Fixed Point Analysis of Single Cell IEEE 802.11e WLANs: Uniqueness, Multistability and Throughput Differentiation

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***Abstract*— We consider the vector fixed point equations arising out of the analysis of the saturation throughput of a single cell IEEE 802.11e (EDCA) wireless local area network with nodes that have different back-off parameters, including different Arbitration InterFrame Space (AIFS) values. We consider balanced and unbalanced solutions of the fixed point equations arising in homogeneous and nonhomogeneous networks. We are concerned, in particular, with (i) whether the fixed point is balanced within a class, and (ii) whether the fixed point is unique. Our simulations show that when multiple unbalanced fixed points exist in a homogeneous system then the time behaviour of the system demonstrates severe short term unfairness (or *multistability*). Implications for the use of the fixed point formulation for performance analysis are also discussed. We provide a condition for the fixed point solution to be balanced within a class, and also a condition for uniqueness. We then provide an extension of our general fixed point analysis to capture AIFS based differentiation, including the concept of virtual collision when there are multiple queues per station; again a condition for uniqueness is established. For the case of multiple queues per node, we find that a model with as many nodes as there are queues, with one queue per node, provides an excellent approximation. An asymptotic analysis of the fixed point is provided for the case in which packets are never abandoned, and the number of nodes goes to ∞. Finally the fixed point equations are used to obtain insights into the throughput differentiation provided by different initial back-offs, persistence factors, and AIFS, for finite number of nodes and for differentiation parameter values similar to those in the IEEE 802.11e standard. Simulation results validate the accuracy of the analysis.**

***Index Terms*— Performance of Wireless LANs, Short term Unfairness, QoS in Wireless LANs, EDCA Analysis**

I. Introduction

A new component of the IEEE 802.11e medium access control (MAC) is an enhanced distributed channel access (EDCA), which provides differentiated channel access to packets by allowing different back-off parameters (see [2]). Several traffic classes are supported, the classes being distinguished by different back-off parameters. Thus, whereas in the legacy DCF all nodes have a single queue, and a single back-off "state machine", all with the same back-off parameters (we say that the nodes are *homogeneous*), in EDCA the nodes can have multiple queues with separate back-off state machines with different parameters, and hence are permitted to be *nonhomogeneous*.

This paper is concerned with the saturation throughput analysis of IEEE 802.11e (EDCA) wireless LANs. We consider a single cell network of IEEE 802.11e nodes (single cell meaning that all nodes are within control channel range of each other), with an ideal channel (without capture, fading or frame error) and assume that packets are lost only due to collision of simultaneous transmissions. For ease of understanding, much of our presentation is for the case in which each node has only one EDCA queue (of some access category). The analysis, however, applies to the general case of multiple EDCA queues (of different access categories) per node and we show this in Section VII.

Much work has been reported on the performance evaluation of EDCA to support differentiated service. Most of the analytical work reported has been based on a decoupling approximation proposed initially by Bianchi ([3]). While keeping this basic decoupling approximation, in [1] Kumar et al. presented a significant simplification and generalisation of the analysis of the IEEE 802.11 back-off mechanism. This analysis led to a certain one dimensional fixed point equation for the collision probability experienced by the nodes in a homogeneous system (i.e., one in which all the nodes have the same back-off parameters). In this paper we consider *multidimensional fixed point equations* for a homogeneous system of nodes, and also for a nonhomogeneous system of nodes. The nonhomogeneity arises due to different initial back-offs, or different back-off multipliers, or different amounts of time that nodes wait after a transmission before restarting their back-off counters (i.e., the AIFS (Arbitration InterFrame Space) mechanism of IEEE 802.11e), or different number of access categories per node.

Our approach in this paper builds upon the one provided in [1]. The main contributions of this paper are the following:

1) We provide examples of homogeneous systems in which, even though a unique balanced fixed point exists (i.e., a solution in which all the coordinates are equal), there can be multiple unbalanced fixed points, thus suggesting *multistability*. We demonstrate by simulation that, in such cases, significant short term unfairness can be observed and the unique balanced fixed point fails to capture the system performance.
2) Next, in the case where the back-off increases multiplicatively (as in IEEE 802.11), we establish a simple sufficient condition for the uniqueness of the solution of the multidimensional fixed point equation in the homogeneous and the nonhomogeneous cases.
3) We perform an analytical study of the throughput differentiation provided by the different back-off mechanisms, b_0, p and AIFS. We do an asymptotic analysis of the service differentiation (with the number of nodes tending to infinity), and also obtain approximate results for a finite number of nodes.

A survey of the literature: There has been much research activity on modeling the performance of IEEE 802.11 and

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in particular of IEEE 802.11e medium access standards. The general approach has been to extend the decoupling approximation introduced by Bianchi ([3]). Without modeling the AIFS mechanism, the extension is straightforward. Only the initial back-off, and the back-off multiplier (*persistence factor*) are modeled. In [4], [5] and [6], such a scheme is studied by extending Bianchi's Markov model per traffic class. In this paper, in Section III, we will provide a generalisation and simplification of this approach. We will then provide examples where nonunique fixed points can exist, demonstrate the consequences of such nonuniqueness, and also conditions that guarantee uniqueness.

The AIFS technique is a further enhancement in IEEE 802.11e that provides a sort of priority to nodes that have smaller values of AIFS. After any successful transmission, whereas high priority nodes (with AIFS = DIFS) wait only for DIFS (DCF Interframe Space) to resume counting down their back-off counters, low priority nodes (with AIFS $>$ DIFS) defer the initiation of countdown for an additional AIFS$-$DIFS slots. Thus a high priority node decrements its back-off counter earlier than a low priority node and also has fewer collisions.

Among the approaches that have been proposed for modeling the AIFS mechanism (for example, [7], [8], [9], [10], [11], [12], [23] and [13]) the ones in [12], [23] and [13] come much closer to capturing the service differentiation provided by the AIFS feature. In [12] the authors propose a Markov model to capture both the different back-off window expansion approach and AIFS. AIFS is modeled by expanding the state-space of the Markov chain to include the number of slots elapsed since the previous transmission attempt on the channel. In [13] the authors observe that the system exists in states in which only nodes of certain access categories can attempt. The approach is to model the evolution of these states as a Markov chain. The transition probabilities of this Markov chain are obtained from the assumed, decoupled attempt probabilities. This approach yields a fixed point formulation. This is the approach we will discuss in Section VI. [23] uses a Markov chain on the number of slots elapsed from the previous transmission to model AIFS based service differentiation. [11] extends the Bianchi's analysis to multiple traffic classes per node case using the Markov chain approach.

We note that the analyses in [12] and [13] are based on Bianchi's approach to modeling the residual back-off by a Markov chain. In this paper, we have extended the simplification reported in [1] (which was for a homogeneous system of nodes) to nonhomogeneous nodes with different back-off parameters and AIFS based priority schemes. Also, we model the case of multiple queues (of different access categories) per node (see [11]). Thus, in our work, we have provided a simplified and integrated model to capture all the essential backoff based service differentiation mechanisms of IEEE 802.11e.

In the previous literature, it is assumed that the collision rate experienced by a queue of any access category is constant over time. There appears to have been no attempt to study the phenomenon of short term unfairness in the fixed point framework. A related work on Ethernet ([25]) identifies short-term unfairness in the system by experimentation and simulation, and suggests modifications to the protocol to eliminate it. Also, all the existing work assumes that the collision probabilities of all the queues with identical access parameters are the same. Thus there appears to have been no earlier work on studying the possibility of unbalanced solutions of the fixed point equations. In addition, the possibility of nonuniqueness of the solution of the fixed point equations arising in the analyses seems to have been missed in the earlier literature. In our work, we study the fixed point equations for IEEE 802.11e networks and take into account all these possibilities.

Outline of the paper: In Section II we review the generalised back-off model that was first presented in [1]. In Section III we develop the multidimensional fixed point equations for the homogeneous and nonhomogeneous cases (without AIFS), and obtain the necessary and sufficient conditions satisfied by the solutions to the fixed point equations. We provide examples in Section IV to show that even in the homogeneous case there can exist multiple unbalanced fixed points and show the consequence of this. In Section V-A, we analyse the fixed point equations for a homogeneous system of nodes and obtain a condition for the existence of only one fixed point. In Sections V-B and VI, we extend the analysis to nonhomogeneous system of nodes, with different back-off parameters (including AIFS). In Section VII we analyse the case of multiple EDCA queues per node. An analytical study of the service differentiation provided by the various access parameters is done in Section VIII. In Section IX, we provide numerical results verifying the validity of the analyses. Section X concludes the paper and discusses future work.

II. The Generalised Back-Off Model

There are n nodes, indexed by $i, 1 \leq i \leq n$. We begin with considering the case in which each node has one EDCA queue. We adopt the notation in [1], whose authors consider a generalisation of the back-off behaviour of the nodes, and define the following back-off parameters (for node i)

$K_i :=$ At the $(K_i + 1)$th attempt either the packet being attempted by node i succeeds or is discarded

$b_{i,k} :=$ The *mean* back-off (in slots) at the kth attempt for a packet being attempted by node i, $0 \leq k \leq K_i$

Definition 2.1: A system of n nodes is said to be **homogeneous**, if all the back-off parameters of the nodes, like, K_i, $b_{i,k}, 0 \leq k \leq K_i$ are the same for all $i, 1 \leq i \leq n$. A system of nodes is called **nonhomogeneous** if the back-off parameters of the nodes are not identical. ■

Remark: IEEE 802.11e permits different back-off parameters to differentiate channel access obtained by the nodes in an attempt to provide QoS. The above definitions capture the possibility of having different CW_{min} and CW_{max} values, different exponential back-off multiplier values and even different number of permitted attempts. For ease of discussion and understanding, we will postpone the topic of AIFS until Section VI. Hence in the discussions up to Section V-B, all the nodes wait only for a DIFS after a busy channel. ■

It has been shown in [1] (and later in [22]) that under the decoupling assumption, introduced by Bianchi in [3], the attempt probability of node i (in a back-off slot, and conditioned on being in back-off) for given collision probability γ_i is given by,

$$G_i(\gamma_i) := \frac{1 + \gamma_i + \cdots + \gamma_i^{K_i}}{b_{i,0} + \gamma_i b_{i,1} + \cdots + \gamma_i^{K_i} b_{i,K_i}} \tag{1}$$

Remarks 2.1:

1) We will assume that $b_{i,\cdot}$ are such that $0 \le G_i(\gamma_i) \le 1$ for all $\gamma_i, 0 \le \gamma_i \le 1$ and $G_i(\gamma_i) < 1$ whenever $\gamma_i > 0$.
2) When the system is homogeneous then we will drop the subscript i from $G_i(\cdot)$, and write the function simply as $G(\cdot)$.

III. The Fixed Point Equation

It is important to note that in the present discussion all rates are conditioned on being in the back-off periods; i.e., we have eliminated all durations other than those in which nodes are counting down their back-off counters, in order to obtain the collision probability γ_i of player i and its attempt probability β_i $(= G_i(\gamma_i))$. Later one brings back the channel activity periods in order to compute the throughput in terms of the attempt probabilities (see [1]). Now consider a nonhomogeneous system of n nodes. Let $\boldsymbol{\gamma}$ be the vector of collision probabilities of the nodes. With the slotted model for the back-off process and the decoupling assumption, the natural mapping of the attempt probabilities of other nodes to the collision probability of a node is given by

$$\gamma_i = \Gamma_i(\beta_1, \beta_2, \ldots, \beta_n) = 1 - \prod_{j=1, j \ne i}^{n} (1 - \beta_j)$$

where $\beta_j = G_j(\gamma_j)$. We could now expect that the equilibrium behaviour of the system will be characterised by the solutions of the following system of equations. For $1 \le i \le n$,

$$\gamma_i = \Gamma_i(G_1(\gamma_1), \cdots, G_n(\gamma_n))$$

We write these n equations compactly in the form of the following multidimensional fixed point equation.

$$\boldsymbol{\gamma} = \boldsymbol{\Gamma}(\mathbf{G}(\boldsymbol{\gamma})) \tag{2}$$

Since $\boldsymbol{\Gamma}(\mathbf{G}(\boldsymbol{\gamma}))$ is a composition of continuous functions it is continuous. We thus have a continuous mapping from $[0,1]^n$ to $[0,1]^n$. Hence by Brouwer's fixed point theorem there exists a fixed point in $[0,1]^n$ for the equation $\boldsymbol{\gamma} = \boldsymbol{\Gamma}(\mathbf{G}(\boldsymbol{\gamma}))$.

