Jointly Optimal Power Control and Routing for a Single Cell, Dense, Ad hoc Wireless Network*

Venkatesh Ramaiyan[a], Anurag Kumar[a], Eitan Altman[b]

***Abstract*—We consider a dense, ad hoc wireless network in a single cell framework, i.e., only one successful transmission is supported at a time. Data packets are sent between source-destination pairs by multihop relaying. We assume that all hops are of length d meters, where d is a design parameter. We consider a multiaccess contention scheme and assume that every node always has data to send, either originated from it or a transit packet (saturation assumption). Our objective is to maximize the transport capacity of the network (measured in bit-meters per second) over power controls (in a fading environment) and over the hop distance d (a routing parameter), subject to an average power constraint.**

We argue that for a dense collection of nodes confined to a small region, single cell operation is nearly optimal. Hence, for a dense ad hoc wireless network operated as a single cell, we study the optimal hop length and power control that maximizes the end-to-end throughput for a given network power constraint. More specifically, for a fading channel and for a fixed transmission time strategy (akin to the IEEE 802.11 TXOP), we find that there exists an intrinsic aggregate bit rate (Θ_{opt} bits per second, depending on the contention mechanism and the channel fading characteristics) carried by the network, when operating at the optimal hop length and power control. The optimal transport capacity is of the form $d_{opt}(\bar{P}_t) \times \Theta_{opt}$ with d_{opt} scaling as $\bar{P}_t^{\frac{1}{\eta}}$, where $\bar{P}_t$ is the available time average transmit power and η is the path loss exponent. Under certain conditions on the fading distribution, we then provide a simple characterisation of the optimal operating point.

***Index Terms*— Optimal Power Control, Self-Organisation, Fixed Transmission Time**

I. Introduction

We consider a large number of stationary nodes that form a multihop ad hoc wireless network. Source-destination pairs are chosen randomly and we assume that the traffic in the network is homogeneous. A distributed multiaccess contention scheme is used in order to schedule transmissions between nodes in the cell; for example, the CSMA/CA based distributed coordination function (DCF) of the IEEE 802.11 standard for wireless local area networks (WLANs). We assume that all nodes can decode all the contention control transmissions (i.e., there are no hidden nodes), and only one successful transmission takes place at any time in the network. In this sense we say that we are dealing with a *single cell* scenario. Thus our work in this paper can be viewed as an extension of the performance analysis presented in [2] and extended in [1]. We further assume that, during the exchange of contention control packets, pairs of communicating nodes are able to estimate the channel fading between themselves and are thus able to perform power control per transmission.

There is a natural tradeoff between using high power and long hop lengths (single hop direct transmission between the source-destination pair), versus using low power and shorter hop lengths (multihop communication using intermediate nodes), with the latter necessitating more packets to be transported in the network. The objective of the present paper is to study optimal routing, in terms of the hop length, and optimal power control for a fading channel, when a single cell network (such as that studied in [1]) is used in a multihop mode. Our objective is to maximise a certain measure of network transport capacity (measured in bit-meters per second; see Section IV), subject to a network power constraint. A network power constraint determines, to a first order, the lifetime of the network.

Situations and considerations such as those that we study could arise in a dense ad hoc sensor network. Ad hoc sensor networks are now being studied as possible replacements for wired measurement networks in large factories. For example, a distillation column in a chemical plant could be equipped with pressure and temperature sensors and valve actuators. The sensors monitor

*This research was supported by the Indo-French Centre for the Promotion of Advanced Research (IFCPAR), under project 2900-IT

[a]Dept. of Electrical Communication Engineering, Indian Institute of Science, Bangalore, 560 012, India; email: rvenkat@ece.iisc.ernet.in, anurag@ece.iisc.ernet.in

[b]INRIA, Sophia-Antipolis, France; email: eitan.altman@sophia.inria.fr

the system and communicate the pressure and temperature values to a central controller which in turn actuates the valves to operate the column at the desired operating point. Direct communication between the sensors and actuators is also a possibility. Such installations could involve hundreds of devices, organised into a single cell ad hoc wireless network because of the physical proximity of the nodes. There would be many flows within the network and there would be multihopping. We wish to address the question of optimal organisation of such an ad hoc network so as to maximise its transport capacity subject to a power constraint. The power constraint relates to the network life-time and would depend on the application. In a factory situation, it is possible that power could be supplied to the devices, hence large power would be available. In certain emergencies, "transient" sensor networks could be deployed for situation management; we use the term "transient" as these networks are supposed to exist for only several minutes or hours, and the devices could be disposable. Such networks need to have large throughputs, but, being transient networks, the power constraint could again be loose. On the other hand sensor networks deployed for monitoring some phenomenon in a remote area would have to work with very small amounts of power, while sacrificing transport capacity. Our formulation aims at providing insights into optimal network operation in a range of such scenarios.

A. Preview of Contributions

We motivate the definition of the transport capacity of the network as the product of the aggregate throughput (in bits per second) and the hop distance (in meters). For random spatio-temporal fading, we seek the power control and the hop distance that jointly optimise the transport capacity, subject to a network average power constraint. For a fixed data transmission time strategy (discussed in Section III-B), we show that the optimal power allocation function has a water pouring form (Section V-A). At the optimal operating point (power control and hop distance) the network throughput (Θ_{opt}, in bits per second) is shown to be a fixed quantity, depending only on the contention mechanism and fading model, but independent of the network power constraint (Section V-B). Further, we show that the optimal transport capacity is of the form $d_{opt}(\bar{P}_t) \times \Theta_{opt}$, with d_{opt} scaling as $\bar{P}_t^{\frac{1}{\eta}}$, where $\bar{P}_t$ is the available time average transmission power, and η is the power law path loss exponent (Theorem V.2). Finally, we provide a condition on the fading density that leads to a simple characterisation of the optimal hop distance (Section V-C).

II. Motivation for Single Cell Operation

In this context, the seminal paper by Gupta and Kumar [4] would suggest that each node should communicate with neighbours as close as possible while maintaining network connectivity. This maximises network transport capacity (in bit-metres per second), while minimising network average power. It has been observed by Dousse and Thiran [5], however, that if, unlike [4], the practical model of bounded received power for finite transmitter power is used, then the increasing interference with an increasing density of simultaneous transmitters is not consistent with a minimum SINR requirement at each receiver. The following argument illustrates that the network transport capacity actually goes to 0, as spatial reuse is increased.

Consider a dense wireless planar network in a square of area A. Let P be the transmit power per node and N be the receiver noise power. Let r denote the spatial coverage radius of each transmitter, i.e., there are $\frac{1}{\pi r^2}$ transmitters in a given unit area. Since P is the maximum signal power received at any given receiver, the SINR achievable per node in such a network is bounded, i.e., $SINR \leq \frac{P}{(N+I_r)}$, where I_r denotes the interference at a node due to spatial reuse. The minimum interference by any simultaneous transmission is bounded below by $\frac{P}{(\sqrt{2A})^{\eta}}$. Hence, $SINR \leq \frac{P}{N+\frac{A}{\pi r^2}\frac{P}{(2A)^{\frac{\eta}{2}}}}$. Observe that reducing the transmit power P only decreases the SINR at a node. The capacity (bits/sec) achieved in such a network is now bounded above by

$$C(r) := \frac{A}{\pi r^2} \log \left(1 + \frac{P}{N + \frac{A}{\pi r^2} \frac{P}{(2A)^{\frac{\eta}{2}}}} \right)$$

Clearly, $C(r)$ is a monotone decreasing function of r for $r \geq 0$ and $C(r) \leq C(0) < \infty$. For a given spatial coverage, r, of the transmitter, we expect that the transmitter-receiver separation is bounded above by r. Then, the transport capacity achieved in the network, is bounded above by $C(r)r$. We see that, $\lim_{r \to 0} rC(r) = 0$. This implies that there exists an optimal $r > 0$ which maximises the transport capacity in the network i.e., the optimum spatial reuse is finite. Further, the maximum transport capacity is bounded above, i.e., $C(r)r \leq C(0)\sqrt{2A} \leq (2A)^{\frac{\eta}{2}}\sqrt{2A}$, independent of the node density or power P. Suppose that the nodes do not have a maximum power constraint but only an average power constraint P. Then a simple TDMA scheme with direct transmissions between the source and the destination with transmit power nP (and hence, an average power P), achieves $\log(n)$ order transport capacity. As seen above, however, with spatial reuse, the system becomes interference limited, and hence, becomes inefficient both for large n and for large P. Thus, we conclude that single cell operation (as defined earlier) is efficient for such networks. In the context of sensor networks, $\log(n)$ scaling has been achieved with maximum node power constraints as well, using cooperative transmission techniques ([6]).

With the above motivation, in this work, we study the transport capacity of power constrained dense ad hoc networks operated as a single cell. More recently, El Gamal and Mammen [7] have shown that, if the transceiver energy at each hop is factored in, then the operating regime studied in [4] is neither energy efficient nor delay optimal. Fewer hops between the transmitter and receiver (and hence, less spatial reuse) reduce the energy consumption and lead to a better throughput-delay tradeoff. While optimal operation of the network might suggest using some spatial reuse (finite, as discussed above), coordinating simultaneous transmissions (in a distributed fashion), in a constrained area, is extremely difficult and the associated time, energy and synchronisation overheads have to be accounted for. In view of the above discussions, in this paper, we assume that the multiple access control (MAC) is such that only one transmitter-receiver pair communicate at any time in the network.

A. Outline of the Paper

In Section III we describe the system model and in Section IV we motivate the objective. We study the transport capacity of a single cell multihop wireless network, operating in the fixed transmission time mode, in Section V. Section VI concludes the paper and discusses future work.

III. The Network Model

There is a dense network of immobile nodes that use multiaccess multihop radio communication to transport packets between various source-destination pairs.

- All nodes use the same contention mechanism with the same parameters (e.g., all nodes use IEEE 802.11 DCF with the same back-off parameters).
- We assume that nodes send control packets (such as RTS/CTS in IEEE 802.11) with a constant power (i.e., power control is not used for the control packets) during contention, and these control packets are decodable by *every* node in the network. As in IEEE 802.11, this can be done by using a low rate, robust modulation scheme and by restricting the diameter of the network. This is the "single cell" assumption, also used in [1], and implies that there can be only one successful ongoing transmission at any time.
- During the control packet exchange, each transmitter learns about the channel "gain" to its intended receiver, and decides upon the power level that is used to transmit its data packet. For example, in IEEE 802.11, the channel gain to the intended receiver could be estimated during the RTS/CTS control packet exchange. Such channel information can then be used by the transmitter to do power control. In our paper, we assume that such channel estimation and power control is possible on a transmission-by-transmission basis.
- In this work, we model only an average power constraint and not a peak power constraint.
- We assume that the traffic is homogeneous in the network and all the nodes have data to send at all times; these could be locally generated packets or transit packets (saturation assumption).

A. Channel Model: Path Loss, Fading and Transmission Rate

The channel gain between a transmitter-receiver pair for a hop is a function of the hop length and the multipath fading "gain" (h). Based on our dense network and traffic homogeneity assumption, we further make the following assumption.

- The nodes self-organise so that all hops are of length d, i.e., a one hop transmission always traverses a distance of d meters. This hop distance, d, will be one of our optimisation variables.

The path loss for a hop distance d is given by $\frac{1}{d^\eta}$, where η is the path loss exponent, chosen depending on the propagation characteristics of the environment (see, for e.g., [15]). This variation of path loss with d holds for $d > d_0$, the far field reference distance; we will assume that this inequality holds ($d > d_0$), and will justify this assumption in the course of the analysis below (see Theorem V.2).

We assume that for each transmitter-receiver pair, the channel gain due to multipath fading may change from transmission to transmission, but remains constant over any packet transmission duration. Since successive transmissions can take place between randomly selected pairs of nodes (as per the outcome of the distributed contention mechanism) we are actually modeling a spatio-temporal fading process. We assume that this fading process is stationary in space and time with some given marginal distribution H. Let the cumulative distribution of H be $A(h)$ (with a p.d.f. $a(h)$), which by our assumption of spatio-temporal stationarity of fading is the same for all transmitter-receiver pairs and for all transmissions. We assume a flat and slow fading channel with additive white

Gaussian noise of power σ^2. And, τ_c, the channel coherence time applicable to all the links in the network, upper bounds the time taken to complete any data transmission in the network. We assume that H and τ_c are independent of the hop distance d.

