise minimizer of p and q is also feasible and Pareto dominates p, thus contradicting the fact that p is Pareto efficient.

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