Consider the i^{th} component of the fixed point equation, i.e.,

$$\gamma_i = 1 - \prod_{1 \le j \le n, j \ne i} (1 - G_j(\gamma_j))$$

or equivalently,

$$(1 - \gamma_i) = \prod_{1 \le j \le n, j \ne i} (1 - G_j(\gamma_j))$$

Multiplying both sides by $(1 - G_i(\gamma_i))$, we get,

$$(1 - \gamma_i)(1 - G_i(\gamma_i)) = \prod_{1 \le j \le n} (1 - G_j(\gamma_j))$$

Thus a *necessary and sufficient condition* for a vector of collision probabilities $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_n)$ to be a fixed point solution is that, for all $1 \le i \le n$,

$$(1 - \gamma_i)(1 - G_i(\gamma_i)) = \prod_{j=1}^{n} (1 - G_j(\gamma_j)) \tag{3}$$

where the right-hand side is seen to be independent of i.

Define $F_i(\gamma) := (1 - \gamma)(1 - G_i(\gamma))$. From Equation 3 we see that if $\boldsymbol{\gamma}$ is a solution of Equation 2, then for all $i, j, 1 \le i, j \le n$,

$$F_i(\gamma_i) = F_j(\gamma_j) \tag{4}$$

Notice that this is only a *necessary condition*. For example, in a homogeneous system of nodes, the vector $\boldsymbol{\gamma}$ such that $\gamma_i = \gamma$ for all $1 \le i \le n$, satisfies Equation 4 for any $0 \le \gamma \le 1$, but not all such points are solutions of the fixed point Equation 2.

Definition 3.1: We say that a fixed point $\boldsymbol{\gamma}$ (i.e., a solution of $\boldsymbol{\gamma} = \boldsymbol{\Gamma}(\mathbf{G}(\boldsymbol{\gamma}))$) is a **balanced** fixed point if $\gamma_i = \gamma_j$ for all $1 \le i, j \le n$; otherwise, $\boldsymbol{\gamma}$ is said to be an **unbalanced fixed point**. ■

Remarks 3.1:

1) It is clear that if there exists an unbalanced fixed point for a homogeneous system, then every permutation is also a fixed point and hence, in such cases, we do not have a unique fixed point.
2) In the homogeneous case, by symmetry, the average collision probability must be the same for every node. If the collision probabilities correspond to a fixed point (see 3, next), then this fixed point will be of the form $(\gamma, \gamma, \cdots, \gamma)$ where γ solves $\gamma = \Gamma(G(\gamma))$ (since $\Gamma_i(\cdot) = \Gamma(\cdot)$ and $G_i(\cdot) = G(\cdot)$ for all $1 \le i \le n$). Such a fixed point of $\gamma = \Gamma(G(\gamma))$ is guaranteed by Brouwer's Fixed Point. The uniqueness of such a balanced fixed point was studied in [1]. We reproduce this result in Theorem 5.1.
3) There is, however, the possibility that even in the homogeneous case, there is an unbalanced solution of $\boldsymbol{\gamma} = \boldsymbol{\Gamma}(\mathbf{G}(\boldsymbol{\gamma}))$. By simulation examples we observe in Section IV that when there exist unbalanced fixed points, the balanced fixed point of the system does not characterise the average performance, even if there exists only one balanced fixed point. In Section V-A, we provide a condition for homogeneous IEEE 802.11 and IEEE 802.11e type nodes (with exponential back-off) under which there is a unique balanced fixed point and no unbalanced fixed point. In such cases, it is now well established, that the unique balanced fixed point accurately predicts the saturation throughput of the system.
4) For the homogeneous case the back-off process can be exactly modeled by a positive recurrent Markov chain (see [1]). Hence the attempt and collision processes will be ergodic and, by symmetry, the nodes will have equal attempt and collision probabilities. In such a situation the existence of multiple unbalanced fixed points will suggest short term unfairness or multistability. We will observe this phenomenon in Section IV.

5) Consider a system of homogeneous nodes having unbalanced solutions for the fixed point equation $\boldsymbol{\gamma} = \boldsymbol{\Gamma}(\mathbf{G}(\boldsymbol{\gamma}))$ (i.e., there exists i, j such that $\gamma_i \neq \gamma_j$), then from Equation 4, we see that $F(\gamma_i) = F(\gamma_j)$, or the function F is many-to-one. Hence for a homogeneous system of nodes, if the function F is one-to-one then there cannot exist unbalanced fixed points. In Section V-B we use this observation to obtain a sufficient condition for the uniqueness of the fixed point in the nonhomogeneous case.

IV. Nonunique Fixed Points and MultiStability: Simulation Examples

A. Example 1

Consider a homogeneous system (let us call it System-I) with $n = 10$ nodes. The function $G(\cdot)$ of the nodes is given by,

$$G(\gamma) = \frac{1 + \gamma + \gamma^2 + \gamma^3 + \ldots}{1 + \gamma + \gamma^2 + \gamma^3 + 64(\gamma^4 + \gamma^5 + \ldots)}$$

The system corresponds to the case where $K = \infty$, $b_0 = b_1 = b_2 = b_3 = 1$ and $b_4 = b_5 = b_6 = \ldots = 64$ (b_i are distributed uniformly over the integers in $[1, CW_i]$ for appropriate CW_i). From the form of function $G(\cdot)$, we can see that a node which is currently at back-off stage 0 is more likely to remain at that stage as it *takes* 4 *successive collisions* to make the attempt rate of the node < 1. Likewise, a node that is in the larger back-off stages $b_4 = b_5 = \cdots = 64$, will retry continuously with mean inter-attempt slots of 64 until it succeeds. Observe that only one node can be at back-off stage 0 at any time. This leads to the apparent multistability of the system.

Figure 1 plots $G(\gamma)$, the corresponding $F(\gamma) = (1-\gamma)(1-G(\gamma))$ and shows the balanced fixed point of the system for $n = 10$ nodes. The balanced fixed point of the system shown in the figure is obtained using the fixed point equation $\gamma = 1 - (1 - G(\gamma))^9$. Observe that the function $F(\cdot)$ is not one-to-one (the function $F(\cdot)$ not being one-to-one does not imply that there exist multiple fixed point solutions; see Remarks 3.1, 5).

Figure 2 shows the existence of unbalanced fixed points for System-I. These fixed points are obtained as follows. Assume that we are interested in fixed points such that $\gamma_1 \neq \gamma_2 = \cdots = \gamma_n$. Given $\gamma_2 = \cdots = \gamma_n$, the attempt probability of the nodes $2, \cdots, n$ is given by $G(\gamma_2)$. Hence, the collision probability of node 1 is given by $\gamma_1 = 1 - (1 - G(\gamma_2))^{n-1}$. The attempt probability of node 1 would then be $G(\gamma_1)$. Using the decoupling assumption, the collision probability of any of the other $n-1$ nodes would then be, $1-(1-G(\gamma_2))^{n-2}(1-G(\gamma_1)) = \gamma_2$. Thus we obtain a fixed point equation for γ_2 (and hence for all the other $\gamma_j, 3 \leq j \leq n$). In Figure 2 we plot $1 - (1 - G(\gamma))^8(1 - G(1 - (1 - G(\gamma))^9))$ (plotted as the line marked with dots), the intersection of which with the "y=x" line shows the solutions for $\gamma_2 (= \cdots = \gamma_n)$. In the same way, we obtain the fixed point equation for γ_1 by eliminating $\gamma_2, \cdots, \gamma_n$ from the multidimensional system of equations. This functiton is plotted in in Figure 2 using pluses

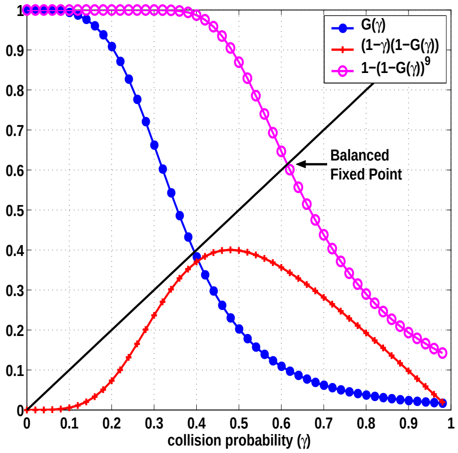

Fig. 1. Example System-I: The balanced fixed point. Plots of $G(\gamma)$, $F(\gamma) = (1-\gamma)(1-G(\gamma))$ and $1-(1-G(\gamma))^9$ vs. the collision probability γ; we also show the "y=x" line.

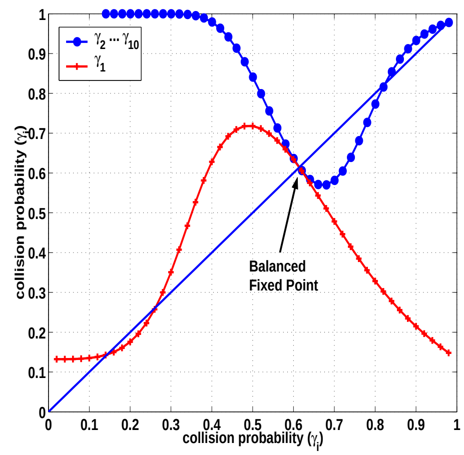

Fig. 2. Example System-I: Demonstration of unbalanced fixed points. Plots of $\gamma_2 = 1-(1-G(\gamma))^8(1-G(1-(1-G(\gamma))^9))$ (the curve drawn with dots and lines) and the function for the fixed point equation for γ_1 (see text) using pluses and lines.

and lines and the intersection of this curve with the "y=x" line shows the corresponding solutions for γ_1. We see that there are three solutions in each case. The smallest values of γ_1 (approx. 0.14) pairs up with the largest value of $\gamma_2 = \cdots = \gamma_n$ (approx. 0.97). Notice that the balanced fixed point of the system is also a fixed point in the plot (compare with Figure 1). Then there is one remaining unbalanced fixed point whose values can be read off the plot. We note that there could exist many other unbalanced fixed points for this system of equations, as we have considered only a particular variety of fixed points that have the property that $\gamma_1 \neq \gamma_2 = \cdots = \gamma_n$.

In order to examine the consequences of multiple unbalanced fixed points we simulated the back-off process with the back-off parameters of System-I. The following remarks summarise our simulation approach in this paper.

Remarks 4.1 (On the Simulation Approach used):

1) We have developed an event-driven simulator written in the "C" language based on the coupled multidimensional backoff process of the various nodes, to compare with

the analytical results. In this simulator, we do not simulate the detailed wireless LAN system (as is done in an ns-2 simulator), but only the backoff slots. We will refer to this as the CMP (Coupled Markov Process) simulator. The main aim of the CMP simulator is to understand the backoff behaviour of the nodes and its dependence on the different backoff parameters. From the point of view of performance analysis, it may also be noted that once the back-off behaviour is correctly modelled the channel activity can easily be added analytically, and thus throughput results can be obtained (see [3] and [1]). Note that, for IEEE 802.11 type networks, a good match between analysis that uses a decoupled Markov model of the back-off process and ns-2 simulations has already been reported in earlier works (see the literature survey in Section I). In addition, for some cases, ns-2 simulations have also been provided in comparison with the CMP simulator and the analytical results.