When a node transmits to another node at a distance d (in the transmitting antenna's far field), using transmitter power P, with channel power gain due to fading, h, then we assume that the transmission rate given by Shannon's formula is achieved over the transmission burst; i.e., the transmission rate is given by

$$C = W \log \left(1 + \frac{hP\alpha}{\sigma^2 d^\eta}\right)$$

where W is the signal bandwidth and α is a constant accounting for any fixed power gains between the transmitter and the receiver. Note that this requires that the transmitter has available several coding schemes of different rates, one of which is chosen for each channel state and power level.

B. Fixed Transmission Time Strategy

We focus on a fixed transmission time scheme, where all data transmissions are of equal duration, independent of the bit rate achieved over the wireless link. This implies that the amount of data that a transmitter sends during a transmission opportunity is proportional to the achieved physical link rate. Let T ($< \tau_c$, the channel coherence time), be the data transmission time. Upon a successful control packet exchange, the channel (between the transmitter, that "won" the contention, and its intended receiver) is reserved for a duration of T seconds independent of the channel state h. This is akin to the "TxOP" (transmission opportunity) mechanism in the IEEE 802.11 standard. Thus, when the power allocated during the channel state h is $P(h)$, and $P(h) > 0$, then data transmission occupies the channel for the duration T seconds, sending $C(h)T$ bits across the channel, where $C(h) = W \log\left(1 + \frac{P(h)h\alpha}{\sigma^2 d^\eta}\right)$. If $P(h) = 0$, we assume that the channel is left idle for the next T seconds.

The optimality of a fixed transmission time scheme, for throughput, as compared to a fixed packet length scheme, can be formally established (see Appendix D), but, due to lack of space, we only provide an intuition here. When using fixed packet lengths, a transmitter may be forced to send the entire packet even if the channel is poor, thus taking longer time and more power. On the other hand, in a fixed transmission time scheme, we send more data when the channel is good and limit our inefficiency when the channel is poor.

IV. Multihop Transport Capacity

Let d denote the hop length and $\{P(h)\}$ a power allocation policy, with $P(h)$ denoting the transmit power used when the channel state is h. We take a simple model for the random access channel contention process. The channel goes through successive contention periods. Each period can be either an idle slot, or a collision period, or a successful transmission with probabilities p_i, p_c and p_s respectively. Under the node saturation assumption, the aggregate bit rate carried by the system, $\Theta_T(\{P(h)\}, d)$, for the hop distance d and power allocation $\{P(h)\}$, is given by (see [2], or [1])

$$\Theta_T(\{P(h)\}, d) := \frac{p_s(\int_0^\infty L(h)\, \mathrm{d}A(h)\,)}{p_i T_i + p_c T_c + p_s(T_o + T)} \tag{1}$$

where $L(h) = C(h)T$, and, T_i, T_c and T_o are the average time overheads associated with an idle slot, collision and data transmission. For e.g., in IEEE 802.11 with the RTS/CTS mechanism being used, a collision takes a fixed time independent of the data transmission rate. We note that $p_i, p_s, p_c, T_i, T_o,$ and T_c depend only on the parameters of the distributed contention mechanism (MAC protocol), and not on any of the decision variables that we consider.

With $\Theta_T(\{P(h)\}, d)$ defined as in (1), we consider $\Theta_T(\{P(h)\}, d) \times d$ as our measure of transport capacity of the network. This measure can be motivated in several ways. $\Theta_T(\{P(h)\}, d)$ is the rate at which bits are transmitted by the network nodes. When transmitted successfully, each bit traverses a distance d. Hence, $\Theta_T(\{P(h)\}, d) \times d$ is the rate of spatial progress of the flow of bits in the network (in bit-metres per second). Viewed alternatively, it is the weighted average of the end-to-end flow throughput with respect to the distance traversed. Suppose that a flow i covers a distance D_i with $\frac{D_i}{d}$ hops (assumed to be an integer for this argument). Let $\beta_i \Theta_T(\{P(h)\}, d)$ be the fraction of throughput of the network that belongs to flow i. Then, $\frac{\beta_i \Theta_T(\{P(h)\}, d)}{\frac{D_i}{d}}$ is the end-to-end throughput for flow i and $\frac{\beta_i \Theta_T(\{P(h)\}, d)}{\frac{D_i}{d}} \times D_i = \beta_i \Theta_T(\{P(h)\}, d) \times d$ is the end-to-end flow throughput for flow i in bit-metres per second. Summing over all the flows, we have $\Theta_T(\{P(h)\}, d) \times d$, the aggregate end-to-flow throughput in bit-metres per second.

With the above motivation, our aim in this paper is to maximise the quantity $\Theta_T(\{P(h)\}, d) \times d$ over the hop distance d and over the power control $\{P(h)\}$, subject to a *network average power constraint*, $\bar{P}$. We use a network power constraint that accounts for the energy used in data transmission as well as the energy overheads associated with communication.

V. Optimising the Transport Capacity

For a given $\{P(h)\}$ and d, and the corresponding throughput $\Theta_T(\{P(h)\}, d)$, the transport capacity in bit-meters per second, which we will denote by $\psi(\{P(h)\}, d)$, is given by

$$\psi(\{P(h)\}, d) := \Theta_T(\{P(h)\}, d) \times d$$

Maximizing $\psi(\cdot, \cdot)$ involves optimizing over d, as well as $\{P(h)\}$. However, we observe that, it would not be possible to vary d with fading, as routes cannot vary at the fading time scale. Hence, we propose to optimize first over $\{P(h)\}$ for a given d, and then optimize over d, i.e., we seek to solve the following problem,

$$\max_{d} \max_{\{\{P(h)\}:\mathcal{P}(\{P(h)\}) \leq \bar{P}\}} \psi(\{P(h)\}, d) \tag{2}$$

where the network average power, $\mathcal{P}(\{P(h)\})$, is given by,

$$\mathcal{P}(\{P(h)\}) := \frac{p_i E_i + p_c E_c + p_s (E_o + T \int_0^\infty P(h) \, \mathrm{d}A(h) \,)}{p_i T_i + p_c T_c + p_s (T_o + T)} \tag{3}$$

E_i, E_c and E_o correspond to the energy overheads associated with an idle period, collision and successful transmission. Thus, E_i denotes the total energy expended in the network over an idle slot, E_c denotes the total average energy expended by the colliding nodes, as well as the idle energy of the idle nodes, and E_o denotes the average energy expended in the successful contention negotiation between the successful transmitter-receiver pair, the receive energy at the receiver (in the radio and in the packet processor), and the idle energy expended by all the other nodes over the time $T_o + T$.

For a given d and power allocation $\{P(h)\}$, define the time average transmission power, $\bar{P}_t(\{P(h)\}, d)$, and the time average overhead power, $\bar{P}_o$ (which does not depend on $\{P(h)\}$ or d), as

$$\begin{aligned} \bar{P}_t(\{P(h)\}, d) &:= \frac{p_s (\int_0^\infty P(h) \, \mathrm{d}A(h) \,) T}{p_i T_i + p_c T_c + p_s (T_o + T)} \\ \bar{P}_o &:= \frac{p_i E_i + p_c E_c + p_s E_o}{p_i T_i + p_c T_c + p_s (T_o + T)} \end{aligned}$$

Then the network power constraint can be rewritten as

$$\bar{P}_t(\{P(h)\}, d) \leq \bar{P} - \bar{P}_o$$

where the right hand side does not depend on $\{P(h)\}$ or d. Observe that $\bar{P}_t (:= \bar{P} - \bar{P}_o)$ is the *time average* transmission power constraint.

A. *Optimization over $\{P(h)\}$ for a fixed d*

Consider the optimization problem

$$\max_{\{\{P(h)\}:\mathcal{P}(\{P(h)\}) \leq \bar{P}\}} \psi(\{P(h)\}, d) \tag{4}$$

The denominators of $\Theta_T(\cdot, \cdot)$ in (1) and of $\mathcal{P}$ in (3) are independent of d and the power control $\{P(h)\}$. Thus, with d fixed, the optimization problem simplifies to maximizing $\int_0^\infty L(h) \, \mathrm{d}A(h)$ or,

$$\int_0^\infty \log \left(1 + \frac{P(h) h \alpha}{\sigma^2 d^\eta} \right) \, \mathrm{d}A(h)$$

subject to the average power contraint,

$$\int_0^\infty P(h) \, \mathrm{d}A(h) \leq \bar{P}_t{}'$$

where $\bar{P}_t{}'$ is given by,

$$\bar{P}_t{}' := \frac{(p_i T_i + p_c T_c + p_s (T_o + T))}{p_s T} \bar{P}_t$$

Notice that $\bar{P}_t{}'$ is also independent of $\{P(h)\}$ or d and is the average transmit power constraint averaged only over the transmission periods.

This is a well-known problem whose optimal solution has the water-pouring form (see [3]). The optimal power allocation function $\{P(h)\}$ is given by

$$P(h) = \left(\frac{1}{\lambda} - \frac{d^\eta \sigma^2}{h \alpha} \right)^+$$

where λ is obtained from the power constraint equation

$$\int_{\frac{\lambda \sigma^2 d^\eta}{\alpha}}^\infty a(h) P(h) dh = \bar{P}_t{}'$$

The optimal power allocation is a nonrandomized policy, where a node transmits with power $P(h)$ every time the channel is in state h (whenever $P(h) > 0$), or leaves the channel idle for h such that $P(h) = 0$.

B. *Optimization over d*

By defining $\Phi(h) := \frac{P(h)}{d^\eta}$, the problem of maximising the throughput over power controls, for a fixed d, becomes

$$\max \int_0^\infty \log \left(1 + \frac{\alpha h}{\sigma^2} \Phi(h) \right) a(h) dh$$

subject to

$$\int_0^\infty \Phi(h) a(h) dh \leq \frac{\bar{P}_t{}'}{d^\eta}$$

Denoting by $\Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$ the optimal value of this problem, the problem of optimisation over the hop-length now becomes

$$\max_d d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right) \tag{5}$$

Theorem V.1: In the problem defined by (5), the objective $d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$, when viewed as a function of d, is continuously differentiable. Further, when the channel fading random variable, H, has a finite mean ($E(H) < \infty$), then

1) $\lim_{d\to 0} d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right) = 0$ and,
2) if in addition, $\eta \geq 2$, $\frac{1}{h^2}a\left(\frac{1}{h}\right)$ is continuously differentiable and $\mathsf{P}(H > h) = O\left(\frac{1}{h^2}\right)$ for large h, then, $\lim_{d\to\infty} d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right) = 0$,

Proof: The proofs of continuous differentiability of $d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$, 1) and 2) are provided in Appendix B ■

Remarks V.1:

1) Under the conditions proposed in Theorem V.1, it follows that $d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$ is bounded over $d \in [0,\infty)$ and achieves its maximum in $d \in (0,\infty)$.
2) When the objective function (5) is unbounded, the optimal solution occurs at $d = \infty$.
3) We note that, in practice, $\eta \geq 2$.

Theorem V.2: The following hold for the problem in (5),

1) Without the constraint $d > d_0$, the optimum hop distance d_{opt} scales as $(\bar{P}_t')^{\frac{1}{\eta}}$.
2) There is a value $\bar{P}_{t\min}'$ such that, for $\bar{P}_t' > \bar{P}_{t\min}'$, $d_{opt} > d_0$, and hence the optimal solution obeys the scaling shown in 1).
3) For $\bar{P}_t' > \bar{P}_{t\min}'$, the optimum power control $\{P(h)\}$ is of the water pouring form and scales as $\bar{P}_t'$.
4) For $\bar{P}_t' > \bar{P}_{t\min}'$, the optimal transport capacity scales as $(\bar{P}_t')^{\frac{1}{\eta}}$.