2) Thus our simulation is programmed as follows. The system evolves over back-off slots. All the nodes are assumed to be in perfect slot synchronisation. The actual coupled evolution of the back-off process is modeled. The back-off distribution is uniform and the residual back-off time is the state for each node. At every slot, depending on the state of the back-off process, there are three possibilities: the slot is idle, there is a successful transmssion, or there is a collision. This causes further evolution of the back-off process.
3) Our simulation approach, which we primarily use to study the back-off behaviour of the nodes, takes few seconds to complete a simulation run, in comparison with the *ns2* simulations which takes any time between few minutes to an hour depending on the number of nodes in the system. The coupled back-off evolution approach we use captures all the essential features of a single cell system with ideal channel (no capture, fading or frame error) and where there is perfect synchronisation among the nodes (which is typical for single cell systems). The simulation provides the attempt rates and collision probabilities directly, which can be used with the throughput formula provided in [1] to obtain the throughput of the nodes.
4) In all our simulations, b_i are distributed uniformly over the integers in $[1, CW_i]$ for appropriate CW_i. We note here that the backoff behaviour of IEEE 802.11e EDCA with the backoff range $[0, CW]$ can be modeled in the same way as IEEE 802.11 DCF with the backoff range $[1, CW + 1]$ and the value of AIFS reduced by 1 (see [13], [21]). Figure 3 shows this equivalence. Thus, the "0 sampling problem" found in IEEE 802.11 DCF is not observed in IEEE 802.11e EDCA.
5) In Figures 4, 7 and 9, for the purpose of reporting the short term unfairness results, the entire duration of simulation is divided into k frames, where the size of each frame is 10,000 slots. The short-term average of the collision probability of each node $j, 1 \leq j \leq n$, is calculated as $\frac{C_j(i)}{A_j(i)}$ where $C_j(i)$ and $A_j(i)$ correspond to the number of collisions and attempts in frame $i, 1 \leq i \leq k$, for node j. The long-term average is similarly calculated as $\frac{1}{n}\sum_{j=1}^{n}\frac{\sum_{i=1}^{k}C_j(i)}{\sum_{i=1}^{k}A_j(i)}$ where n is the number of nodes. Notice that the long-term average collision rate is a batch biased average of the short-term collision rates. Hence, when looking at the graphs, it will be incorrect to visually average the short-term collision rate plots in an attempt to obtain the long-term average collision rate. This is because when a node is shown to have a low collision probability, it is the one that is attempting every slot (while the other nodes attempt with a mean gap of 64 slots), and hence it sees a low probability of collision. In this case $A_j(\cdot)$ is large and $C_j(\cdot) \ll A_j(\cdot)$. On the other hand, when a node is shown to have a high collision probability it is attempting at an average rate of $\frac{1}{64}$ and almost all its attempts collide with the node that is then attempting in every slot. In this case $A_j(\cdot)$ is small and $C_j(\cdot) \approx 1$. Thus, in obtaining the overall average, it is essential to account for the large variation in $A_j(\cdot)$ between the two cases. ■

In Figure 4 we plot a (simulation) snap shot of the short term average collision probability of 2 of the 10 nodes of System-I and the average collision probability of the nodes (The average is calculated over all frames and all nodes. Since the nodes are identical, the average collision probability is the same for all the nodes). Observe that the short term average has a huge variance around the long term average. It is evident that over 1000's of slots one node or the other monopolises the channel (and the remaining nodes see a collision probability of 1 during those slots). This could be described as multistability. A look into the fairness index (see Figure 10) plotted as a function of the frame size used to calculate throughput suggests that System-I exhibits significant unfairness in service even over reasonably large time intervals.

Implication for the use of the balanced fixed point: Notice also that the average collision rate shown in Figure 4 is about 0.25, whereas the balanced fixed point shown in Figure 1 shows a collision probability of about 0.62. *Hence we see that in this case, where there are multiple fixed points, the balanced fixed point does not capture the actual system performance.*

B. Example 2

Let us now consider yet another homogeneous example (let us call it System-II) with $n = 20$ nodes. The function $G(\cdot)$ of the nodes is given by,

$$G(\gamma) = \frac{1 + \gamma + \gamma^2 + \cdots + \gamma^7}{1 + 3\gamma + 9\gamma^2 + 27\gamma^3 + \cdots + 2187\gamma^7}$$

The system corresponds to the case where $K = 7$, $b_0 = 1$, $p = 3$ and $b_k = p^k b_0$ for all $0 \leq k \leq K$. (b_i are uniformly distributed in [1, CW_i] for appropriate CW_i) We notice that in this example the way the back-off expands is similar to the way it expands in the IEEE 802.11 standard, except that the initial back-off is very small (1 slot) and the multiplier is 3, rather than 2. Figure 5 plots $G(\gamma)$, the corresponding $F(\gamma) = (1 - \gamma)(1 - G(\gamma))$ and the balanced fixed point of the system for $n = 20$ nodes. The balanced fixed point of the

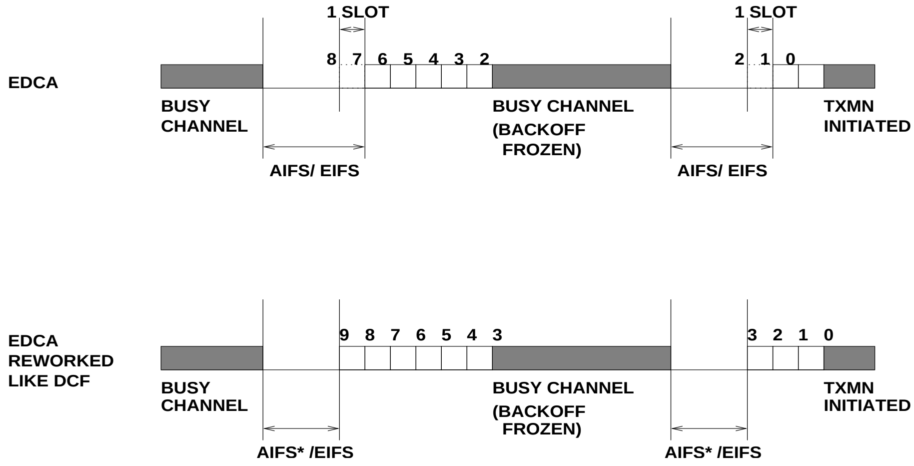

Fig. 3. IEEE 802.11e EDCA backoff compared with a "DCF like" scheme with $AIFS^* := AIFS - aSlotTime$ and $BC^* := BC + 1$, where $BC = 8$ is the residual backoff counter value. We note that because of the way IEEE 802.11e EDCA decrements backoff, [0,CW] is actually equivalent to [1, CW+1] (for AIFS appropriately defined).

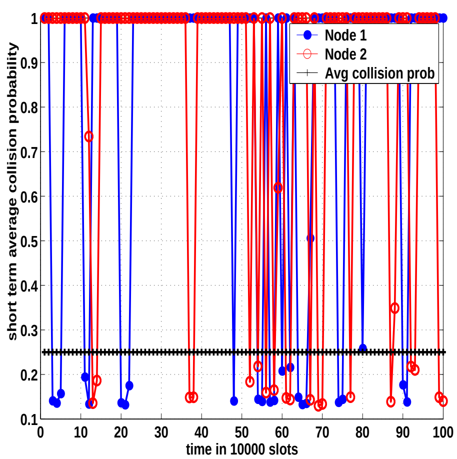

Fig. 4. Example System-I: Snap-shot of short term average collision probability of 2 of the 10 nodes. Also plotted is the average collision probability of the nodes (averaged *over all frames and nodes*). The 95% confidence interval for the average collision probability lies within 0.7% of the mean value.

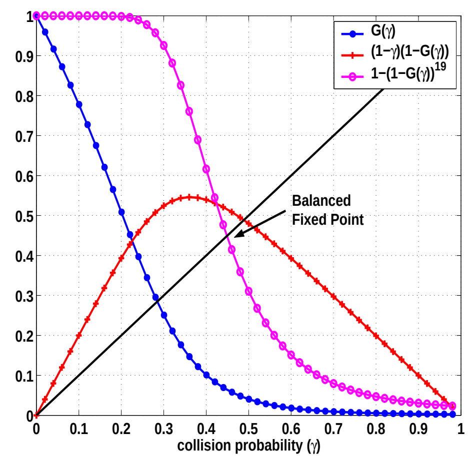

Fig. 5. Example System-II: The balanced fixed point. Plots of $G(\gamma)$, $F(\gamma) = (1-\gamma)(1-G(\gamma))$ and $1-(1-G(\gamma))^{19}$ vs. the collision probability γ; the line "y=x" is also shown. Notice that the function F is not one-to-one.

system shown in the figure is obtained using the fixed point equation $\gamma = 1 - (1 - G(\gamma))^{19}$.

As in the case of System-I, Figure 6 shows the existence of multiple unbalanced fixed points for System-II. The fixed points we have shown correspond to the case where $\gamma_1 \neq \gamma_2 = \cdots = \gamma_n$ and are obtained just as discussed for System-I.

Figure 7 plots a snap shot of the short term average collision probability (from simulation) of 2 of the 20 nodes and the average collision probability of the nodes (same for all the nodes). Observe that the short term averages vary a lot as compared to the long term average, suggesting multistability. Again, as in the case of System-I, comparing the average collision probability with the balanced fixed point of the system in Figure 5, we see that the fixed point does not capture the actual system performance.

Discussion of Examples 1 and 2: From the simulation examples, we can make the following inferences.

1) When there are multiple unbalanced fixed points in

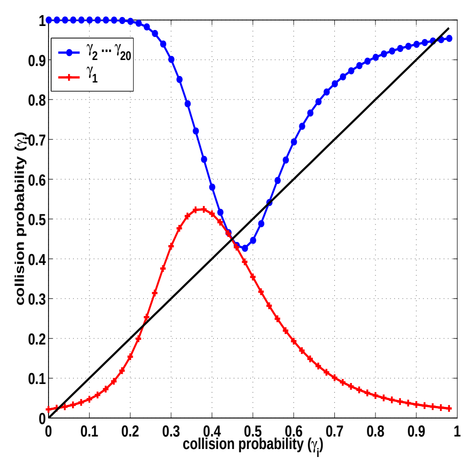

Fig. 6. Example System-II: Demonstration of unbalanced fixed points. Plots of $\gamma_2 = 1 - (1-G(\gamma))^{18}(1 - G(1-(1-G(\gamma))^{19}))$ (the curve drawn with dots and lines) and the function for the fixed point equation for γ_1 (see text) using pluses and lines.

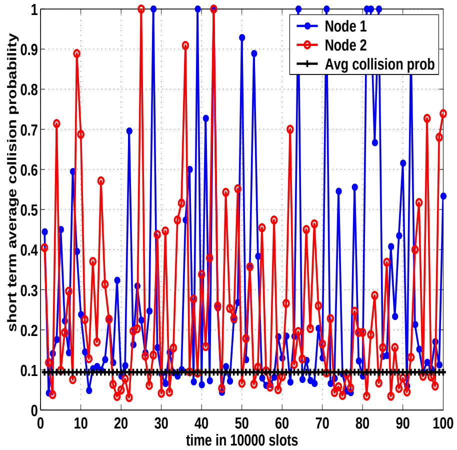

Fig. 7. Example System-II: Snap-shot of short term average collision probability of 2 of the 20 nodes. The average collision probability is also plotted in the figure (averaged *over all slots and nodes*). The 95% confidence interval for the average collision rate lies within 0.7% of the mean value.

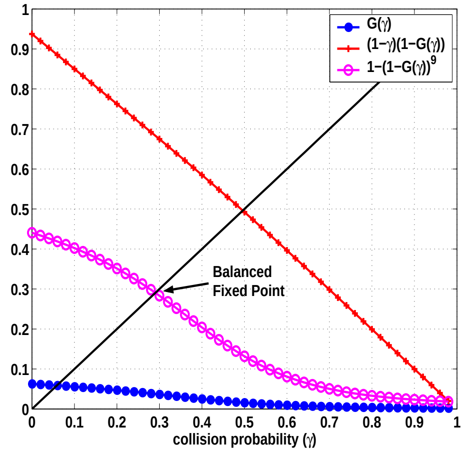

Fig. 8. Example System-III: Plots of $G(\gamma)$, $F(\gamma) = (1-\gamma)(1-G(\gamma))$ and $1-(1-G(\gamma))^9$ vs. the collision probability γ; the line "y=x" is also shown.

a homogeneous system then the system can display multistability, which manifests itself as significant short term unfairness in channel access.

2) When there are multiple unbalanced fixed points in a homogeneous system then the collision probability obtained from the balanced fixed point may be a poor approximation to the long term average collision probability.

Similar conclusions can be drawn for nonhomogeneous systems when the system of fixed point equations have multiple solutions. ■

It appears that the existence of multiple-fixed points is a consequence of the form of the $G(\cdot)$ function in the above examples, where $G(\cdot)$ is similar to a switching curve; see, for example, Figure 1 where there is a very high attempt probability at low collision probabilities and a very low attempt probability at high collision probabilities.

C. Example 3

Consider a homogeneous system in which back-off increases multiplicatively as in IEEE 802.11 DCF (let us call it System-III), with $n = 10$ nodes. The function $G(\cdot)$ is given by,

$$G(\gamma) = \frac{1+\gamma+\gamma^2+\ldots+\gamma^7}{16+32\gamma+64\gamma^2+\ldots+2048\gamma^7}$$

The system corresponds to the case where $K = 7$, $p = 2$ and $b_0 = 16$ and $b_k = p^k b_0$ for all $0 \leq k \leq K$ (b_i are uniformly distributed in $[1, CW_i]$ for appropriate CW_i). These parameters are similar to those used in the IEEE 802.11 standard. Figure 8 plots $G(\cdot)$, the corresponding $F(\gamma) = (1-\gamma)(1-G(\gamma))$ and the unique balanced fixed point of the system. (Notice that F is one-to-one and uniqueness of the fixed point will be proved in Section V-A.) The balanced fixed point of the system is obtained using the fixed point equation $\gamma = 1-(1-G(\gamma))^9$. The balanced fixed point yields a collision probability of approximately 0.29.