Proof:

1) Let d_{opt} be optimal for $\bar{P}_t' > 0$. We claim that, for $x > 0$, $x^{\frac{1}{\eta}} d_{opt}$ is optimal for the power constraint $x\bar{P}_t'$. For suppose this was not so, it would mean that there exists $d > 0$ such that
$$\left(x^{\frac{1}{\eta}} d_{opt}\,\Gamma\left(\frac{x\bar{P}_t'}{(x^{\frac{1}{\eta}} d_{opt})^\eta}\right)\right) < d\,\Gamma\left(\frac{x\bar{P}_t'}{d^\eta}\right)$$
or, equivalently,
$$\left(d_{opt}\,\Gamma\left(\frac{\bar{P}_t'}{d_{opt}^\eta}\right)\right) < x^{-\frac{1}{\eta}} d\,\Gamma\left(\frac{\bar{P}_t'}{(x^{-\frac{1}{\eta}} d)^\eta}\right)$$
which contradicts the hypothesis that d_{opt} is optimal for $\bar{P}_t'$.
2) Using the path loss model $\frac{P}{d^\eta}$, we see that for $d < d_0$, the received power is scaled more than P, due to the factor $\frac{P}{d^\eta}$, and an d_0^η factor in α, i.e., the model over-estimates the received power and the transport capacity. Hence, the achieved transport capacity for $d < d_0$ is definitely less than $d\,\Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$. The result now follows from the scaling result in 1).
3) It follows from 1) that, if $\bar{P}_t'$ scales by a factor x, then the optimum d scales by $x^{\frac{1}{\eta}}$, so that, at the optimum, $\frac{\bar{P}_t'}{d^\eta}$ is unchanged. Hence the optimal $\{\Phi(h)\}$ is unchanged, which means that $\{P(h)\}$ must scale by x. The water pouring form is evident.
4) Again, by 1) and 2), if $\bar{P}_t'$ scales by a factor x, then the optimum d scales by $x^{\frac{1}{\eta}}$, so that, at the optimum, $\frac{\bar{P}_t'}{d^\eta}$ is unchanged. Thus $\Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$ is unchanged, and the optimal transport capacity scales as the optimum d, i.e., by the factor $x^{\frac{1}{\eta}}$.

■

Remarks V.2:

The above theorem yields the following observations for the fixed transmission time model.

1) As an illustration, with $\eta = 3$, in order to double the optimal transport capacity, we need to use 2^3 times the $\bar{P}_t'$. This would result in a considerable reduction in network lifetime, assuming the same battery energy.
2) We observe that as the power constraint $\bar{P}_t'$ scales, the optimal bit rate carried in the network, $\Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$, stays constant, but the optimal transport capacity increases since the optimal hop length increases. Further, because of the way the optimal power control and the optimal hop length scale together, the nodes transmit at the same *physical bit rate* in each fading state; see the proof of Theorem V.2 part 3). ■

C. *Characterisation of the Optimal d*

By the results in Theorem V.1 we can conclude that the optimal solution of the maximisation in (5) lies in the set of points for which the derivative of $d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$ is zero. For fixed $\bar{P}_t'$, define $\pi(d) := \frac{\bar{P}_t'}{d^\eta}$. Differentiating $d \times \Gamma(\pi(d))$, we obtain, (see Appendix A for the details)

$$\frac{\partial}{\partial d}(d\,\Gamma(\pi(d)) \quad = \quad \Gamma(\pi(d)) - \eta\pi(d)\lambda(\pi(d))$$

where $\lambda(\pi)$ is the Lagrange multiplier for the optimisation problem that yields $\Gamma(\pi(d))$. Since d appears only

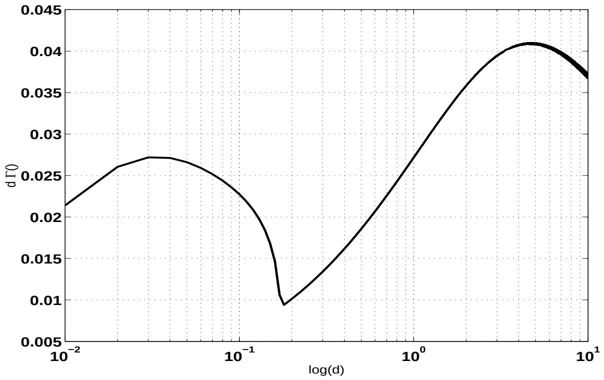

Fig. 1. Plot of $d \times \Gamma\left(\frac{1}{d^3}\right)$ (linear scale) vs. d (log scale) for a channel with two fading states h_1, h_2. The fading gains are $h_1 = 100$ and $h_2 = 0.5$, with probabilities $a_{h_1} = 0.01 = 1 - a_{h_2}$. The function has 3 non-trivial stationary points.

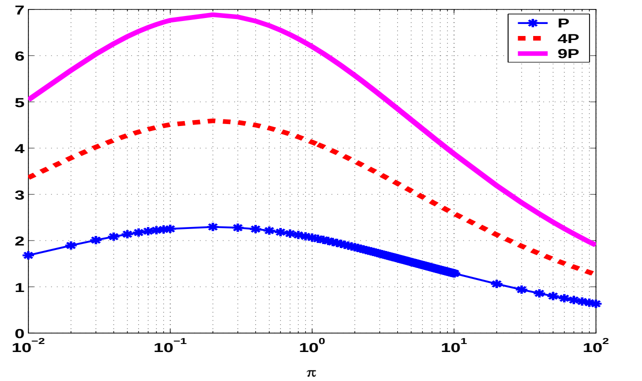

Fig. 2. Plot of $d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$ (linear scale) vs. $\pi \left(= \frac{\bar{P}_t'}{d^\eta}\right)$ (log scale) for a fading channel (with exponential distribution). We consider 3 power levels ($\bar{P}_t{}'$, $4\bar{P}_t{}'$ and $9\bar{P}_t{}'$) and $\eta = 2$. The function has a unique optimum $\pi_{opt}(\pi_{opt} \approx 0.2)$ for all the 3 cases.

via $\pi(d)$, we can view the right hand side as a function of π. We are interested in the zeros of the above expression. Clearly, $\pi = 0$ is a solution. This solution corresponds to the case $d = \infty$; However, we are interested only in solutions of $d \in (0, \infty)$, and hence, we seek positive solutions of π of $\Gamma(\pi) - \eta\pi\lambda(\pi) = 0$.

Remarks V.3: The above analysis has been done for a continuously distributed fading random variable H. The analysis can be done for a discrete valued fading distribution as well, and we provide this analysis in Appendix C. The following example then illustrates that, in general, the function $\Gamma(\pi) - \eta\pi\lambda(\pi) = 0$ can have multiple solutions. Consider a fading distribution that takes two values: $h_1 = 100$ and $h_2 = 0.5$, with probabilities $a_{h_1} = 0.01 = 1 - a_{h_2}$. Figure 1 plots $d \times \Gamma\left(\frac{1}{d^3}\right)$ for the system with $\eta = 3$. Notice that there are 3 stationary points other than the trivial solution $d = \infty$ (which is not shown in the figure). Also, the maximising solution is not the first stationary point (the stationary point close to 0). If, on the other hand, $a_{h_1} = 0.001 = 1 - a_{h_2}$, we again have 3 stationary points, but the optimal solution now is the first stationary point. ■

More generally, and still pursuing the discrete case, let $\mathcal{H}$ denote the set of fading states when the fading random variable is discrete with a finite number of values; $|\mathcal{H}|$ denotes the cardinality of $\mathcal{H}$.

Theorem V.3: There are at most $2|\mathcal{H}| - 1$ stationary points of $d\,\Gamma(\pi(d))$ in $0 < d < \infty$.

Proof: See Appendix C for the related analysis and the proof of this theorem. ■

We conclude from the above discussion that it is difficult to characterise the optimal solution when there are multiple stationary points. Hence we seek conditions for a unique positive stationary point, which must then be the maximising solution. In Appendix A, we have shown that the equation characterising the stationary points, $\Gamma(\pi) - \eta\pi\lambda(\pi) = 0$, can be rewritten as

$$\int_0^1 (\log(y) - \eta(y-1)) \frac{\lambda^2}{y^2} f\left(\frac{\lambda}{y}\right) dy = 0 \qquad (6)$$

for $f(x) := a\left(\frac{\sigma^2 x}{\alpha}\right)\frac{\sigma^2}{\alpha}$, the density of the random variable $\frac{\alpha H}{\sigma^2}$. Notice that π does not appear in this expression. The solution directly yields the Lagarange multiplier of the throughput maximisation problem for the optimal value of hop length. The following theorem guarantees the existence of atmost one stationary point of (6).

Theorem V.4: If for any $\lambda_1 > \lambda_2 > 0$, $\frac{f\left(\frac{\lambda_2}{y}\right)}{f\left(\frac{\lambda_1}{y}\right)}$ is a strictly monotonic decreasing function of y, then the objective function $d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$ has at most one stationary point $d_{opt}, 0 < d_{opt} < \infty$.

Proof: The proof follows from Lemmas A.1, and A.2 in Appendix A. ■

Corollary V.1: If H has an exponential distribution and $\eta \geq 2$, then the objective in the optimisation problem of (5) has a unique stationary point $d_{opt} \in (0, \infty)$, which achieves the maximum.

Proof: $a(h)$ is of the form $\mu e^{-\mu h}$ and the monotonicity hypothesis in Theorem V.4 holds for $a(h)$. Also, from Theorem V.1, we see that $\lim_{d \to 0} d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right) = 0$ and $\lim_{d \to \infty} d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right) = 0$. ■

Remarks V.4:
1) Hence, for $\eta \geq 2$, for the Rayleigh fading model there exists a unique stationary point which corresponds to the optimal operating point.
2) For $\bar{P}_t{}' > \bar{P}'_{t\min}$, and for the conditions in Theorem V.1 and V.4, let π_{opt} denote the unique stationary point of (6). Then define $\Gamma(\pi_{opt}) = \Theta_{opt}$. It follows from Theorem V.2 that the optimal trans-

port capacity takes the form $\left(\frac{\bar{P}_t{}'}{\pi_{opt}}\right)^{\frac{1}{\eta}}\Theta_{opt}$, where Θ_{opt} depends on $a(h)$ and the MAC parameters but not on $\bar{P}$ (or $\bar{P}_t$).
3) Figure 2 numerically illustrates our results for Rayleigh fading and $\eta = 2$. Scaling $\bar{P}_t{}'$ by 4 scales the transport capacity from 2.3 to 4.6, i.e., by $4^{\frac{1}{\eta}} = \sqrt{4}$ and similarly for scaling $\bar{P}_t{}'$ by 9.

The uniqueness results guarantees that a distributed implementation of the optimization problem, if it converges, shall converge to the unique stationary point, which is the optimal solution.

VI. Conclusion

In this paper we have studied a problem of jointly optimal power control and self-organisation in a single cell, dense, ad hoc multihop wireless network. The self-organisation is in terms of the hop distance used when relaying packets between source-destination pairs.

We formulated the problem as one of maximising the transport capacity of the network subject to an average power constraint. We showed that, for a fixed transmission time scheme, there corresponds an intrinsic aggregate packet carrying capacity at which the network operates at the optimal operating point, independent of the average power constraint. We also obtained the scaling law relating the optimal hop distance to the power constraint, and hence relating the optimal transport capacity to the power constraint (see Theorem V.2). Because of the way the power control and the optimal hop length scale, the optimal physical bit rate in each fading state is invariant with the power constraint. In Theorem V.4 we provide a characterisation of the optimal hop distance in cases in which the fading density satisfies a certain monotonicity condition.

One motivation for our work is the optimal operation of sensor networks. If a sensor network is supplied with external power, or if the network is not required to have a long life-time, then the value of the power constraint, $\bar{P}$, can be large, and a long hop distance will be used, yielding a large transport capacity. On the other hand, if the sensor network runs on batteries and needs to have a long life-time then $\bar{P}$ would be small, yielding a small hop length. In both cases the optimal aggregate bit rate carried by the network would be the same.

Future work on this topic will include developing a distributed algorithm for nodes to adapt themselves towards the optimal operating point, and studying the effect of spatial reuse, mobility of nodes on multihop communications.

We observed that for dense wireless networks, single cell operations with simple TDMA schemes are throughput optimal (and also delay optimal, $O(1)$). This motivated us to study single cell operations for power constrained dense ad hoc networks. We expect that single cell operations may be efficient and optimal as well for many ad hoc and sensor network scenarios. A thorough study on such scenarios involving node power P, area A and the number of nodes n would help design better wireless systems with simpler operations.

References

[1] A. Kumar, E. Altman, D. Miorandi and M. Goyal, New Insights from a fixed point analysis of single cell IEEE 802.11 Wireless LANs, Proc Infocom, 2005.

[2] G. Bianchi, Performance Analysis of the IEEE 802.11 Distributed Coordination Function, IEEE JSAC, 2000.

[3] A.J. Goldsmith and P.P. Varaiya, Capacity of Fading Channels with Channel Side Information, IEEE Trans on IT, 1997.

[4] P. Gupta and P.R. Kumar, The Capacity of Wireless Networks, IEEE Trans on IT, 2000.

[5] O. Dousse and P. Thiran, Connectivity vs capacity in dense adhoc networks, Proc IEEE Infocom, 2004.

[6] Hesham El Gamal, On the scaling laws of dense wireless sensor networks: the data gathering channel, ToIT, 2005.

[7] Abbas El Gamal and James Mammen, Optimal Hopping in Ad hoc Wireless Networks, Proc IEEE Infocom, 2006.

[8] S. Toumpis and A.J. Goldsmith, Capacity Region for Wireless Ad Hoc Networks, IEEE Trans on Wireless Communications, 2003.

[9] T. Elbatt and A. Ephremides, Joint scheduling and power control for wireless ad-hoc networks, Proc Infocom, 2002.

[10] R. Cruz and A.V. Santhanam, Optimal Routing, Link Scheduling and Power Control in Multi-hop Wireless Networks, Proc Infocom, 2003.

[11] M.J. Neely, E. Modiano, and C.E. Rohrs, Dynamic Power Allocation and Routing for Time Varying Wireless Networks, Proc Infocom, 2003.

[12] R. Bhatia and M. Kodialam, On power efficient communication over multi-hop wireless networks: Joint routing, scheduling and power control, Proc Infocom, 2004.

[13] B. Radunovic and J.Y. Le Boudec, Optimal Power Control, Scheduling and Routing in UWB Networks, IEEE JSAC, 2004.

[14] F. Cali, M. Conti and E. Gregori, IEEE 802.11 Protocol: Design and Performance Evaluation of an Adaptive Backoff Mechanism, IEEE JSAC, 2000.

[15] T.S. Rappaport, Wireless Communications: Principles & Practice, Prentice Hall, 2002.

[16] J.P. Aubin, Applied Functional Analysis, John Wiley & Sons, 2000.

Appendix

A. Stationary Points of $d \times \Gamma(\pi(d))$

Recall that we defined $\pi(d) := \frac{\bar{P}_t{}'}{d^\eta}$. Further, $\Gamma(\pi(d))$ was defined by

$$\Gamma(\pi(d)) := \max \int_0^\infty \log\left(1 + \frac{\alpha h}{\sigma^2}\frac{P(h)}{d^\eta}\right) a(h)dh \quad (7)$$

where the maximum is over all power controls $\{P(h)\}$ satisfying the constraint

$$\int_0^\infty \frac{P(h)}{d^\eta} a(h) dh \leq \pi(d) \tag{8}$$

For ease of notation, let us use the substitution $x := \frac{\alpha h}{\sigma^2}$. Write $\xi(x) := \xi(\frac{\alpha h}{\sigma^2}) = \frac{P(h)}{d^\eta}$ and $f(x) := a\left(\frac{\sigma^2 x}{\alpha}\right)\frac{\sigma^2}{\alpha}$. Note that $f(\cdot)$ is the probability density of the random variable $X := \frac{\alpha H}{\sigma^2}$. Then, equations (7) and (8) can be rewritten as

$$\Gamma(\pi) = \max \int_0^\infty \log(1 + x\xi(x)) f(x) dx$$

and

$$\int_0^\infty \xi(x) f(x) dx \leq \pi$$

This optimisation problem is one of maximising a convex functional of $\{\xi(x)\}$, subject to a linear constraint. The optimal solution of the problem has water-pouring form, and the optimal solution is given by,

$$\xi(x) = \left(\frac{1}{\lambda(\pi)} - \frac{1}{x}\right)^+$$

where $\lambda(\pi)$ is obtained from

$$\int_{\lambda(\pi)}^\infty \left(\frac{1}{\lambda(\pi)} - \frac{1}{x}\right) f(x) dx = \pi$$

Further, the derivative of the optimum value $\Gamma(\pi)$, w.r.t. π, i.e., $\frac{\partial\Gamma(\pi)}{\partial\pi} = \lambda(\pi)$ (see Aubin [16]).

Let us now reintroduce the dependence on d, and consider the problem of optimising $d \times \Gamma(\pi(d))$ over d. Differentiating $d\,\Gamma(\pi(d))$ w.r.t. d, we get,

$$\begin{aligned}
\frac{\partial}{\partial d}(d\,\Gamma(\pi(d)) &= \Gamma(\pi(d)) + d\,\frac{\partial}{\partial d}\Gamma(\pi(d)) \\
&= \Gamma(\pi(d)) + d\,\frac{\partial\Gamma}{\partial\pi}(\pi(d)) \times \frac{\partial\pi(d)}{\partial d} \\
&= \Gamma(\pi(d)) + d\,\Gamma'(\pi(d)) \times \frac{-\eta\bar{P}_t{}'}{d^{\eta+1}} \\
&= \Gamma(\pi(d)) - \eta\pi(d)\Gamma'(\pi(d))
\end{aligned}$$

where $\Gamma'(\pi) := \frac{\partial\Gamma(\pi)}{\partial\pi}$. Substituting $\Gamma'(\pi) = \lambda(\pi)$, we have,

$$\frac{\partial}{\partial d}(d\Gamma(\pi(d)) = \Gamma(\pi(d)) - \eta\pi(d)\lambda(\pi(d)) \tag{9}$$

The stationary points of $d \times \Gamma(\pi(d))$ are now obtained by equating the right hand side of (9) to zero. Note that since d appears in this equation only as $\pi(d)$, we need only study the roots of the equation

$$\Gamma(\pi) - \eta\pi\lambda(\pi) = 0 \tag{10}$$

We now proceed to obtain a characterisation of the stationary points. Substituting the optimal solution in the expression of $\Gamma(\pi)$ and $\lambda(\pi)$, and suppressing the argument π in $\lambda(\pi)$, we get,

$$\Gamma(\pi) = \int_\lambda^\infty \log\left(\frac{x}{\lambda}\right) f(x) dx \tag{11}$$

with λ being given by

$$\pi = \int_\lambda^\infty \left(\frac{1}{\lambda} - \frac{1}{x}\right) f(x) dx \tag{12}$$

Using the substitution $z = \frac{1}{x}$, $l = \frac{1}{\lambda}$, and defining $g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right)$, we get,

$$\Gamma(\pi) = \int_0^l \log\left(\frac{l}{z}\right) g(z) dz \tag{13}$$

with l (actually, $l(\pi)$) being given by

$$\pi = \int_0^l (l - z)\, g(z) dz \tag{14}$$

We note that $g(\cdot)$ is the density of the random variable $Z := \frac{1}{X} = \frac{\sigma^2}{\alpha H}$.

For a function $t(\cdot)$ of the random variable Z, define the operators $\mathsf{E}_l(\cdot)$ and $\mathsf{G}_l(\cdot)$ as

$$\begin{aligned}
\mathsf{E}_l(t(Z)) &:= \frac{\int_0^l t(z) g(z) dz}{\int_0^l g(z) dz} \\
\mathsf{G}_l(t(Z)) &:= \int_0^l t(z) g(z) dz
\end{aligned}$$

Lemma A.1: The roots of (10) are equivalent to obtaining the roots of the equation

$$\eta\mathsf{G}_{\frac{1}{\lambda}}(\lambda Z - 1) = \mathsf{G}_{\frac{1}{\lambda}}(\log(\lambda Z)) \tag{15}$$

with π then being given by

$$\pi = \int_\lambda^\infty \left(\frac{1}{\lambda} - \frac{1}{x}\right) f(x) dx$$

Proof: Using the definitions of $\mathsf{E}_l(\cdot)$ and $\mathsf{G}_l(\cdot)$, (13) and (14) simplify to

$$\Gamma(\pi) = \log(l)\mathsf{P}(Z \leq l) - \mathsf{G}_l(\log(Z)) \tag{16}$$

$$\pi = l\mathsf{P}(Z \leq l) - \mathsf{G}_l(Z) \tag{17}$$

(17) provides the l (actually $l(\pi)$) to be substituted in (16). Substituting for $\Gamma(\pi)$ (from (16)), and for l (from (17)), into the right hand side of (9), dividing across by $\mathsf{P}(Z \leq l)$, and using the definition of $\mathsf{E}_l(\cdot)$, we have,

$$\log\left(\frac{\pi + \mathsf{G}_l(Z)}{\mathsf{P}(Z \leq l)}\right) - \mathsf{E}_l(\log(Z)) - \frac{\eta\pi}{\pi + \mathsf{G}_l(Z)} = 0$$

$$\log\left(\frac{\pi}{\mathsf{P}(Z \leq l)} + \mathsf{E}_l(Z)\right) - \mathsf{E}_l(\log(Z)) - \frac{\eta\pi}{\pi + \mathsf{G}_l(Z)} = 0$$

$$\log\left[\left(\frac{\pi}{\mathsf{G}_l(Z)} + 1\right)\mathsf{E}_l(Z)\right] + \log\left(e^{-\mathsf{E}_l(\log(Z))}\right) - \frac{\eta\pi}{\pi + \mathsf{G}_l(Z)} = 0$$

Rearranging terms, we get,

$$\log\left(\frac{\pi + \mathsf{G}_l(Z)}{\mathsf{G}_l(Z)}\right) + \log\left(\mathsf{E}_l(Z)e^{-\mathsf{E}_l(\log(Z))}\right) - \frac{\eta\pi}{\pi + \mathsf{G}_l(Z)} = 0$$

Denote $b_l := \log\left(\mathsf{E}_l(Z)e^{-\mathsf{E}_l(\log(Z))}\right)$. Then, we have,

$$\log\left(\frac{\pi + \mathsf{G}_l(Z)}{\mathsf{G}_l(Z)}\right) + b_l - \frac{\eta\pi}{\pi + \mathsf{G}_l(Z)} = 0$$

Using (17), we have

$$\frac{\mathsf{G}_l(Z)}{\pi + \mathsf{G}_l(Z)} = \frac{\mathsf{G}_l(Z)}{l\mathsf{P}(Z \leq l)} = \frac{\mathsf{E}_l(Z)}{l}$$

which, with the previous equation, yields

$$\log\left(\frac{l}{\mathsf{E}_l(Z)}\right) + b_l - \eta\left(1 - \frac{\mathsf{E}_l(Z)}{l}\right) = 0$$

Recall that l is actually $l(\pi)$. We now find that π appears in the equation only as $l(\pi)$. Hence we can view this as an equation in the variable $l(= \frac{1}{\lambda})$. Rearranging terms, we get

$$-\log\left(\frac{\mathsf{E}_l(Z)}{l}\right) + \eta\frac{\mathsf{E}_l(Z)}{l} = -(b_l - \eta)$$

Exponentiating both sides, and substituting back for b_l, yields

$$\frac{\mathsf{E}_l(Z)}{l}e^{-\eta\frac{\mathsf{E}_l(Z)}{l}} = \mathsf{E}_l(Z)e^{-\mathsf{E}_l(\log(Z))}e^{-\eta}$$

On cancelling $\mathsf{E}_l(Z)$, and transposing terms, we next obtain

$$e^{-\eta\left(\frac{\mathsf{E}_l(Z)}{l} - 1\right)} = e^{-\mathsf{E}_l\left(\log\left(\frac{Z}{l}\right)\right)}$$

or,

$$e^{-\eta\left(\mathsf{E}_l\left(\frac{Z-l}{l}\right)\right)} = e^{-\mathsf{E}_l\left(\log\left(\frac{Z}{l}\right)\right)}$$

Taking log on both sides, we have,

$$\eta\mathsf{E}_l\left(\frac{Z-l}{l}\right) = \mathsf{E}_l\left(\log\left(\frac{Z}{l}\right)\right)$$

In terms of $\mathsf{G}_l(\cdot)$, this is equivalent to

$$\eta\mathsf{G}_l\left(\frac{Z-l}{l}\right) = \mathsf{G}_l\left(\log\left(\frac{Z}{l}\right)\right)$$

which is the desired result after writing $l = \frac{1}{\lambda}$. ■

We next address the question of a *unique* positive solution of (15). The following lemma guarantees the existence of a unique positive solution, when $f(\cdot)$, the density of $\frac{\alpha H}{\sigma^2}$, satisfies a certain monotonicity condition.