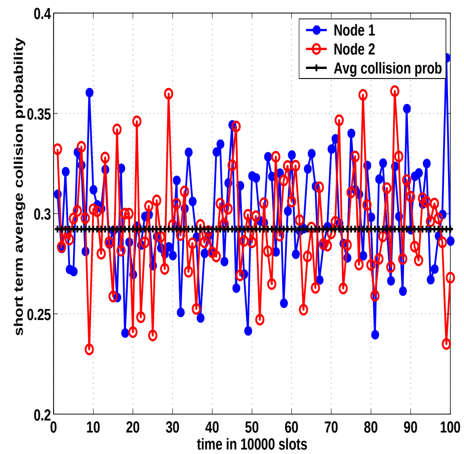

Fig. 9. Example System-III: Snap-shot of short term average collision probability of 2 of the 10 nodes. Also plotted is the average collision probabilty obtained by the nodes. The 95% confidence interval of the average collision rate lies within 0.2% of the mean value.

Figure 9 plots a snap shot of the short term average collision probability (from simulation) of 2 of the 10 nodes and the average collision probability of the nodes of the Example System-III. Notice that the short term average collision rate is close to the average collision rate (the vertical scale in this figure is much finer than in the corresponding figures for System-I and System-II). Also, the average collision rate matches well with the balanced fixed point solution obtained in Figure 8.

Remark: Thus we see that in a situation in which there is a unique fixed point not only is there lack of multistability, but also the fixed point solution yields a good approximation to the long run average behaviour. ■

D. Short Term Fairness in Examples 1, 2, 3

Figure 10 plots the throughput fairness index $\frac{1}{n}\frac{(\sum_{i=1}^{n}\tau_i)^2}{\sum_{i=1}^{n}\tau_i^2}$ (where τ_i is the average throughput of node i over the measurement frame, see [18]) against the frame size used to measure throughput. The fairness index is obtained for each frame and

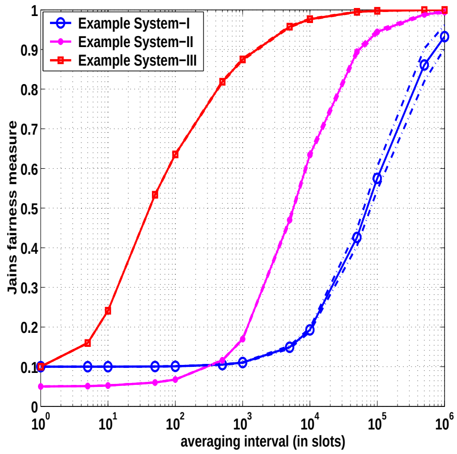

Fig. 10. Throughput fairness index is plotted against the number of slots used to measure throughput. The dotted lines mark the 95% confidence interval.

is averaged over the duration of the simulation. Also plotted in the figure is the 95% confidence interval. We note that values of this index will lie in the interval $[0, 1]$, and smaller values of the index correspond to greater unfairness between the nodes. The performance of all the three example systems are compared. Notice that Example System-III (similar to IEEE 802.11 DCF) has the best fairness properties. The system achieves fairness of 0.9 over 1000's of slots. However, for Example System-I and II, similar performance is achieved only over 1,000,000 and 100,000 slots. The unfairness of Example Systems-I and II can be attributed to their apparent multistability.

In Section V we establish conditions for the uniqueness of the solutions to the multidimensional fixed point equation.

V. Analysis of the Fixed Point

A. The Homogeneous Case

The following two results are adopted from [1].

Lemma 5.1: $G(\gamma)$ is nonincreasing in γ if $b_k, k \geq 0$, is a nondecreasing sequence. In that case, unless $b_k = b_0$ for all k, $G(\gamma)$ is strictly decreasing in γ. ■

Theorem 5.1: For a homogeneous system of nodes, $\Gamma(G(\gamma)) : [0, 1] \to [0, 1]$, has a unique fixed point if $b_k, k \geq 0$, is a nondecreasing sequence. ■

Remark: The fixed point $(\gamma, \gamma, \cdots, \gamma)$ is the unique balanced fixed point for $\boldsymbol{\gamma} = \boldsymbol{\Gamma}(\mathbf{G}(\boldsymbol{\gamma}))$. From Equation 4, we see that a *necessary* condition for the existence of unbalanced fixed points in a homogeneous system of nodes is that the function $F(\gamma) = (1 - \gamma)(1 - G(\gamma))$ needs to be many-to-one. In other words, if the function $(1 - \gamma)(1 - G(\gamma))$ is one-to-one and if $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a solution of the system $\boldsymbol{\gamma} = \boldsymbol{\Gamma}(\mathbf{G}(\boldsymbol{\gamma}))$, then $\gamma_i = \gamma_j$ for all i, j. ■

Consider the exponentially increasing back-off case for which $G(\cdot)$ is given by,

$$G(\gamma) = \frac{1 + \gamma + \gamma^2 + \ldots + \gamma^K}{b_0(1 + p\gamma + p^2\gamma^2 + \ldots + p^K\gamma^K)} \quad (5)$$

Clearly, $G(\gamma)$ is a continuously differentiable function and so is $F(\gamma) = (1-\gamma)(1-G(\gamma))$. The following simple lemma is a consequence of the mean value theorem.

Lemma 5.2: $F(\gamma)$ is one-to-one in $0 \leq \gamma \leq 1$ *if* $F^{'}(\gamma) \neq 0$ for all $0 \leq \gamma \leq 1$. ■

Remarks 5.1:
When $F(\cdot)$ is one-to-one in $0 \leq \gamma \leq 1$ and $G(\cdot)$ is such that $0 \leq G(\gamma) \leq 1$ for all $0 \leq \gamma \leq 1$, the following hold
(i) $F(\gamma) = 0$ iff $\gamma = 1$,
(ii) $F(0) > 0$, and
(iii) $F(\gamma)$ is a decreasing function of γ. ■
Now the derivative of F is

$$F^{'}(\gamma) = -1 + G(\gamma) - G^{'}(\gamma)(1 - \gamma)$$

Lemma 5.3: If $K \geq 1, p \geq 2$ and $G(\cdot)$ is as in Equation 5, then $G^{'}(\gamma) < 0$ and $|G^{'}(\gamma)| \leq \frac{2p}{b_0}$ for all $0 \leq \gamma \leq 1$. ■

Clearly, $G(\gamma) \leq \frac{1}{b_0}$ and $1 \geq (1 - \gamma) \geq 0$ for all $0 \leq \gamma \leq 1$. Substituting into the expression for $F^{'}(\gamma)$, we get,

$$F^{'}(\gamma) \leq -1 + \frac{1 + 2p}{b_0}$$

Thus, if in addition to the other condition in Lemma 5.3, if $b_0 > 1 + 2p$, then $F^{'}(\gamma) < 0$ and the following result holds by virtue of the remark following Theorem 5.1.

Theorem 5.2: For a function $G(\cdot)$ defined as in Equation 5 if $K \geq 1, p \geq 2$ and $b_0 > 2p + 1$, then the system $\boldsymbol{\gamma} = \boldsymbol{\Gamma}(\mathbf{G}(\boldsymbol{\gamma}))$ has a unique fixed point which is balanced. ■

Remark: It can be shown that if Lemma 5.3 holds for $G(\cdot)$ as in Equation 5 it also holds for any case in which $b_k = p^k b_0$ for $0 \leq k \leq m \leq K$ and $b_k = p^m b_0$ for $m < k \leq K$. The latter situation closely matches the IEEE 802.11 standard (with $b_0 = 16, p = 2, K = 7, m = 5$). Hence a homogeneous IEEE 802.11 WLAN has a unique fixed point which is also balanced. In general, if the function $G(\cdot)$ is arbitrary (as in Equation 1) but monotone decreasing, there exists a unique balanced fixed point for the system as long as the function $(1 - \gamma)(1 - G(\gamma))$ is one-to-one.

B. The Nonhomogeneous Case

In this section, we will extend our results to systems with nonhomogeneous nodes. AIFS will be introduced in Section VI. Nonhomogeneity is introduced by using different values of b_0, p and K in different nodes.

Consider a nonhomogeneous system of n nodes, with $G_i(\cdot)$ a monotonically decreasing function and $F_i(\gamma) := (1-\gamma)(1-G_i(\gamma))$ being one-to-one for all i. Let there be two fixed point solutions $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ for the above system (see Section III for the fixed point equations), and there exists $k, 1 \leq k \leq n$, such that $\gamma_k \neq \lambda_k$. From the necessary condition (Equation 4) we require that, for all i, and for some $J_1 > 0$ and $J_2 > 0$ (clearly, $J_1, J_2 \neq 0$, see Remarks 5.1),

$$(1 - \gamma_i)(1 - G_i(\gamma_i)) = J_1$$
$$(1 - \lambda_i)(1 - G_i(\lambda_i)) = J_2$$

Since $(1-\gamma)(1-G_i(\gamma))$ is one-to-one, applying this to γ_k and λ_k, we require $J_1 \neq J_2$. Without loss of generality, assume

$J_1 < J_2$. Hence, $\gamma_i > \lambda_i$ for all i (see Remarks 5.1). Using Equation 3 we have,

$$\begin{aligned} \lambda_i &= 1 - \prod_{j \neq i} (1 - G_j(\lambda_j)) \\ &\geq 1 - \prod_{j \neq i} (1 - G_j(\gamma_j)) \\ &= \gamma_i \end{aligned}$$

a contradiction. Hence, it must be that $J_1 = J_2$ or there exists a unique fixed point.

Notice that the arguments above immediately imply the following result.

Theorem 5.3: If $G_i(\gamma)$ is a decreasing function of γ for all i and $(1-\gamma)(1-G_i(\gamma))$ is a strictly monotone function on $[0, 1]$, then the system of equations $\beta_i = G_i(\gamma_i)$ and $\gamma_i = \Gamma_i(\beta_1, \ldots, \beta_i, \ldots, \beta_n)$ has a unique fixed point. ■

Where nodes use exponentially increasing back-off, the next result then follows.

Theorem 5.4: For a system of nodes $1 \leq i \leq n$, with $G_i(\cdot)$ as in Equation 5, that satisfy $K_i \geq 1,\ p_i \geq 2$ and $b_{0_i} > 2p_i{+}1$, there a exists a unique fixed point for the system of equations, $\gamma_i = 1 - \prod_{j \neq i}(1 - G_j(\gamma_j))$ for $1 \leq i \leq n$. ■

Remark: The above result has relevance in the context of the IEEE 802.11e standard where the proposal is to use differences in back-off parameters to differentiate the throughputs obtained by the various nodes. While Theorem 5.4 only states a sufficient condition, it does point to a caution in choosing the back-off parameters of the nodes.

Figure 11 compares the collision probability obtained using the fixed point analysis for a homogeneous system, with ns-2 simulation and the CMP simulator. The plot shows 3 different cases, Priority 0, 1 and 2, corresponding to the IEEE 802.11e EDCA default settings for AC_VO, AC_VI and AC_BE."

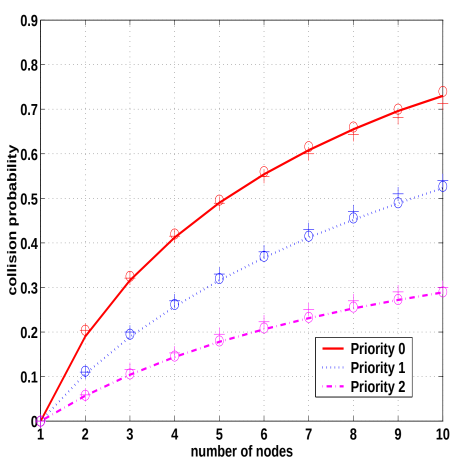

Fig. 11. Plots of collision probability for a homogeneous system of nodes. Three different cases are considered, Priority 0 (AC_VO), Priority 1 (AC_VI) and Priority 2 (AC_BE). The lines correspond to the fixed point analysis, the "+" correspond to the ns-simulations and "O" correspond to the CMP simulator. The 95% confidence interval lies within 1% of the simulation estimate.

VI. Analysis of the AIFS Mechanism

Our approach for obtaining the fixed point equations when the AIFS mechanism is included is the same as the one developed in [13]. However, we develop the analysis in the more general framework introduced in [1] and extended here in Section III. We show that under the condition that $F(\cdot)$ is one-to-one there exists a unique fixed point for this problem as well. The analysis is presented here for two different AIFS class case, but can be extended to any number of classes. Also in this section, we consider only the case in which there is one queue (of an AIFS class) in each node. Extension to the case of multiple queues per node is done in Section VII.