Lemma A.2: (15) has at most one positive solution if for any $\lambda_1 > \lambda_2 > 0$, $\frac{f\left(\frac{\lambda_2}{y}\right)}{f\left(\frac{\lambda_1}{y}\right)}$ is a strictly monotone decreasing function of y.

Proof: Expanding $\mathsf{G}_{\frac{1}{\lambda}}(\cdot)$, (15) becomes,

$$\eta\int_0^{\frac{1}{\lambda}}(\lambda z - 1)g(z)dz - \int_0^{\frac{1}{\lambda}}\log(\lambda z)g(z)dz = 0$$

Rewriting the equation in terms of $f(\cdot)$, we have,

$$\int_0^{\frac{1}{\lambda}}(\eta(\lambda z - 1) - \log(\lambda z))\frac{1}{z^2}f\left(\frac{1}{z}\right)dz = 0$$

In this last equation change the variable to $y := \lambda z$, yielding

$$\int_0^1(\log(y) - \eta(y-1))\frac{\lambda^2}{y^2}f\left(\frac{\lambda}{y}\right)dy = 0 \qquad (18)$$

Define $c(y) := (\log(y) - \eta(y-1))\frac{1}{y^2}$ and $b_\lambda(y) := f\left(\frac{\lambda}{y}\right)$. Thus, we are interested in a positive λ that solves

$$\int_0^1 c(y)b_\lambda(y)dy = 0$$

Observe that $\lim_{y\to 0} c(y) = -\infty$ and $c(1) = 0$. Further, there exists a unique y' such that $c(y) \leq 0$ for all $0 \leq y \leq y'$ and $c(y) \geq 0$ for all $y' \leq y \leq 1$. Since $b_\lambda(y) \geq 0$ for all y and λ, we have $c(y)b_\lambda(y) \leq 0$ for all $0 \leq y \leq y'$ and $c(y)b_\lambda(y) \geq 0$ for all $y' \leq y \leq 1$.

Consider λ_1, λ_2 such that $\lambda_1 > \lambda_2 > 0$. By hypothesis, $\frac{b_{\lambda_2}(y)}{b_{\lambda_1}(y)}$ is a strictly monotone decreasing function of y. Hence, $\frac{c(y)b_{\lambda_2}(y)}{c(y)b_{\lambda_1}(y)}$ is also a strictly monotone decreasing function of y. We then have,

$$\frac{\int_0^{y'}|c(y)|b_{\lambda_2}(y)dy}{\int_0^{y'}|c(y)|b_{\lambda_1}(y)dy} = \frac{\int_0^{y'}|c(y)|\frac{b_{\lambda_2}(y)}{b_{\lambda_1}(y)}b_{\lambda_1}(y)dy}{\int_0^{y'}|c(y)|b_{\lambda_1}(y)dy}$$

$$> \frac{b_{\lambda_2}(y')}{b_{\lambda_1}(y')},$$

And,

$$\frac{\int_{y'}^1 c(y) b_{\lambda_2}(y)dy}{\int_{y'}^1 c(y) b_{\lambda_1}(y)dy} = \frac{\int_{y'}^1 c(y) \frac{b_{\lambda_2}(y)}{b_{\lambda_1}(y)} b_{\lambda_1}(y)dy}{\int_{y'}^1 c(y) b_{\lambda_1}(y)dy}$$

$$< \frac{b_{\lambda_2}(y')}{b_{\lambda_1}(y')}$$

Hence,

$$\frac{\int_0^{y'} |c(y)| b_{\lambda_2}(y)dy}{\int_0^{y'} |c(y)| b_{\lambda_1}(y)dy} > \frac{\int_{y'}^1 c(y) b_{\lambda_2}(y)dy}{\int_{y'}^1 c(y) b_{\lambda_1}(y)dy}$$

i.e.,

$$\frac{\int_0^{y'} |c(y)| b_{\lambda_2}(y)dy}{\int_{y'}^1 c(y) b_{\lambda_2}(y)dy} > \frac{\int_0^{y'} |c(y)| b_{\lambda_1}(y)dy}{\int_{y'}^1 c(y) b_{\lambda_1}(y)dy}$$

i.e., the ratio of the negative area of the integral to the positive area of the integral is a strictly monotonic function of λ. Hence, as λ decreases, the integral can cross 0 at most once, or, there exists at most one (non-trivial) solution for (18). ■

B. Additional Proofs for the Continuous Fading Case

Lemma B.3: $d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$ is continuously differentiable with respect to d.

Proof: $\Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$ and $\lambda(\pi(d))$ are continuous function of $\pi(d)$, and $\pi(:= \frac{\bar{P}_t'}{d^\eta})$ itself is a continous function of d. Hence, from (9), we see that $d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)$ is a continously differentiable function of d. ■

Remarks B.1: The following are some points to note about the characterisation in Lemma A.1.

1) Note that both sides of (15) are negative quantities, and $\mathsf{G}_{\frac{1}{\lambda}}(\lambda Z - l) \geq \mathsf{G}_{\frac{1}{\lambda}}(\log(\lambda Z))$ for all $\lambda \geq 0$.
2) $\mathsf{G}_{\frac{1}{\lambda}}(\lambda Z - 1)$ and $\mathsf{G}_{\frac{1}{\lambda}}(\log(\lambda Z))$ are concave increasing functions of λ. The following is a proof for $\mathsf{G}_{\frac{1}{\lambda}}(\log(\lambda Z))$. Let $\lambda = \alpha\lambda_1 + (1-\alpha)\lambda_2$ for $0 < \alpha < 1$ and $\lambda_1 < \lambda < \lambda_2$ (equivalently $\frac{1}{\lambda_1} > \frac{1}{\lambda} > \frac{1}{\lambda_2}$). We then have,

$$\mathsf{G}_{\frac{1}{\lambda}}(\log(\lambda Z)) = \mathsf{G}_{\frac{1}{\lambda}}(\log((\alpha\lambda_1 + (1-\alpha)\lambda_2)Z))$$

$$= \int_0^{\frac{1}{\lambda}} \log((\alpha\lambda_1 + (1-\alpha)\lambda_2)z)g(z)dz$$

From the concavity of the logarithm function, we have,

$$\geq \int_0^{\frac{1}{\lambda}} (\alpha \log(\lambda_1 z) + (1-\alpha)\log(\lambda_2 z))g(z)dz$$

$$= \alpha \int_0^{\frac{1}{\lambda}} \log(\lambda_1 z)g(z)dz + (1-\alpha)\int_0^{\frac{1}{\lambda}} \log(\lambda_2 z)g(z)dz$$

$$= \alpha\left(\int_0^{\frac{1}{\lambda_1}} \log(\lambda_1 z)g(z)dz - \int_{\frac{1}{\lambda}}^{\frac{1}{\lambda_1}} \log(\lambda_1 z)g(z)dz\right)$$

$$+ (1-\alpha)\left(\int_0^{\frac{1}{\lambda_2}} \log(\lambda_2 z)g(z)dz + \int_{\frac{1}{\lambda_2}}^{\frac{1}{\lambda}} \log(\lambda_2 z)g(z)dz\right)$$

$$\geq \alpha \int_0^{\frac{1}{\lambda_1}} \log(\lambda_1 z)g(z)dz + (1-\alpha)\int_0^{\frac{1}{\lambda_2}} \log(\lambda_2 z)g(z)dz$$

The last inequality follows from the fact that in the integral $\int_{\frac{1}{\lambda}}^{\frac{1}{\lambda_1}} (\cdot)dz$, the function $\log(\lambda_1 z)$ is negative and in the integral $\int_{\frac{1}{\lambda_2}}^{\frac{1}{\lambda}} (\cdot)dz$, the function $\log(\lambda_2 z)$ is positive. Hence, we have,

$$\mathsf{G}_{\frac{1}{\lambda}}(\log((\alpha\lambda_1 + (1-\alpha)\lambda_2)Z)) \geq$$

$$\alpha \mathsf{G}_{\frac{1}{\lambda_1}}(\log(\lambda_1 Z)) + (1-\alpha)\mathsf{G}_{\frac{1}{\lambda_2}}(\log(\lambda_2 Z))$$

which completes the proof.

3) Differentiating $G_{\frac{1}{\lambda}}(\cdot)$, we have,
   - $\frac{\partial}{\partial\lambda}\mathsf{G}_{\frac{1}{\lambda}}(\log(\lambda Z)) = \frac{1}{\lambda}P(Z \leq \frac{1}{\lambda})$
   - $\frac{\partial}{\partial\lambda}\mathsf{G}_{\frac{1}{\lambda}}(\lambda Z - 1) \leq \frac{\partial}{\partial\lambda}\mathsf{G}_{\frac{1}{\lambda}}(\log(\lambda Z))$
   - $\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}\mathsf{G}_{\frac{1}{\lambda}}(\lambda Z - 1) = E(Z)$
   - $\lim_{\lambda\to\infty}\frac{\partial}{\partial\lambda}\mathsf{G}_{\frac{1}{\lambda}}(\lambda Z - 1) = 0$
   - $\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}\mathsf{G}_{\frac{1}{\lambda}}(\log(\lambda Z)) = \infty$
   - $\lim_{\lambda\to\infty}\frac{\partial}{\partial\lambda}\mathsf{G}_{\frac{1}{\lambda}}(\log(\lambda Z)) = 0$

■

Lemma B.4: If H (or equivalently $X := \frac{H\alpha}{\sigma^2}$) has a finite mean, then $\lim_{d\to 0} d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right) = 0$.

Proof: From (14), we have,

$$\int_0^l (l-z)g(z)dz = \pi$$

where l is in fact $l(\pi)$. Talking l outside the integral and rewriting the integral as an expectation, we get,

$$l\int_0^l \left(1 - \frac{z}{l}\right) g(z)dz = \pi$$

$$lE_Z\left(1 - \frac{Z}{l}\right)^+ = \pi$$

or,

$$E_Z\left(1-\frac{Z}{l}\right)^+ = \frac{\pi}{l}$$

By Monotonce Convergence Theorem, we have,

$$\lim_{l\to\infty} E_Z\left(1-\frac{Z}{l}\right)^+ \uparrow 1$$

Since $l \to \infty$ as $d \to 0$ (or $\pi \to \infty$), we have,

$$\lim_{\pi\to\infty} \frac{l(\pi)}{\pi} = 1 \tag{19}$$

Consider now, $\lim_{d\to 0} d \times \Gamma(\pi(d))$, or equivalently, $\lim_{\pi\to\infty} \pi^{-\frac{1}{\eta}}\Gamma(\pi)$. Observe that,

$$0 \le \pi^{-\frac{1}{\eta}}\Gamma(\pi) = \pi^{-\frac{1}{\eta}} E_Z\left(-\log\left(\frac{Z}{l(\pi)}\right)\right)^+$$

where the last equality is obtained from (13). Expanding the term inside the expectation, we have,

$$= \pi^{-\frac{1}{\eta}} E_Z\left(\log\left(\frac{1}{Z}\right) + \log\left(\frac{l(\pi)}{\pi}\right) + \log(\pi)\right)^+$$

Using the inequality $\log\left(\frac{1}{z}\right) \le \frac{1}{z}$ for $z \ge 0$, we have,

$$\le \pi^{-\frac{1}{\eta}} E_Z\left(\frac{1}{Z} + \log\left(\frac{l(\pi)}{\pi}\right) + \log(\pi)\right)^+$$

Since $E_Z\left(\frac{1}{Z}\right) < \infty$ (by hypothesis on EH), $\eta > 0$ and using (19), we have the right hand side of the above expression $\to 0$ as $\pi \to \infty$, which implies $\lim_{\pi\to\infty} \pi^{-\frac{1}{\eta}}\Gamma(\pi) = 0$, or

$$\lim_{d\to 0} d\,\Gamma(\pi(d)) = 0$$

■

Lemma B.5: Let $\eta \ge 2$, $\frac{1}{x^2} f\left(\frac{1}{x}\right)$ be continuously differentiable and $\lim_{x\to 0} \frac{1}{x^2} f\left(\frac{1}{x}\right) = 0$. Then $\frac{\partial}{\partial d}\left(d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)\right) \le 0$ as $d \to \infty$.

Proof: From (9) and the proof of Lemma A.1, we see that,

$$\begin{aligned}\frac{\partial}{\partial d}(d\,\Gamma(\pi(d))) &= \Gamma(\pi(d)) - \eta\pi(d)\lambda(\pi(d))\\ &\propto \int_0^{\frac{1}{\lambda}} (\eta(\lambda z - 1) - \log(\lambda z))\frac{1}{z^2} f\left(\frac{1}{z}\right) dz\\ &\propto \int_0^1 (\eta(y-1) - \log(y))\frac{1}{y^2} f\left(\frac{\lambda}{y}\right) dy\end{aligned}$$

where the last equation is obtained by substituting $y = \lambda z$. We know that that as d increases, λ increases. Define $b(y) := \eta(y-1) - \log(y)$. For $\eta > 1$, there exists a y' (depending on η) such that $b(y) \ge 0$ for $0 \le y \le y'$ and $b(y) \le 0$ for $y' \le y \le 1$, also $b(1) = 0$. Continuing with the right hand side expression above,

$$\begin{aligned}&= \int_0^{y'} (\eta(y-1) - \log(y))\frac{1}{y^2} f\left(\frac{\lambda}{y}\right) dy\\ &+ \int_{y'}^1 (\eta(y-1) - \log(y))\frac{1}{y^2} f\left(\frac{\lambda}{y}\right) dy\end{aligned}$$

Further,

$$\int_0^1 (\eta(y-1) - \log(y))dy = 1 - \frac{\eta}{2}$$

Hence, for $\eta \ge 2$, the integral $\int_0^1 b(y)dy$ is non-positive.

Let $g(x) := \frac{1}{x^2} f\left(\frac{1}{x}\right)$. Then $g(x)$ is continuously differentiable function and $\lim_{x\to 0} g(x) = 0$ by hypothesis. Define z_0 as

$$z_0 := \sup\{z : g(x) = 0, 0 \le x \le z\}$$

If $z_0 > 0$, then, we see that for λ sufficiently large,

$$\int_0^{y'} (\eta(y-1) - \log(y))\frac{1}{y^2} f\left(\frac{\lambda}{y}\right) dy = 0$$

This is because for λ sufficiently large, $\frac{1}{y^2} f\left(\frac{\lambda}{y}\right) = 0$ for $0 \le y \le y'$. Hence, $\lim_{d\to\infty} \frac{\partial}{\partial d}\left(d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)\right) \le 0$.

If $z_0 = 0$, we then have $g'(z) \ge 0$ in a small neighbourhood of 0 (since g is continuously differentiable by hypothesis). Hence, the function $g(z)$ is a monotonic increasing function in an ϵ neighbourhood of 0, i.e., $g(0) < g(z) \le g(z') \le g(\epsilon)$ for all $0 < z < z' < \epsilon$. Hence for all sufficiently large λ, $\frac{1}{y^2} f\left(\frac{\lambda}{y}\right)$ is a monotone increasing function of y in $[0,1]$. Hence,

$$\begin{aligned}&\int_0^{y'} (\eta(y-1) - \log(y))\frac{1}{y^2} f\left(\frac{\lambda}{y}\right) dy +\\ &\int_{y'}^1 (\eta(y-1) - \log(y))\frac{1}{y^2} f\left(\frac{\lambda}{y}\right) dy\\ &\le \left(\frac{1}{y'}\right)^2 f\left(\frac{\lambda}{y'}\right) \int_0^{y'} (\eta(y-1) - \log(y))dy +\\ &+ \left(\frac{1}{y'}\right)^2 f\left(\frac{\lambda}{y'}\right) \int_{y'}^1 (\eta(y-1) - \log(y))dy\\ &= \left(\frac{1}{y'}\right)^2 f\left(\frac{\lambda}{y'}\right) (1 - \frac{\eta}{2}) \le 0\end{aligned}$$

for $\eta \ge 2$. Thus, $\frac{\partial}{\partial d}\left(d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)\right) \le 0$ as $d \to \infty$. ■

Lemma B.6: Let $\eta \geq 2$ and $\frac{1}{x^2} f\left(\frac{1}{x}\right)$ be continuously differentiable. If for large x, $\mathsf{P}(X > x) = O(\frac{1}{x^2})$ (or equivalently for $H = \frac{\sigma^2 X}{\alpha}$), then $\lim_{d\to\infty} d \times \Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right) = 0$.

Proof: Let $\mathsf{P}(X > x) = O(\frac{1}{x^2})$ (for large x). i.e.,

$$\int_x^\infty f(x)dx = O(\frac{1}{x^2})$$

Now substitute for $z = \frac{1}{x}$. We then have,

$$\int_0^{\frac{1}{x}} \frac{1}{z^2} f\left(\frac{1}{z}\right) dz = O(\frac{1}{x^2})$$

Define $g(z) := \frac{1}{z^2} f\left(\frac{1}{z}\right)$. We claim that $g(0) = 0$. If $g(0) \neq 0$, we then have $g(z) \geq \epsilon$ for all $0 \leq z < \delta$ for some δ. Then,

$$\int_0^{\frac{1}{x}} g(z)dz \geq \epsilon \frac{1}{x}$$

for all $\frac{1}{x} \leq \delta$, which is a contradiction. Hence $\lim_{z\to 0} g(z) = 0$ or $\lim_{z\to 0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 0$.

We know from (9) that

$$\frac{\partial}{\partial d}(d\,\Gamma(\pi(d))) = \Gamma(\pi(d)) - \eta\pi(d)\lambda(\pi(d))$$

Now from Lemma B.5, we see that, for $\eta \geq 2$,

$$\Gamma(\pi(d)) - \eta\pi(d)\lambda(\pi(d)) \quad \leq \quad 0$$

for large d, or as $d \to \infty$,

$$\Gamma(\pi(d)) \leq \eta\pi(d)\lambda(\pi(d))$$

Multiplying by d on both the sides, we have,

$$d\,\Gamma(\pi(d)) \leq \eta\pi(d)\lambda(\pi(d))d = \eta\bar{P}_t' \frac{\lambda(\pi(d))}{d^{\eta-1}} \qquad (20)$$

Since $\frac{\partial}{\partial d}\left(d\,\Gamma\left(\frac{\bar{P}_t'}{d^\eta}\right)\right) \leq 0$ as $d \to \infty$, the function $d\ \Gamma(\pi(d))$ is monotonic decreasing for $d \to \infty$. Hence, if $\lim_{d\to\infty} d\ \Gamma(\pi(d)) \neq 0$, it implies that $\lim_{d\to\infty} d\,\Gamma(\pi(d)) \geq \epsilon > 0$, which, using (20), implies that $\frac{\lambda(\pi(d))}{d^{\eta-1}} \geq \epsilon$ or $\lambda(\pi(d)) \geq \epsilon d^{\eta-1}$ for $d \to \infty$.

From (12),

$$\int_\lambda^\infty \left(\frac{1}{\lambda} - \frac{1}{x}\right) f(x)dx = \frac{\bar{P}_t'}{d^\eta}$$

ignoring the negative term, we have,

$$\frac{1}{\lambda}\int_\lambda^\infty f(x)dx \quad \geq \quad \frac{\bar{P}_t'}{d^\eta}$$

or,

$$\int_\lambda^\infty f(x)dx \quad \geq \quad \frac{\bar{P}_t'}{d^\eta}\lambda$$

Using $\lambda(\pi(d)) \geq \epsilon d^{\eta-1}$, when $\lim_{d\to\infty} d\ \Gamma(\pi(d)) \neq 0$, we have,

$$\begin{aligned} &\geq \frac{\bar{P}_t'}{d^\eta}\epsilon d^{\eta-1} \\ &= \bar{P}_t'\epsilon\frac{1}{d} \end{aligned}$$

But we have

$$\int_\lambda^\infty f(x)dx = \mathsf{P}(X > \lambda) = O(\frac{1}{\lambda^2}) \leq O(\frac{1}{d^{2\eta-2}})$$

Hence we find that $O\left(\frac{1}{d}\right) \leq O\left(\frac{1}{d^{2\eta-2}}\right)$, which is a contradiction for $\eta \geq 2$. Hence, $\lim_{d\to\infty} d \times \Gamma\left(\frac{\bar{P}_t}{d^\eta}\right) = 0$. ∎

C. Discrete Fading States

The optimization problem (4) for the discrete fading state case, simplifies to

$$\begin{aligned} \max \quad & \sum_{h\in\mathcal{H}} a_h \ln\left(1 + \left(\frac{\alpha h}{\sigma^2}\right)\frac{P(h)}{d^\eta}\right) \\ \text{subject to} \quad & \sum_{h\in\mathcal{H}} a_h P(h) \leq \bar{P}_t' \end{aligned} \qquad (21)$$

For notational convenience, let us index the set of fading states, $\mathcal{H}$, in descending order by the index $i, 1 \leq i \leq |\mathcal{H}|$, i.e., $h_1 > h_2 > h_3 > \cdots$. Further, denote

$$a_{h_i} = a_i,\ x_i = \frac{\alpha h_i}{\sigma^2}, \text{and } \xi_i = \frac{P(h_i)}{d^\eta}$$

Also, denote

$$\Pi = \frac{\bar{P}_t'}{d^\eta}$$

We will later recall that, for each power constraint $\bar{P}_t'$, Π is a function of d. Using this new notation and change of variables, we obtain the problem

$$\begin{aligned} \max \quad & \sum_i a_i \ln(1 + x_i\xi_i) \\ \text{subject to} \quad & \sum_i a_i\xi_i \leq \Pi \end{aligned} \qquad (22)$$

We have the maximisation of a concave mapping from $\mathbb{R}^{|\mathcal{H}|}$ to $\mathbb{R}$ subject to a linear constraint. The KKT conditions are necessary and sufficient, and the following "water pouring" form of the optimal solution is well known. There exists $\lambda(\Pi) > 0$, such that, for $1 \leq i \leq |\mathcal{H}|$,

$$\xi_i = \left(\frac{1}{\lambda(\Pi)} - \frac{1}{x_i}\right)^+$$

with $\lambda(\Pi)$ being given by

$$\sum_{\{i: \frac{x_i}{\lambda(\Pi)} > 1\}} a_i \left(\frac{1}{\lambda(\Pi)} - \frac{1}{x_i} \right) = \Pi$$

Defining, for $1 \leq k \leq |\mathcal{H}|$,

$$p_k = a_1 + a_2 + \cdots + a_k, \text{ and } \alpha_k = \sum_{i=1}^{k} \frac{a_i}{x_i}$$

and $\Pi_0 = 0, \Pi_{|\mathcal{H}|} = \infty$, the Lagrange multiplier, $\lambda(\Pi)$, is given by

$$\lambda(\Pi) = \left(\frac{1}{p_k} (\alpha_k + \Pi) \right)^{-1} \tag{23}$$

for $\Pi_{k-1} < \Pi \leq \Pi_k$ when $1 \leq k \leq |\mathcal{H}| - 1$, and for $\Pi_{|\mathcal{H}|-1} < \Pi < \infty$ when $k = |\mathcal{H}|$. Here the break-points $\Pi_k, 1 \leq k \leq |\mathcal{H}| - 1$, are obtained by equating the values of $\lambda(\Pi)$ on either sides of the break-points, and are expressed as

$$\Pi_k = \left(\frac{\frac{\alpha_{k+1}}{p_{k+1}} - \frac{\alpha_k}{p_k}}{\frac{1}{p_k} - \frac{1}{p_{k+1}}} \right)$$

The denominator of this expression is clearly > 0, and a little algebra shows that, since $x_{k+1} > x_i, 1 \leq i \leq k$, the numerator is also > 0.