Let us begin by recalling the basic idea of AIFS based service differentiation (see [14]). In legacy DCF, a node decrements its back-off counter, and then attempts to transmit only after it senses an idle medium for more than a DCF interframe space (DIFS). However, in EDCA (Enhanced Distributed Channel Access), based on the access category of a node (and its AIFS value), a node attempts to transmit only after it senses the medium idle for more than its AIFS. Higher priority nodes have smaller values of AIFS , and hence obtain a lower average collision probability, since these nodes can decrement their back-off counters, and even transmit, in slots in which lower priority nodes (waiting to complete their AIFSs) cannot. Thus, *nodes of higher priority (lower AIFS) not only tend to transmit more often but also have fewer collisions compared to nodes of lower priority (larger AIFS).* The model we use to analyze the AIFS mechanism is quite general and accomodates the actual nuances of AIFS implementations (see [16] for how AIFS and DIFS differs) when the AIFS parameter values and the sampled back-off values are suitably adjusted. See Figure 3 on how the actual AIFS can be modeled using a "DCF like" scheme.

A. The Fixed Point Equations

Let us consider two classes of nodes of two different priorities. The priority for a class is supported by using AIFS as well as b_0, p and K. All the nodes of a particular priority have the same values for all these parameters. There are $n^{(1)}$ nodes of Class 1 and $n^{(0)}$ nodes of Class 0. Class 1 corresponds to a higher priority of service. The AIFS for Class 0 exceeds the AIFS of Class 1 by l slots. Thus, after every transmission activity in the channel, while Class 0 nodes wait to complete their AIFS, Class 1 nodes can attempt to transmit in those l slots. Also, if there is any transmission activity (by Class 1 nodes) during those l slots, then again the Class 0 nodes wait for another additional l slots compared to the Class 1 nodes, and so on.

As in [3] and [1], we need to model only the evolution of the back-off process of a node (i.e., the back-off slots after removing any channel activity such as transmissions or collisions) to obtain the collision probabilities. For convenience, let us call the slots in which only Class 1 nodes can attempt as *excess AIFS* slots, which will correspond to the subscript EA in the notation. In the *remaining* slots (corresponding to the subscript R in the notation) nodes of either class can attempt. Let us view such groups of slots, where different sets of nodes contend for the channel, as different *contention periods*. Let us define

$\beta_i^{(1)}$:= the attempt probability of a Class 1 node for all $i, 1 \leq$

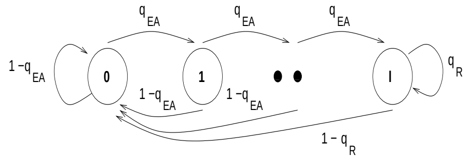

Fig. 12. AIFS differentiation mechanism: Markov model for remaining number of AIFS slots.

$i \leq n^{(1)}$, in the slots in which a Class 1 node can attempt (i.e., all the slots)

$\beta_i^{(0)}$:= the attempt probability of a Class 0 node for all $i, 1 \leq i \leq n^{(0)}$, in the contention periods during which Class 0 nodes can attempt (i.e., slots that are not Excess AIFS slots)

Note that in making these definitions we are modeling the attempt probabilities for Class 1 as being constant over all slots, i.e., the Excess AIFS slots and the remaining slots. This simplification is just an extension of the basic decoupling approximation, and has been shown to yield results that match well with simulations (see [13]). We provide results using our simulation approach in Section IX.

Now the collision probabilities experienced by nodes will depend on the contention period (*excess AIFS* or *remaining* slots) that the system is in. The approach is to model the evolution over contention periods as a Markov Chain over the states $(0, 1, 2, \cdots, l)$, where the state s, $0 \leq s \leq (l-1)$, denotes that an amount of time equal to s slots has elapsed since the end of the AIFS for Class 1. These states correspond to the *excess AIFS* period in which only Class 1 nodes can attempt. In the *remaining* slots, when the state is $s = l$, all nodes can attempt.

In order to obtain the transition probabilities for this Markov chain we need the probability that a slot is idle. Using the decoupling assumption, the idle probability in any slot during the *excess AIFS* period is obtained as,

$$q_{EA} = \prod_{i=1}^{n^{(1)}} (1 - \beta_i^{(1)}) \qquad (6)$$

Similarly, the idle probability in any of the remaining slots is obtained as,

$$q_R = \prod_{i=1}^{n^{(1)}} (1 - \beta_i^{(1)}) \prod_{j=1}^{n^{(0)}} (1 - \beta_j^{(0)}) \qquad (7)$$

The transition structure of the Markov chain is shown in Figure 12. As compared to [13], we have used a simplification that the maximum contention window is much larger than l. If this were not the case then some nodes would certainly attempt before reaching l. In practice, l is small (e.g., 1 slot or 5 slots; see [2]) compared to the maximum contention window.

Let $\pi(EA)$ be the stationary probability of the system being in the *excess AIFS* period; i.e., this is the probability that the above Markov chain is in states 0, or 1, or $\cdots$, or $(l-1)$. In addition, let $\pi(R)$ be the steady state probability of the system being in the remaining slots, i.e., state l of the Markov chain. Solving the balance equations for the steady state probabilities, we obtain,

$$\pi(EA) = \frac{1 + q_{EA} + q_{EA}^2 + \cdots + q_{EA}^{l-1}}{1 + q_{EA} + q_{EA}^2 + \cdots + q_{EA}^{l-1} + \frac{q_{EA}^l}{1-q_R}}$$

$$\pi(R) = \frac{\frac{q_{EA}^l}{1-q_R}}{1 + q_{EA} + q_{EA}^2 + \cdots + q_{EA}^{l-1} + \frac{q_{EA}^l}{1-q_R}} \qquad (8)$$

The average collision probability of a node is then obtained by averaging the collision probability experienced by a node over the different contention periods. The average collision probability for Class 1 nodes is given by, for all i, $1 \leq i \leq n^{(1)}$,

$$\gamma_i^{(1)} = \pi(EA)(1 - \prod_{j=1, j\neq i}^{n^{(1)}} (1 - \beta_j^{(1)})) + \pi(R)(1 - (\prod_{j=1, j\neq i}^{n^{(1)}} (1 - \beta_j^{(1)}) \prod_{j=1}^{n^{(0)}} (1 - \beta_j^{(0)}))) \qquad (9)$$

Similarly, the average collision probability of a Class 0 node is given by, for all i, $1 \leq i \leq n^{(0)}$,

$$\gamma_i^{(0)} = 1 - (\prod_{j=1}^{n^{(1)}} (1 - \beta_j^{(1)}) \prod_{j=1, j\neq i}^{n^{(0)}} (1 - \beta_j^{(0)})) \qquad (10)$$

Our analysis in the remaining section now generalises the analysis of [13] and also establishes uniqueness of the fixed point and the property that the fixed point is balanced over nodes in the same class. Define $G^{(1)}(\cdot)$ and $G^{(0)}(\cdot)$ as in Equation 1 (except that the superscripts here denote the class dependent back-off parameters, with nodes within a class having the same parameters). Then the average collision probability obtained from the previous equations can be used to obtain the attempt rates by using the relations

$$\beta_i^{(1)} = G^{(1)}(\gamma_i^{(1)}), \text{ and } \beta_j^{(0)} = G^{(0)}(\gamma_j^{(0)}) \qquad (11)$$

for all $1 \leq i \leq n^{(1)}$, $1 \leq j \leq n^{(0)}$. We obtain fixed point equations for the collision probabilities by substituting the attempt probabilities from Equation 11 into Equations 9 and 10 (and also into Equations 6 and 7). We have a continuous mapping from $[0,1]^{n^{(1)}+n^{(0)}}$ to $[0,1]^{n^{(1)}+n^{(0)}}$. It follows from Brouwer's fixed point theorem that there exists a fixed point.

B. Uniqueness of the Fixed Point

Lemma 6.1: If $F^{(\cdot)}$ is one-to-one, then collision probabilities of all the nodes of the same class are identical; i.e., the fixed points are balanced within each class. ■

Theorem 6.1: The set of Equations 9, 10 and 11 (together with 8, 6 and 7), representing the fixed point equations for the AIFS model, has a unique solution if the corresponding functions $G^{(1)}$ and $G^{(0)}$ are monotone decreasing and $F^{(1)}$ and $F^{(0)}$ are one-to-one. ■

Remark: It follows from the earlier results in this paper (see, for example, Theorem 5.2) that if $G^{(0)}(\cdot)$ and $G^{(1)}(\cdot)$ are of the form in Equation 5, and if $K^{(i)} \geq 1, p^{(i)} \geq 2$, and $b_0^{(i)} > 2p^{(i)} + 1$, for $i = 0, 1$, then the fixed point will be unique.

C. Simulation Results (Fixed Point Analysis, CMP and ns-2 Simulations)

Although the numerical accuracy of the fixed point analysis has been reported before (see [3], [13]), for completeness, in Figures 13 and 14, we compare the collision probability obtained using the fixed point analysis with ns-2 simulation and the CMP simulator. Figure 13 plots the collision probabilities of AC_VO (access category for voice; the high priority) nodes and AC_BE (access category for best-effort traffic, e.g., TCP; the low priority) nodes, with the number of AC_BE nodes fixed to 4. Figure 14 plots the collision probabilities of AC_VI (access for video; the high priority) nodes and AC_BE (the low priority) nodes with the number of AC_BE nodes fixed to 12. AC_VO, AC_VI and AC_BE correspond to the IEEE 802.11e EDCA access categories. As observed in the plots, the AIFS model works very well whenever $l \ll CW_{min}$ of the traffic classes. Additional plots comparing the analysis with the CMP simulator have been provided (figures 15, 16 and 17) in support of our analysis.

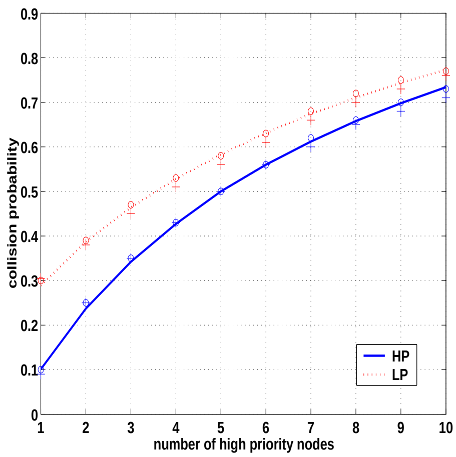

Fig. 13. Plots of collision probability of HP - Priority 0 (AC_VO) nodes and LP - Priority 2 (AC_BE) nodes with the number of Priority 2 nodes fixed to 4. The lines correspond to the fixed point analysis, the "$+$" correspond to the ns-simulations and "o" correspond to the CMP simulator. The 95% confidence interval lies within 1% of the simulation estimate.

Remarks 6.1 (AIFS Differentiation and Multistability): It has been observed that (see Section VIII) as the number of nodes in the system increases, AIFS provides non-preemptive service to high priority nodes, starving the low priority nodes. This may lead to long periods of time when high priority nodes get serviced while the low priority nodes wait. We capture this behaviour using the Markov model in Figure 12. This cannot be viewed as multistability (as seen in Section IV), because AIFS always gives preferential access to the high priority nodes, while starving the low priority nodes, and never the other way. Further, in our analysis on AIFS, the attempt probability $\beta^{(i)}$ of a class i corresponds to only those slots in which class i can attempt (rather than all slots). The variation in attempt rate and collision probability, due to AIFS, is captured using the Markov model shown in Figure 12.

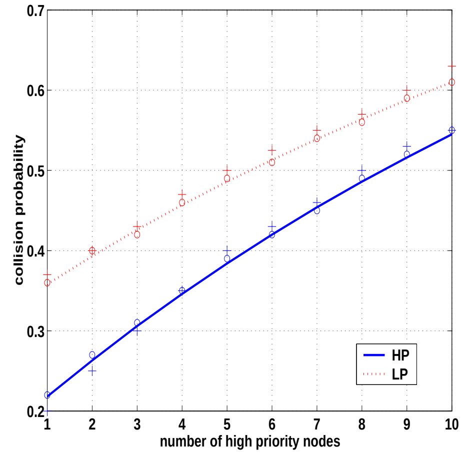

Fig. 14. Plots of collision probability of HP - Priority 1 (AC_VI) nodes and LP - Priority 2 (AC_BE) nodes with the number of Priority 2 nodes fixed to 12. The lines correspond to the fixed point analysis, the "$+$" correspond to the ns-simulations and "o" correspond to the CMP simulator. The 95% confidence interval lies within 1% of the simulation estimate.

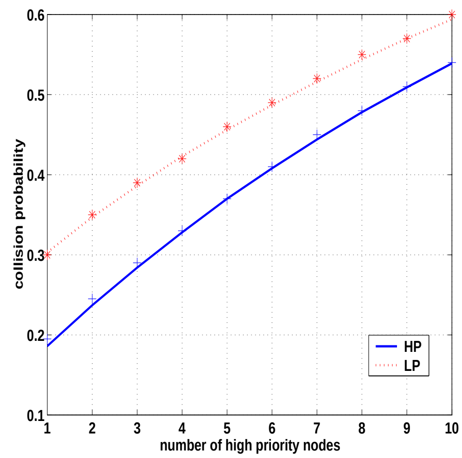

Fig. 15. Plots of collision probability of HP - Priority 1 (AC_VI) nodes and LP - Priority 2 (AC_BE) nodes with the number of Priority 2 nodes fixed to 8. The lines correspond to the fixed point analysis and the symbols correspond to the CMP simulator. The 95% confidence interval lies within 1% of the simulation estimate.