For each Π, let us denote the optimal value of the problem defined by (22) by $\Gamma(\Pi)$. We infer that

$$\frac{\partial \Gamma}{\partial \Pi} = \lambda(\Pi)$$

Now, fixing the power constraint $\bar{P}_t{}'$, and reintroducing the dependence on d, we recall that $\Pi(d) = \frac{\bar{P}_t{}'}{d^\eta}$, and hence conclude that

$$\frac{\partial \Gamma}{\partial d} = \lambda(\Pi(d)) \left(\frac{-\eta \bar{P}_t{}'}{d^{\eta+1}} \right)$$

Define $d_0 = \infty, d_{|\mathcal{H}|} = 0$, and, for $1 \leq k \leq |\mathcal{H}| - 1$, define

$$d_k^\eta = \bar{P}_t{}' \cdot \left(\frac{\frac{1}{p_k} - \frac{1}{p_{k+1}}}{\frac{\alpha_{k+1}}{p_{k+1}} - \frac{\alpha_k}{p_k}} \right)$$

Note that $0 = d_{|\mathcal{H}|} < d_{|\mathcal{H}|-1} < \cdots < d_2 < d_1 < d_0 = \infty$. Now, substituting for $\lambda(\Pi(d))$ from (23) and integrating, yields the following result

Theorem C.1: For given $\bar{P}_t{}'$, the optimal value $\Gamma(d)$ of the problem defined by (21) has the following characterisation.

1) The derivative of $\Gamma(d)$ w.r.t. d is given by

$$\frac{\partial \Gamma}{\partial d} = \frac{1}{d} \left(\frac{-\eta p_k \bar{P}_t{}'}{\alpha_k d^\eta + \bar{P}_t{}'} \right) \tag{24}$$

for $d_k \leq d < d_{k-1}$ when $1 \leq k \leq |\mathcal{H}| - 1$, and for $0 < d < d_{|\mathcal{H}|-1}$ when $k = |\mathcal{H}|$.

2) $\frac{\partial \Gamma}{\partial d}$ is a negative, continuous and increasing function of d. In particular $\Gamma(d)$ is a decreasing, and convex function of d.

3) The function $\Gamma(d)$ is given by

$$\Gamma(d) = p_k \ln \left(\alpha_k + \frac{\bar{P}_t{}'}{d^\eta} \right) \gamma_k \tag{25}$$

for $d_k \leq d < d_{k-1}$ when $1 \leq k \leq |\mathcal{H}| - 1$, and for $0 < d < d_{|\mathcal{H}|-1}$ when $k = |\mathcal{H}|$, with the constants of integration γ_k being given as follows.

$$\gamma_1 = \frac{1}{\alpha_1} = \frac{x_1}{a_1}$$

and, for $2 \leq k \leq \mathcal{H}$, γ_k is obtained recursively as

$$\gamma_k = \frac{\left(\left(\alpha_{k-1} + \frac{\bar{P}_t{}'}{d_{k-1}^\eta} \right) \gamma_{k-1} \right)^{\left(\frac{p_{k-1}}{p_k} \right)}}{\alpha_k + \frac{\bar{P}_t{}'}{d_{k-1}^\eta}}$$

Proof: (25) is obtained by integrating the derivative in (24) over each segment of its definition. The integration constants γ_k are obtained by equating $\Gamma(d)$ on either sides of the break-points of the argument d. ∎

1) Optimisation over d: Using Theorem C.1, we conclude that we need to look at the stationary points of $\Gamma(d)d$. To this end, consider the solutions of

$$\Gamma(d) + d\,\Gamma'(d) = 0$$

Reintroducing the variable $\Pi = \frac{\bar{P}_t{}'}{d^\eta}$, and canceling p_k, we need the solutions of

$$\ln \left(1 + \frac{\Pi}{\alpha_k} \right) \alpha_k \gamma_k - \frac{\eta \Pi}{\alpha_k + \Pi} = 0$$

for $\Pi_{k-1} < \Pi \leq \Pi_k$ when $1 \leq k \leq |\mathcal{H}| - 1$, and for $\Pi_{|\mathcal{H}|-1} < \Pi < \infty$ when $k = |\mathcal{H}|$, with the break-points $\Pi_k, 1 \leq k \leq |\mathcal{H}|$, as given earlier. Let us write $\frac{\Pi}{\alpha_k + \Pi} = 1 - \frac{1}{1 + \frac{\Pi}{\alpha_k}}$, define $b_k = \ln \alpha_k \gamma_k$ (observe that $b_1 = 0$), and, for given k, use the new variable

$$y = \frac{1}{1 + \frac{\Pi}{\alpha_k}}$$

Note that, for $0 < \Pi < \infty$, $1 > y > 0$. Define $\delta_k = \frac{1}{1 + \frac{\Pi_k}{\alpha_k}}$. Then we seek the solutions of

$$\ln \frac{1}{y} + b_k - \eta (1 - y) = 0$$

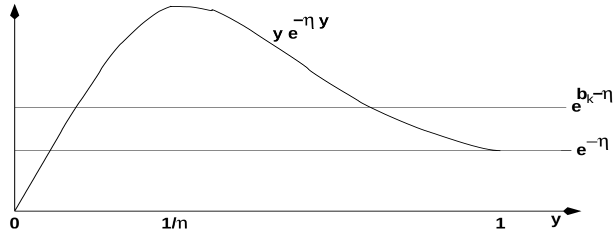

Fig. 3. The stationary points of $\Gamma(d)d$ lie among the intersections of the curve $ye^{-\eta y}$ and lines $e^{b_k-\eta}, 1 \leq k \leq |\mathcal{H}|$, in the interval $0 < y < 1$. Here the plot is drawn for $\eta = 3$.

for $\delta_k \leq y < \delta_{k-1}$, for each $k, 1 \leq k \leq |\mathcal{H}|$; note that $\delta_0 = 1$, and $\delta_{|\mathcal{H}|} = 0$. The equations can be written more simply as

$$e^{b_k-\eta} = ye^{-\eta y},$$

and are depicted in Figure 3. At this point we can conclude the following

Theorem C.2: 1) $\lim_{d\to 0}\Gamma(d)d = 0$

2) There are at most $2|\mathcal{H}| - 1$ stationary points of $\Gamma(d)d$ in $0 < d < \infty$.

Proof: 2) follows from the arguments just before the theorem statement, since each line $e^{(b_k-\eta)}$, for $2 \leq k \leq |\mathcal{H}|$, has at most two intersections with $ye^{-\eta y}$, in $0 < y < 1$, and $e^{-\eta}$ has only one such intersection. ∎

D. Fixed Transmission Time vs Fixed Packet Size

In this section, we will formally establish that fixed transmission time schemes are more throughput efficient compared to fixed packet size schemes, for a given average power constraint. We will prove this result in a general framework, without explicitly modelling the underlying MAC, the power control schemes used or the channel fading distribution.

Data Transmission Model: In a fixed transmission time scheme, all data transmissions (with positive rate) are of a fixed amount of time T, independent of the channel state h and the power used. Earlier, in our work (see Section III-B), we assumed that, when the channel fade is poor (and hence $P(h) = 0$), the channel is left idle for the next T seconds. Further, the optimal power control policy for such a system was found to be a non-randomized policy, where a node transmits with constant power $P(h)$ every time the channel is in state h (see Section V-A). Here, we will allow the possibility of the channel *being relinquished when bad* with a fixed time overhead $\leq T$. We consider a spatio-temporal fading process with successive transmitter-receiver pairs being selected by a distributed multiaccess contention mechanism. Hence, relinquishing the channel might improve throughput, as successive fade levels might have little correlation. The optimal policy for such a MAC could be a randomized policy. Hence, we will allow a randomized power control, i.e., for a channel state h, the transmitter chooses a power P_h according to some distribution. In a fixed packet size scheme, all data transmissions (with positive rate) carry a fixed amount of data L independent of the channel state h and the power control used. Here as well, we will allow the possibility of a randomized power control and the posibility of relinquishing the channel with a fixed time overhead (when the channel fade is poor).

Optimality Criterion: The *throughput* optimality of a data transmission scheme is established either by comparing the energy required to send a certain amount of bits in a given time or by comparing the amount of bits sent with a given amount of energy in a given time. (We will discuss more about this optimality criterion in Remark D.2). We study a data transmission scheme by considering two data transmissions of positive rates, in some arbitrary channel states with gains h_1 and h_2 and with applied powers P_{h_1} and P_{h_2}. We do not make any assumption on the probabilities of h_1 and h_2, and about the power control policy which yields the powers P_{h_1} and P_{h_2}.

For a given power control scheme (h, P_h), we will then assume that the transmission rate given by Shannon's formula is achieved over the transmission burst; i.e., the transmission rate is given by

$$C_h = W\log(1 + hP_h)$$

We have absorbed the factor $\frac{\alpha}{\sigma^2 d^\eta}$ in to the term h (since d is fixed in this discussion). Hence, the time durations taken to transmit the L bits during the channel states h_1 and h_2 (with the powers P_{h_1} and P_{h_2}) are given by $T_{h_1} := \frac{L}{W\log(1+h_1P_{h_1})}$ and $T_{h_2} := \frac{L}{W\log(1+h_2P_{h_2})}$. Then, the total time occupied by these two transmissions is

$$T_P = \frac{L}{W\log(1 + h_1P_{h_1})} + \frac{L}{W\log(1 + h_2P_{h_2})} \quad (26)$$

spending an amount of energy equal to

$$E_P = \frac{LP_{h_1}}{W\log(1 + h_1P_{h_1})} + \frac{LP_{h_2}}{W\log(1 + h_2P_{h_2})} \quad (27)$$

Define $L_P := 2 \times L$ as the amount of bits sent in time T_P using an energy E_P in channel states h_1 and h_2.

Lemma D.7: Let $h_1 > h_2$. For a fixed packet size scheme, if P_{h_1} and P_{h_2} are applied powers during channel states h_1 and h_2, then having $h_1P_{h_1} \geq h_2P_{h_2}$ is throughput optimal.