VII. Multiple Access Categories per Node

In this section we further generalize our fixed point analysis to include the possibility of multiple access categories (or queues) per node. We consider n nodes and c_i access categories (ACs) per node i; the ACs can be of either AIFS class (for simplicity, we consider only two AIFS classes) and $c_i = c_i^{(1)} + c_i^{(0)}$ (the superscripts refering to the AIFS classes as before). The ACs in a node need not have the same $G(\cdot)$. Since there are multiple ACs per node, each with its own back-off process, it is possible that two or more ACs in a node complete their back-offs at the same slot. This is then called *Virtual Collision*, and is resolved in favour of the queue with the highest *Collision Priority* in the node. We label the ACs from 1 to c_i, with AC 1 corresponding to the highest collision priority in the node and AC c_i corresponding to the least collision priority. Unlike the single access category per node case where a collision is caused whenever any two

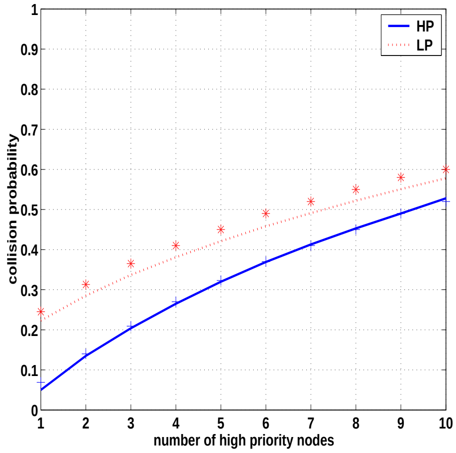

Fig. 16. Plots of collision probability of HP - Priority 1 (AC_VI) nodes and LP - Priority 3 (AC_BK) nodes with the number of Priority 3 nodes fixed to 4. The lines correspond to the fixed point analysis and the symbols correspond to the CMP simulator. The 95% confidence interval lies within 1% of the simulation estimate.

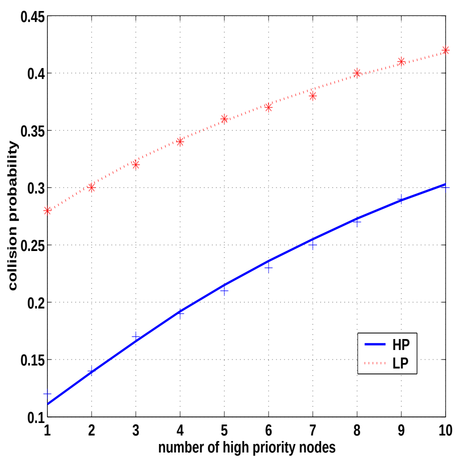

Fig. 17. Plots of collision probability of HP - Priority 2 (AC_BE) nodes and LP - Priority 3 (AC_BK) nodes with the number of Priority 3 nodes fixed to 8. The lines correspond to the fixed point analysis and the symbols correspond to the CMP simulator. The 95% confidence interval lies within 1% of the simulation estimate.

nodes (equivalently, ACs) attempt in a slot, here, a AC sees a collision in a slot only when a AC of some other node or a higher priority AC of the same node attempts in that slot. A low priority AC of a node cannot cause collision to a higher priority AC in the same node. In Section VII-A we will study multiple access categories per node without AIFS (i.e., all the ACs wait only for DIFS) and consider AIFS later in Section VII-B.

We assume that, in a node (say i), the AIFS of Class 0 ACs (with $c_i^{(0)}$ ACs) exceeds the AIFS of the higher priority Class 1 ACs (with $c_i^{(1)}$ ACs) by l slots. This assumption conforms with the way access categories are defined in the IEEE 802.11e standard. Also, when collision priorities are interchanged with AIFS priorities, the actual performance of the system would be hard to characterise.

A. Without AIFS

Let $\gamma_{i,j}$ be the collision probability of AC j of node i and $\beta_{i,j}$ be the attempt probability of AC j of node i, when the AC can attempt. The fixed point equations for this system are, for all $i = 1, \cdots, n$ (and $j = 1, \cdots, c_i$),

$$\beta_{i,j} = G_{i,j}(\gamma_{i,j}) \quad (12)$$

$$\gamma_{i,j} = 1 - \prod_{m=1}^{j-1}(1-\beta_{i,m}) \prod_{\{k=1,k\neq i\}}^{n} \prod_{l=1}^{c_k}(1-\beta_{k,l}) \quad (13)$$

where $G_{i,j}(\cdot)$ depend on the back-off parameters of AC j of node i. The term $\prod_{m=1}^{j-1}(1-\beta_{i,m})$ in the above equation corresponds to the higher priority ACs in the same node. Observe that the $G_{i,j}(\cdot)$ definition allows the possibility of different back-off parameters (b_0, p, K) within a node.

Theorem 7.1: The fixed point equations in $\boldsymbol{\gamma}$, obtained by substituting Equations 12 in Equations 13 has a unique fixed point when $G_{i,j}$ is monotone decreasing and $F_{i,j}(\gamma) := (1-\gamma)(1-G_{i,j}(\gamma))$ is one-to-one for all $i = 1, \cdots, n$ and $j = 1, \cdots, c_i$. ■

B. With AIFS

In this section, we analyse the system where nodes have ACs of either AIFS class (the case where there are only Class 1 ACs can be modeled using the approach in Section VII-A). Define for $1 \leq i \leq n$, $1 \leq j \leq c_i$, $C_{i,j} \in \{0, 1\}$ to be the AIFS class of AC j in node i. Writing the fixed point equations for i, j s.t. $C_{i,j} = 1$, we obtain,

$$\begin{aligned}\gamma_{i,j} = {} & 1 - (\pi(EA) \prod_{m=1}^{j-1}(1-\beta_{i,m}) \prod_{\{k=1,k\neq i\}}^{n} \prod_{\{1\leq l\leq c_k : C_{k,l}=1\}} (1-\beta_{k,l}) \\ & + \pi(R) \prod_{m=1}^{j-1}(1-\beta_{i,m}) \prod_{\{k=1,k\neq i\}}^{n} \prod_{l=1}^{c_k}(1-\beta_{k,l})) \quad (14)\end{aligned}$$

and for i, j s.t. $C_{i,j} = 0$, we obtain,

$$\gamma_{i,j} = 1 - \prod_{m=1}^{j-1}(1-\beta_{i,m}) \prod_{\{k=1,k\neq i\}}^{n} \prod_{l=1}^{c_k}(1-\beta_{k,l}) \quad (15)$$

and $\beta_{i,j} = G_{i,j}(\gamma_{i,j})$. $\pi(EA)$ and $\pi(R)$ are defined as before (see Equation 8), with q_{EA} and q_R defined as

$$q_{EA} = \prod_{k=1}^{n} \prod_{\{1\leq l\leq c_k : C_{k,l}=1\}} (1-\beta_{k,l})$$

$$q_R = \prod_{k=1}^{n} \prod_{l=1}^{c_k}(1-\beta_{k,l}) \quad (16)$$

Theorem 7.2: The fixed point equations (14) and (15) have a unique solution when $G_{i,j}$ are monotone decreasing and $F_{i,j}(\cdot)$ are one-to-one for all $i = 1, \cdots, n$ and for each i, $j = 1, \cdots, c_i$. ■

VIII. Throughput Differentiation: An Analytical Study

It should be noted that all the results in this section are for the fixed point solution. Hence, when we use the term "collision probability" and "attempt rate" it is only in so far as a good match between the fixed point analysis and simulation has already been reported in earlier literature (see Section I).

We will consider two alternatives for K, the maximum retransmission attempts allowed for a packet, namely $K = \infty$ and K finite. In this section, for the finite K case, the form of the function $G(\gamma)$, for all γ, $0 \le \gamma \le 1$ is,

$$G(\gamma) = \frac{1+\gamma+\gamma^2+\ldots+\gamma^K}{b_0(1+p\gamma+p^2\gamma^2+\ldots+p^K\gamma^K)} \quad (17)$$

It is clear that for finite K the attempt rate of a node is lower bounded, and hence as the number of nodes increases to infinity the collision probability of any node goes to 1. Hence, for this case, we will obtain insights regarding performance differentiation only for a finitely large number of nodes. For the infinite K case, however, we will study (as in [1]) the asymptotics of performance differentiation as the number of nodes tends to ∞. In the $K = \infty$ case, the function $G(\gamma)$ simplifies to,

$$G_\infty(\gamma) = \begin{cases} \frac{(1-\gamma p)}{b_0(1-\gamma)} & 0 \le \gamma < \frac{1}{p} \\ 0 & \gamma \ge \frac{1}{p} \end{cases} \quad (18)$$

In the nonhomogeneous case we will write $G^{(1)}_\infty(\gamma)$ and $G^{(0)}_\infty(\gamma)$. For the homogeneous case with $K = \infty$, the (balanced fixed point) asymptotic analysis as $n \to \infty$ was performed in [1].

Consider a set of nodes, divided into two classes, Class 1 and Class 0, with Class 1 corresponding to a higher priority of service. *For simplicity, we assume that $n^{(1)}$ and $n^{(0)}$, the number of nodes of Class 1 and Class 0 respectively, are related as, $n^{(1)} = \alpha n, n^{(0)} = (1-\alpha)n$ for some n and* $\alpha, 0 < \alpha < 1$. Let $\gamma^{(1)}(K,n)$ and $\beta^{(1)}(K,n)$ be the fixed point solutions for the collision probability and attempt rate of a Class 1 node for a given K and total number of nodes n. Similarly, let $\gamma^{(0)}(K,n)$ and $\beta^{(0)}(K,n)$ be the corresponding values for a Class 0 node.

We will study three cases:

Case 1: $b_0^{(1)} < b_0^{(0)}$, $p^{(1)} = p^{(0)} = p$, $AIFS^{(1)} = AIFS^{(0)} = DIFS$

Case 2: $b_0^{(1)} = b_0^{(0)} = b_0$, $p^{(1)} < p^{(0)}$, $AIFS^{(1)} = AIFS^{(0)} = DIFS$

Case 3: $b_0^{(1)} = b_0^{(0)} = b_0, p^{(1)} = p^{(0)} = p, AIFS^{(1)} < AIFS^{(0)}$

Note that in the analysis in earlier sections, we used the Binomial model for the number of attempts in a slot. With $n \to \infty$, in this section, we will use the Poisson batch model for the number of attempts in a slot (as in [1]).

A. Case 1: Differentiation by b_0

1) $K = \infty$, Asymptotic Analysis as $n \to \infty$: With the random number of attempts of each class in a back-off slot being modeled as Poisson distributed, the collision probabilities $\gamma^{(\cdot)}(\infty,n)$ and the attempt rates $\beta^{(\cdot)}(\infty,n)$ are related by

$$\begin{aligned} \gamma^{(1)}(\infty,n) &= 1 - e^{-((n^{(1)}-1)\beta^{(1)}(\infty,n)+n^{(0)}\beta^{(0)}(\infty,n))} \\ \gamma^{(0)}(\infty,n) &= 1 - e^{-(n^{(1)}\beta^{(1)}(\infty,n)+(n^{(0)}-1)\beta^{(0)}(\infty,n))} \end{aligned} \quad (19)$$

Substituting $\beta^{(\cdot)}(\infty,n) = G^{(\cdot)}_\infty(\gamma^{(\cdot)}(\infty,n))$ in the above equations gives the desired fixed point equations governing the system. Trivially, we see that,

$$(1-\gamma^{(1)}(\infty,n))e^{-\beta^{(1)}(\infty,n)} = (1-\gamma^{(0)}(\infty,n))e^{-\beta^{(0)}(\infty,n)} \quad (20)$$

Lemma 8.1: For $i \in \{0,1\}$, $F^{(i)}_\infty(\gamma) := (1-\gamma)e^{-G^{(i)}_\infty(\gamma)}$ is one-to-one for all $\gamma, 0 \le \gamma \le 1$ if $b_0^i \ge 2p+1$. ■

Theorem 8.1: In Case 1, with $K = \infty$, when $F^{(i)}_\infty$ is one-to-one for $i \in \{0,1\}$,

1) $\gamma^{(1)}(\infty,n) < \gamma^{(0)}(\infty,n)$ for all n
2) $\lim_{n\to\infty} \gamma^{(1)}(\infty,n) \uparrow \frac{1}{p}, \lim_{n\to\infty} \gamma^{(0)}(\infty,n) \uparrow \frac{1}{p}$
3) $\lim_{n\to\infty}(n^{(1)}\beta^{(1)}(\infty,n) + n^{(0)}\beta^{(0)}(\infty,n)) \uparrow \ln(\frac{p}{p-1})$ ■

Theorem 8.2: In Case 1, with $K = \infty$, the ratio of the throughputs of Class 1 and Class 2 converges to $\frac{b_0^{(0)}-p}{b_0^{(1)}-p}$ as $n \to \infty$. ■

Remark: Thus, for example, if $b_0^{(1)} = 16$, $b_0^{(0)} = 32$, and $p = 2$ then the ratio of the Class 1 to Class 0 node throughput will be approximately $30/14$ for large n.

2) Finite K, Approximate Analysis for Large n: With finite K, as the number of nodes increases, the collision probability of either class increases to 1 (since the attempt rate is lower bounded) and $G^{(\cdot)}$ is small (since it decreases like $\frac{1}{b_0 p^{K+1}}$, see Equation 17). Then the difference between the collision probabilities (we drop the arguments K and n in the following)

$$\begin{aligned} \gamma^{(1)} - \gamma^{(0)} &= (G^{(0)}(\gamma^{(0)}) - G^{(1)}(\gamma^{(1)})) \\ &\quad (1-G^{(0)}(\gamma^{(0)}))^{(n^{(0)}-1)}(1-G^{(1)}(\gamma^{(1)}))^{(n^{(1)}-1)} \end{aligned}$$

also becomes insignificant. Hence, we can assume that $\gamma^{(1)} \approx \gamma^{(0)}$. For equal packet length transmission, the ratio of the throughputs of a Class 1 node to a Class 0 node corresponds to the ratio of their success probabilities, hence the throughput ratio is given by,

$$\frac{G^{(1)}(\gamma^{(1)})(1-G^{(1)}(\gamma^{(1)}))^{n^{(1)}-1}(1-G^{(0)}(\gamma^{(0)}))^{n^{(0)}}}{G^{(0)}(\gamma^{(0)})(1-G^{(1)}(\gamma^{(1)}))^{n^{(1)}}(1-G^{(0)}(\gamma^{(0)}))^{n^{(0)}-1}} = \frac{\frac{G^{(1)}(\gamma^{(1)})}{(1-G^{(1)}(\gamma^{(1)}))}}{\frac{G^{(0)}(\gamma^{(0)})}{(1-G^{(0)}(\gamma^{(0)}))}} \quad (21)$$

Using $\gamma^{(1)} \approx \gamma^{(0)}$, writing this as γ, and using the fact that $G^{(\cdot)}(\gamma) \approx 0$ for large n, we have,

$$(21) \approx \frac{\frac{G^{(1)}(\gamma)}{(1-G^{(1)}(\gamma))}}{\frac{G^{(0)}(\gamma)}{(1-G^{(0)}(\gamma))}} \approx \frac{G^{(1)}(\gamma)}{G^{(0)}(\gamma)} = \frac{b_0^{(0)}}{b_0^{(1)}}$$

It follows that when service differentiation is provided by the back-off window, for a large number of nodes, the throughput ratio roughly corresponds to $\frac{b_0^{(0)}}{b_0^{(1)}}$, which, for large values of $b_0^{(0)}$ and $b_0^{(1)}$ is almost that same as that obtained for the asymptotic analysis with $K = \infty$ in Theorem 8.2

Remark: For finite K case, this observation (throughput ratio is approximately equal to $\frac{b_0^{(0)}}{b_0^{(1)}}$) is well known. This result has been shown analytically (using similar approximations) and also has been observed in simulations (see [6], [12] and [15]). It has been observed in [1] that for a given number of nodes, n, there will exist a $K(n)$ such that the system performance will not vary much for all $K > K(n)$. Hence, an asymptotic analysis would suffice for such cases. Moreover, we have obtained this result in a much more general setting, using the function $G(\cdot)$.

B. Case 2: Differentiation by p

It may be noted that in the current version of IEEE 802.11e standard this mechanism no longer exists [2].

1) $K = \infty$, Asymptotic Analysis as $n \to \infty$: The fixed point equation governing the collision probability and the attempt rate is the same as Equation 19. The following theorem summarizes the main results for Case 2.

Theorem 8.3: In Case 2, with $K = \infty$, when $F_\infty^{(i)}$ is one-to-one for $i \in \{0, 1\}$, the following hold:

1) $\gamma^{(1)}(\infty, n) < \gamma^{(0)}(\infty, n)$ for all n
2) $\lim_{n\to\infty} \gamma^{(1)}(\infty, n) \uparrow \frac{1}{p^{(1)}}, \lim_{n\to\infty} \gamma^{(0)}(\infty, n) \uparrow \frac{1}{p^{(1)}}$
3) $\lim_{n\to\infty} n^{(1)}\beta^{(1)}(\infty, n) \uparrow \ln(\frac{p^{(1)}}{p^{(1)}-1})$
4) $\lim_{n\to\infty} n^{(0)}\beta^{(0)}(\infty, n) = 0$ ■

Remark: Thus we see that, with $K = \infty$ and a large number of nodes, unlike initial back-off based differentiation, the persistence factor based differentiation completely suppresses the class with the larger value of p. ■

2) Finite K, Approximate Analysis for Large n: For finite K, with the approximation $\gamma^{(1)} \approx \gamma^{(0)}$ and the fact that $G^{(\cdot)}(\gamma^{(\cdot)}) \approx 0$, the throughput ratio approximates to $\frac{(1+p^{(0)}\gamma+p^{(0)2}\gamma^2+\ldots+p^{(0)K}\gamma^K)}{(1+p^{(1)}\gamma+p^{(1)2}\gamma^2+\ldots+p^{(1)K}\gamma^K)}$ (see Equation 21). Hence, as the collision probability of the system increases with load, the ratio of the throughputs of Class 1 to Class 0 also increases (depending on $p^{(1)}$, $p^{(0)}$ and the value of K). We note that as $n \to \infty$, the throughput ratio for the finite K case is finite, unlike the asymptotic case ($K = \infty$). However, the ratio tends to infinity when we consider $K \to \infty$.

C. Case 3: Differentiation by AIFS

1) $K = \infty$, Asymptotic Analysis for $n \to \infty$: In this case service differentiation is provided only by AIFS and we let $G_\infty^{(1)} = G_\infty^{(0)} = G_\infty$ (i.e., the back-off parameters b_0 and p are the same). With the assumption that the number of attempts in each slot is Poisson distributed, the fixed point equations for the AIFS model are (see Equations 9 and 10)

$$\begin{aligned}\gamma^{(1)}(\infty, n) &= \pi(EA)(1 - e^{-(n^{(1)}-1)\beta^{(1)}(\infty,n)}) + \\ &\quad \pi(R)(1 - e^{-(n^{(1)}-1)\beta^{(1)}(\infty,n)-n^{(0)}\beta^{(0)}(\infty,n)}) \\ \gamma^{(0)}(\infty, n) &= (1 - e^{-n^{(1)}\beta^{(1)}(\infty,n)-(n^{(0)}-1)\beta^{(0)}(\infty,n)})\end{aligned}$$

Theorem 8.4: In Case 3, with $K = \infty$, when $F_\infty^{(i)}$ is one-to-one for $i \in \{0, 1\}$,

1) $\gamma^{(1)}(\infty, n) < \gamma^{(0)}(\infty, n)$ for all n
2) $\lim_{n\to\infty} \gamma^{(1)}(\infty, n) \uparrow \frac{1}{p}, \lim_{n\to\infty} \gamma^{(0)}(\infty, n) \uparrow \frac{1}{p}$
3) $\lim_{n\to\infty} n^{(1)}\beta^{(1)}(\infty, n) \uparrow \ln(\frac{p}{p-1})$
4) $\lim_{n\to\infty} n^{(0)}\beta^{(0)}(\infty, n) = 0$ ■

Remark: Again we see that using AIFS for differentiation, when $K = \infty$ and large n, completely suppresses the class with the larger value of AIFS. Observe that Parts 3 and 4 of Theorem 8.4 imply that the individual node attempt ratio $\frac{\beta^{(1)}(\infty,n)}{\beta^{(0)}(\infty,n)}$ goes to ∞ as $n \to \infty$. Some insight into this result will be obtained from the analysis in the following section.

2) Finite K, Approximate Analysis:

Lemma 8.2: In Case 3 for finite K, with $l = 1$, if the fixed point collision probabilities are $\gamma^{(1)}$ and $\gamma^{(0)}$, then the ratio of the throughputs of Class 1 to Class 0 is given by

$$\frac{\frac{G^{(1)}(\gamma^{(1)})}{(1-G^{(1)}(\gamma^{(1)}))}}{\frac{G^{(0)}(\gamma^{(0)})}{(1-G^{(0)}(\gamma^{(0)}))}} \frac{1}{q_R}$$

■

Using this result and approximating $(1 - G^{(i)}(\gamma^{(i)})) \approx 1$ as before, the ratio of throughput equals

$$\frac{\frac{G^{(1)}(\gamma^{(1)})}{(1-G^{(1)}(\gamma^{(1)}))}}{\frac{G^{(0)}(\gamma^{(0)})}{(1-G^{(0)}(\gamma^{(0)}))}} \frac{1}{q_R} \approx \frac{G^{(1)}(\gamma^{(1)})}{G^{(0)}(\gamma^{(0)})} \frac{1}{q_R} \tag{22}$$

For general l, we can expect a factor like $\frac{1}{q_R^l}$ in the previous expression. For low loads, when q_R is not close to 0, the dominating term in the previous expression is $\frac{G^{(1)}(\gamma^{(1)})}{G^{(0)}(\gamma^{(0)})}$. At high loads, both the terms contribute to throughput differentiation depending on the values of $n^{(1)}$ and $n^{(0)}$.

IX. Numerical Study and Discussion

In Figure 18 we plot throughput ratios obtained from a simulation of the coupled back-off processes of two classes of nodes (the simulation approach is explained in Remarks 4.1). We note that this is the throughput ratio if the packet sizes of the two classes are equal. If the packet sizes are unequal then we only need to multiply the throughput ratio plotted here by the ratio of the packet lengths of the two classes. Also plotted is the analytical results obtained from our fixed point approach. The following remarks help in interpreting the results in Figure 18.

Remarks 9.1:

1) Consider AIFS based differentiation. For finite K the attempt rates are bounded below, and the term $\frac{G^{(1)}(\gamma^{(1)})}{G^{(0)}(\gamma^{(0)})}$ is bounded, but as $(n^{(1)}+n^{(0)}) \to \infty$ the idle probability $q_R \to 0$ ensuring (see Equation 22) that the individual node throughput ratio goes to ∞ for finite K as well (similar to the asymptotic results in Theorem 8.4). In addition, when $n^{(1)}$ increases, $\pi(EA)$ increases to 1. Hence, the lower priority nodes (with larger AIFS) rarely get a chance to attempt and the throughput ratio goes to infinity; this is demonstrated by the simulation results in Figure 18, plots with $+$ and $\star$. When $n^{(1)}$ is kept

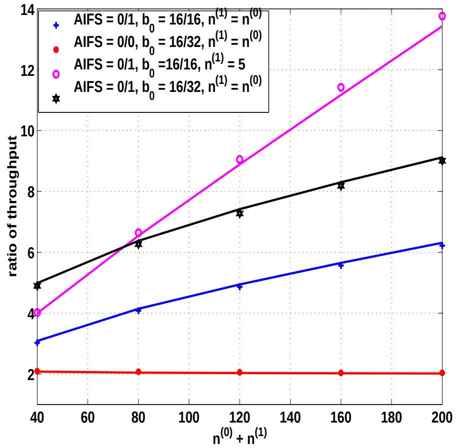

Fig. 18. Ratio of the throughput of a Class 1 (higher priority) node to the throughput of a Class 0 node (lower priority). Analysis results (solid lines) and simulation results (symbols). Four cases are considered: $+$: differentiation only by AIFS with equal number of nodes, $n^{(1)} = n^{(0)}$; $\star$: differentiation by AIFS and by b_0 with equal number of nodes, $n^{(1)} = n^{(0)}$; $\bullet$: differentiation only by b_0 with equal number of nodes, $n^{(1)} = n^{(0)}$; $\circ$: differentiation only by AIFS with, $5 = n^{(1)} \ll n^{(0)}$. In all cases $p = 2$ and $K = 7$ for either class. For the simulation results, the 95% confidence interval lies within 1% of the average value.