Proof: Suppose that $h_1P_{h_1} < h_2P_{h_2}$. Then,

$$\log(1 + h_1P_{h_1}) < \log(1 + h_2P_{h_2})$$

Find power controls $\tilde{P}_{h_1}$ and $\tilde{P}_{h_2}$ such that

$$\begin{aligned} \log(1+h_1P_{h_1}) &= \log(1+h_2\tilde{P}_{h_2}) \quad (28)\\ \log(1+h_2P_{h_2}) &= \log(1+h_1\tilde{P}_{h_1}) \quad (29) \end{aligned}$$

or, equivalently,

$$\begin{aligned} h_1P_{h_1} &= h_2\tilde{P}_{h_2} \quad (30)\\ h_2P_{h_2} &= h_1\tilde{P}_{h_1} \quad (31) \end{aligned}$$

With the power control scheme $(h_1, \tilde{P}_{h_1}), (h_2, \tilde{P}_{h_2})$, the total time occupied in the transmissions of $2 \times L$ bits during the channel states h_1 and h_2 is,

$$\begin{aligned} T_{\tilde{P}} &= \frac{L}{W\log(1+h_1\tilde{P}_{h_1})} + \frac{L}{W\log(1+h_2\tilde{P}_{h_2})}\\ &= T_P \end{aligned}$$

(from (28) and (29)). Now, consider the energy spent to transmit these $2 \times L$ bits, i.e.,

$$E_{\tilde{P}} = \frac{L\tilde{P}_{h_1}}{W\log(1+h_1\tilde{P}_{h_1})} + \frac{L\tilde{P}_{h_2}}{W\log(1+h_2\tilde{P}_{h_2})}$$

Substituting for $\tilde{P}_{h_1}$ and $\tilde{P}_{h_2}$ from (30) and (31), we have,

$$E_{\tilde{P}} = \frac{1}{h_1}\frac{Lh_2P_{h_2}}{W\log(1+h_2P_{h_2})} + \frac{1}{h_2}\frac{Lh_1P_{h_1}}{W\log(1+h_1P_{h_1})}$$

Rearranging the terms, we have,

$$\begin{aligned} E_{\tilde{P}} &= \frac{1}{h_2}\frac{Lh_1P_{h_1}}{W\log(1+h_1P_{h_1})} + \frac{1}{h_1}\frac{Lh_2P_{h_2}}{W\log(1+h_2P_{h_2})}\\ &< \frac{1}{h_1}\frac{Lh_1P_{h_1}}{W\log(1+h_1P_{h_1})} + \frac{1}{h_2}\frac{Lh_2P_{h_2}}{W\log(1+h_2P_{h_2})}\\ &= \frac{LP_{h_1}}{W\log(1+h_1P_{h_1})} + \frac{LP_{h_2}}{W\log(1+h_2P_{h_2})}\\ &= E_P \end{aligned}$$

where the inequality follows from the fact that

$$\begin{aligned} &\frac{Lh_1P_{h_1}}{W\log(1+h_1P_{h_1})}\left(\frac{1}{h_2}-\frac{1}{h_1}\right)\\ &\quad < \frac{Lh_2P_{h_2}}{W\log(1+h_2P_{h_2})}\left(\frac{1}{h_2}-\frac{1}{h_1}\right) \end{aligned}$$

since $h_1 > h_2$ and $h_1P_{h_1} < h_2P_{h_2}$ (by assumption) and the fact that $\frac{x}{\log(1+x)}$ is strictly monotone increasing.

It follows that an optimal power control must have $h_1P_{h_1} \geq h_2P_{h_2}$. ■

Remark: From Lemma D.7, we see that, when $h_1 > h_2$, $C_{h_1} := W\log(1+h_1P_{h_1}) \geq W\log(1+h_2P_{h_2}) =: C_{h_2}$, or equivalently, $T_{h_1} \leq T_{h_2}$. ■

We will now provide a comparison of the fixed packet scheme with a fixed transmission time scheme and show the optimality of the fixed transmission time schemes. The comparison is done under the following assumption.

- The channel has the same marginal fading distribution, whenever sampled by a transmitter, for either schemes. This is a reasonable assumption as we consider spatio-temporal fading, with successive transmissions from possibly different source-destination pairs chosen by the distributed multiaccess contention scheme.

For the fixed packet size scheme, $L_P := 2 \times L$ bits were transmitted in $T_P (= T_{h_1} + T_{h_2})$ time (see (26)) with an amount of energy equal to E_P (see (27)), in two channel samples h_1 and h_2. A reasonable comparison would be to find the throughput of a fixed transmission time scheme for a total duration of T_P seconds involving two data transmissions with channel samples h_1 and h_2 of equal duration $T = \frac{T_P}{2}$ and a total energy of E_P. We will assume that P_{h_1} and P_{h_2}, the power used for the fixed packet size scheme are such that $T_{h_1} \leq T_{h_2}$ (see Lemma D.7). Hence, we have $T_{h_1} \leq T \leq T_{h_2}$, or, the fixed transmission time scheme spends relatively more time on a better channel. Clearly, its throughput is better than the fixed packet size scheme for the same energy constraint, as seen below.

Let $P_{t_{h_1}}$ and $P_{t_{h_2}}$ be the optimal power control for the fixed transmission time strategy such that

$$E_T := P_{t_{h_1}}T + P_{t_{h_2}}T = P_{h_1}T_{h_1} + P_{h_2}T_{h_2} = E_P$$

We have,

$$L_P = 2L = T_{h_1}W\log(1+h_1P_{h_1}) + T_{h_2}W\log(1+h_2P_{h_2})$$

Expanding the left hand side, we have,

$$\begin{aligned} 2L &= T_{h_1}W\log(1+h_1P_{h_1}) + (T_{h_2}-T)W\log(1+h_2P_{h_2})\\ &+ TW\log(1+h_2P_{h_2}) \end{aligned}$$

Using $h_1 > h_2$, we get,

$$\begin{aligned} 2L &\leq T_{h_1}\log(1+h_1P_{h_1}) + (T_{h_2}-T)\log(1+h_1P_{h_2})\\ &+ T\log(1+h_2P_{h_2})\\ &\leq T\log(1+h_1P_{t_{h_1}}) + T\log(1+h_2P_{t_{h_2}})\\ &=: L_T \end{aligned}$$

where the last inequality follows from the fact that $(h_1, P_{t_{h_1}})$ and $(h_2, P_{t_{h_2}})$ is the optimal power control scheme for the fixed transmission time scheme with time $T_P (= 2 \times T)$ and energy $E_T (= E_P)$.

Remarks D.2: For $L(t)$ defined as the amount of bits sent upto time t, and $E(t)$ defined as the total energy spent upto time t, the average throughput (Θ) and the average power ($\bar{P}$) of the system are, in general, defined as

$$\Theta := \liminf_{t\to\infty} \frac{L(t)}{t}$$

$$\bar{P} := \limsup_{t\to\infty} \frac{E(t)}{t}$$

Under additional assumptions on the fading process and the power control scheme used, the expressions are simplified as an ensemble average (for example, see (1) and (3) for a fixed transmission time scheme). In this section, the optimality of the schemes have been shown directly, by comparing the amount of bits transmitted for a particular sample of channel for a given amount of time and energy, or by comparing the amount of energy used to transmit a given amount of bits for a particular sample of channel in a given amount of time. For example, the argument provided here directly translates to an argument with the ensemble average for the discrete fading case. This approach is not only straightforward, but also is very general. ■

In the remaining part of this section, we will establish a property of the optimal solution of a fixed packet size scheme.

Lemma D.8: Let $h_1 > h_2$. For a fixed packet size scheme, if P_{h_1} and P_{h_2} correspond to a optimal power control for channel states h_1 and h_2, then $P_{h_1} \leq P_{h_2}$.

Proof: Suppose that $P_{h_1} > P_{h_2}$. For E_p and T_p defined as before, define $P := \frac{E_p}{T_p}$. Clearly, $P_{h_1} > P > P_{h_2}$. Since P_{h_1} and P_{h_2} correspond to optimal solutions, we require,

$$\sum_{i=1,2} \frac{L}{W\log(1+h_iP_{h_i})} \leq \sum_{i=1,2} \frac{L}{W\log(1+h_iP)}$$

P_{h_1} and P_{h_2} are the optimal throughput solutions of the problem for the given time and energy constraint. When the power allocated to channel state h_1 is increased beyond P_{h_1} (to infinity), then for the same energy constraint, the time required to transmit $2\times L$ bits increases to infinity. Hence, there will exist power controls $P(h_1)$ and $P(h_2)$ ($P(h_1) > P(h_2)$) satisfying the energy constraint with the throughput same as that of P, i.e.,

$$\sum_{i=1,2} \frac{L}{W\log(1+h_iP(h_i))} = \sum_{i=1,2} \frac{L}{W\log(1+h_iP)}$$

Since the two power controls $P(\cdot)$ and P satisfy the energy constraint, we also have,

$$\begin{aligned} P(h_1)\frac{L}{W\log(1+h_1P(h_1))} &+ P(h_2)\frac{L}{W\log(1+h_2P(h_2))} \\ &= P\frac{L}{W\log(1+h_1P)} + P\frac{L}{W\log(1+h_2P)} \end{aligned}$$

Ignoring L and W, we will use the following notation for simplicity.

$$\begin{aligned} x_1 &= \frac{1}{\log(1+h_1P)} \\ x_2 &= \frac{1}{\log(1+h_2P)} \\ y_1 &= \frac{1}{\log(1+h_1P(h_1))} \\ y_2 &= \frac{1}{\log(1+h_2P(h_2))} \end{aligned}$$

Notice that $y_1 < x_1 < x_2 < y_2$. The throughput constraint can be rewritten as, $x_1 + x_2 = y_1 + y_2$. And the energy constraint can be rewritten as,

$$\begin{aligned} &x_1\frac{1}{h_1}(\exp^{\frac{1}{x_1}}-1) + x_2\frac{1}{h_2}(\exp^{\frac{1}{x_2}}-1) \\ &= y_1\frac{1}{h_1}(\exp^{\frac{1}{y_1}}-1) + y_2\frac{1}{h_2}(\exp^{\frac{1}{y_2}}-1) \end{aligned}$$

or,

$$\begin{aligned} &\frac{1}{h_1}[y_1(\exp^{\frac{1}{y_1}}-1) - x_1(\exp^{\frac{1}{x_1}}-1)] \\ &= \frac{1}{h_2}[x_2(\exp^{\frac{1}{x_2}}-1) - y_2(\exp^{\frac{1}{y_2}}-1)] \end{aligned} \tag{32}$$

Define $f := x(\exp^{\frac{1}{x}}-1)$. We know that f is a convex decreasing function. And if we can show that

$$|f'(x_1)| > \frac{h_1}{h_2}|f'(x_2)|$$

then, clearly the equality in the energy constraint (32) cannot hold for $y_1 - x_1 = x_2 - y_2$. Differentiating f, we have,

$$f'(x) = \exp^{\frac{1}{x}}\left(1-\frac{1}{x}\right) - 1$$

Reverting back to the original variables, we have,

$$\begin{aligned} f'(x_1) &= (1+h_1P)(1-\log(1+h_1P)) - 1 \\ f'(x_2) &= (1+h_2P)(1-\log(1+h_2P)) - 1 \end{aligned}$$

Without loss of generality, assume $h_2 = 1$. Hence, we need to show that

$$\left|\frac{f'(x_1)}{f'(x_2)}\right| > h_1$$

or,

$$\frac{(1+h_1P)(1-\log(1+h_1P))-1}{(1+P)(1-\log(1+P))-1} > h_1$$

rewriting, we have,

$$\begin{aligned} 1 - \log(1+h_1P) + h_1P - h_1P\log(1+h_1P) - 1 \\ < h_1[1-\log(1+P)+P-P\log(1+P)-1] \end{aligned}$$

(the change of sign is because the derivative is negative). Simplifying further, we need to show,

$$-\log(1+h_1P)(1+h_1P) < -h_1\log(1+P)(1+P)$$

or,

$$\log(1+h_1P)(1+h_1P) > h_1\log(1+P)(1+P)$$

But the function $(1+p)\log(1+p)$ is a convex increasing function with the derivative greater than 1, implying that the above inequality is infact true. This proves that we cannot find $P(h_1)$ and $P(h_2)$ (satisfying the energy constraint) which has the same throughput as P.

Hence, for an optimal power control policy $P_{h_1} \le P_{h_2}$ when ever $h_1 > h_2$.

■

Corollary .1: For a fixed packet size scheme, if P_h^1 and P_h^2 correspond to an optimal power control policy for a given channel state h, both positive, then $P_h^1 = P_h^2$.

Remark: For the fixed transmission time scheme, the optimal power control has the water pouring form, with more power allocated for a better channel. In contrast, the optimal power control for a fixed packet size scheme suggests more power for a poorer channel, thus leading to a reduction in the efficiency of the system. However, we note that, the rate allocated for a better channel state is always greater than a poorer channel in either case. ■