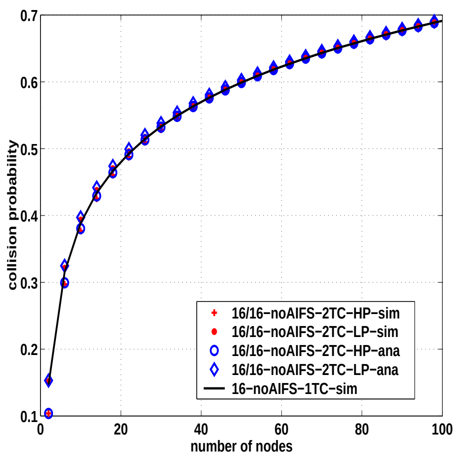

Fig. 19. Collision probability of high priority AC (HP) and low priority AC (LP) in a system of nodes with two ACs. Both simulation (sim) and analysis (ana) are plotted. The back-off parameters of both the ACs (in all the nodes) are identical with $b_0 = 16$ and AIFS = DIFS. Also plotted is the collision probability (obtained from simulation) for single AC per node case with same back-off parameters and twice the number of nodes. In all the cases $p = 2$ and $K = 7$. For the simulation results, the 95% confidence interval lies within 1% of the mean value.

constant and $n^{(0)}$ is increased (which is more typical), the collision probability of Class 0 nodes increases to 1 and their success probability tends to 0. However, the collision probability of Class 1 nodes remains much less than 1 depending on the value of $n^{(1)}$ and hence again the throughput ratio tends to ∞ (see Figure 18, plots with $\circ$). Figure 18 also shows the throughput ratio when only b_0 is used for differentiation (plots with $\bullet$); notice that, as shown earlier, the throughput ratio is just the reciprocal of the ratios of the initial back-off durations, and does not change with n.

2) For Case 3, in general, $\gamma^{(1)}$ and $\gamma^{(0)}$ are different, unlike

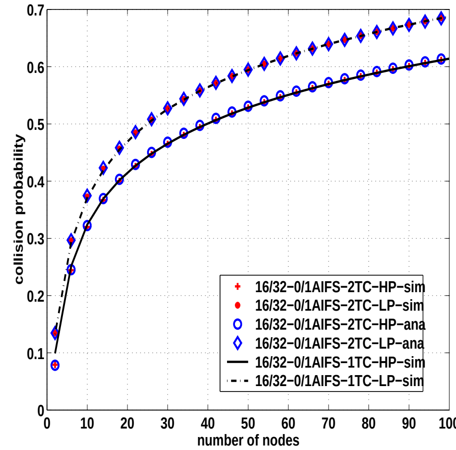

Fig. 20. Collision probability of high priority AC (HP) and low priority AC (LP) in a system of nodes with two ACs. Both simulation (sim) and analysis (ana) are plotted. For the high priority AC, $b_0 = 16$ and AIFS = DIFS, while for the low priority AC we have $b_0 = 32$ and AIFS = DIFS + 1 slot. Also plotted is the collision probability (from simulation) of two classes of nodes when the two ACs of a node are considered as independent ACs in separate nodes. In all the cases $p = 2$ and $K = 7$. For the simulation results, the 95% confidence interval lies within 1% of the mean value.

in Cases 1 and 2. This is captured by the first term in the expression $\frac{G^{(1)}(\gamma^{(1)})}{G^{(0)}(\gamma^{(0)})} \frac{1}{q_R}$.

3) Notice that similar results for AIFS hold even when the functions $G^{(1)}$ and $G^{(0)}$ are not identical (see Figure 18, plot with $\star$). A comparison between the plots with $+$ and $\star$ in Figure 18 shows the effect of using both b_0 and AIFS for throughput differentiation. The b_0 based differentiation causes the entire curve to shift up (in favour of the higher priority class), and AIFS still causes the ratio to increase with increasing n. ■

Figures 19 and 20 plot performance results for the multiple ACs per node case. In Figure 19, we consider a set of homogeneous nodes each with two access categories. The back-off parameters for either AC are the same ($b_0 = 16, p = 2, K = 7$ and AIFS = DIFS). The figure plots the collision probability of the higher priority (HP) AC and the low priority (LP) AC in simulation as well as the analysis. Also plotted in comparison is the collision probability (from simulation) for the single AC per node case with twice the number of nodes. Notice that, except for small n, the performance of the high priority AC and the low priority AC are almost identical (the back-off parameters are identical), and close to the performance of the single AC per node case (see Remark 9.2 below).

In Figure 20, we again consider a set of nodes each with two access categories. The higher priority AC has $b_0 = 16$ and AIFS = DIFS, while the low priority AC has $b_0 = 32$ and AIFS = DIFS + 1 slot. $p = 2$ and $K = 7$ for either case. Figure 20 plots the collision probability of the high priority AC and the low priority AC from simulation as well as the analysis. Also plotted is the collision probability for the two classes of nodes (from simulation) obtained by modeling the two ACs in a node as independent ACs in separate nodes. Notice again that except for small n, the performance of the multiple queue per node case is close to the performance of the single queue case.

Remarks 9.2: The above observations from Figures 19 and 20 can be understood as follows. From the fixed point equations in Section VII, we see that for the high priority AC in any node, only one term corresponding to the low priority AC of the same node is missing (for the systems in Figures 19 and 20 with two ACs), in comparison to the case in which all the ACs are in $2n$ separate nodes. Hence, as n increases, the effect this single AC in the same node diminishes, and the performance of the multiple queue per node case coincides with the performance of the single queue per node case each with one of the original ACs.

X. Summary

In this paper we have studied a multidimensional fixed point equation arising from a model of the back-off process of the EDCA access mechanism in IEEE 802.11e Wireless LANs. Our first concern was the consequences of the nonuniqueness of the fixed point solution and conditions for uniqueness. We demonstrated via examples of homogeneous systems that even when the balanced fixed point is unique, the existence of unbalanced fixed points coexists with the observation of severe short term unfairness in simulations. Further, in such examples the balanced fixed point solution does not capture the long run average behaviour of the system. With these observations in mind, we concluded that it is desirable to have systems in which there is a unique fixed point, even for a nonhomogeneous system.

We have provided simple sufficient conditions on the node back-off parameters that guarantee that a unique fixed point exists. We have shown that the default IEEE 802.11 parameters satisfy these sufficient conditions. The IEEE 802.11e standard motivated us to consider the nonhomogeneous case, and in this case our results suggest certain *safe* ranges of parameters that guarantee the uniqueness of the fixed point while providing service differentiation.

Using the fixed point analysis, we were also able to obtain insights into how the different back-off parameters provide throughput differentiaton between the nodes in a nonhomogeneous system. We observed that using initial back-off window, in general, a fixed throughput ratio can be achieved. On the other hand, using p and $AIFS$ the service can be significantly biased towards the high priority class, with the differentiation increasing in favour of the high priority class as the load in the system increases. We also observed that the effect of collision priority, where there are multiple access categories per node, decreases when the number of nodes increases.

This paper is concerned with the saturation throughput analysis of an IEEE 802.11e single cell WLAN without fading and capture. We have developed a general framework to analyse single cell systems with capture in [19]. Extending to a multi-cell scenario, in [20], the performance analysis of IEEE 802.11 networks comprising intefering co-channel cells was studied using the fixed point approach.

The fixed point approach is simply a heuristic that is found to work well in some cases. Our work in this paper suggests where it might not work and where it might work. In a recent work [17], the authors have proved that for random backoff algorithms, when the number of sources grow large, the system is indeed decoupled, providing a theoretical justification of decoupling arguments used in the analysis.

XI. Acknowledgements

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References

[1] Anurag Kumar, Eitan Altman, Daniele Miorandi and Munish Goyal, New Insights from a fixed point analysis of single cell IEEE 802.11 Wireless LANs, *Proceedings of the IEEE Infocom*, 2005.

[2] IEEE 802.11 WG, IEEE Std 802.11e, IEEE Standard for Information technology. Telecommunications and Information Exchange between Systems - Local and Metropolitan Area Networks. Specific requirements Part 11: Wireless LAN Medium Access Control (MAC) and Physical Layer (PHY) Specifications, Amendment 8: Medium Access Control (MAC) Quality of Service Enhancements, 2005.

[3] G. Bianchi, Performance Analysis of the IEEE 802.11 Distributed Coordination Function, *IEEE Journal on Selected Areas in Communications*, Vol. 18, No. 3, Pages: 535-547, March, 2000.

[4] Yang Xiao, An Analysis for Differentiated Services in IEEE 802.11 and IEEE 802.11e Wireless LANs, *Proc. of IEEE ICDCS'04*, 2004.

[5] Yang Xiao, Backoff-based Priority Schemes for IEEE 802.11, *Proc. of IEEE ICC'03*, 2003.

[6] Bo Li and Roberto Battiti, Performance Analysis of An Enhanced IEEE 802.11 Distributed Coordination Function Supporting Service Differentiation, *"Quality for all", QofIS'03, LNCS 2811, Springer-Verlag Berlin*, Pages: 152-161, 2003.

[7] Yang Xiao, Enhanced DCF of IEEE 802.11e to Support QoS, *Proc. of IEEE WCNC'03*, 2003.

[8] Yunli Chen, Qing-An Zeng and Dharma P. Agrawal, Performance Analysis of IEEE 802.11e Enhanced Distributed Coordination Function, *The 11th IEEE International Conference on Networks (ICON'03)*, Pages: 573-578, 2003.

[9] Yu-Liang Kuo, Chi-Hung Lu, Eric Hsiao-Kuang Wu, Gen-Huey Chen and Yi-Hsien Tseng, Performance analysis of the enhanced distributed coordination function in the IEEE 802.11e, *IEEE 58th Vehicular Technology Conference, VAC 2003-Fall*, Vol. 5, Pages: 3488-3492, 2003.

[10] H. Zhu and I. Chlamtac, An analytical model for IEEE 802.11e EDCF differential services, *ICCCN'03*, 2003.

[11] Zhen-ning Kong, Danny H. K. Tsang, Brahim Bensaou and Deyun Gao, Performance Analysis of IEEE 802.11e Contention-Based Channel Access, *IEEE Journal on Selected Areas in Communications*, Vol. 22, Issue 10, Pages: 2095-2106, December, 2004.

[12] J. Zhao, Z. Guo, Q. Zhang and W. Zhu, Performance study of MAC for service differentiation in IEEE 802.11, *GLOBECOM'02*, Pages: 787-791, 2002.

[13] Robinson, J.W. and Randhawa, T.S., Saturation Throughput Analysis of IEEE 802.11e Enhanced Distributed Coordination Function, *IEEE Journal on Selected Areas in Communications*, Vol. 22, Issue 5, Pages: 917-928, June, 2004.

[14] Stefan Mangold, Sunghyun Choi, Peter May, Ole Klein, Guido Hiertz and Lothar Stibor, IEEE 802.11e Wireless LAN for Quality of Service, *Proc. European Wireless (EW'02)*, February, 2002.

[15] Jianhua He, Lin Zheng, Zhongkai Yang and Chun Tung Chou, Performance analysis and service differentiation in IEEE 802.11 WLAN, *28th Annual IEEE International Conference on Local Computer Networks (LCN '03)*, Pages: 691 - 697, 2003.

[16] Bianchi G., Tinnirello I. and Scalia L., Understanding 802.11e Contention-Based Prioritization Mechanisms and Their Coexistence with Legacy 802.11 Stations, *IEEE Network*, Vol. 19, Issue 4, July/August, 2005.

[17] Bordenave C., McDonald D., Proutiere A., Random Multi-access Algorithms: A Mean Field analysis, *Allerton conference on Communication, Control and Computing*, 2005.

[18] R. Jain, D. Chiu and W. Hawe, A quantitative measure of fairness and discrimination for resource allocation in shared computer systems, *DEC Research Report TR-301*, September 1984.

[19] Venkatesh Ramaiyan and Anurag Kumar, Fixed Point Analysis of the Saturation Throughput of IEEE 802.11 WLANs with Capture, *Proceedings of the National Conference on Communications (NCC)*, 2005.
[20] Manoj K Panda, Anurag Kumar and S H Srinivasan, Saturation Throughput Analysis of a System of Interfering IEEE 802.11 WLANs, *IEEE International Symposium on a World of Wireless, Mobile and Multimedia Networks (WoWMoM)*, 2005.
[21] M. Garetto and C.-F. Chiasserini Performance Analysis of the 802.11 Distributed Coordination Function under Sporadic Traffic, Technical Report, 2004.
[22] G.Bianchi and I. Tinnirello, Remarks on IEEE 802.11 DCF Performance Analysis, IEEE Communications Letters, Vol.9 ,No.8, August 2005.
[23] I. Tinnirello and G. Bianchi, On the accuracy of some common modeling assumptions for EDCA analysis, CITSA 2005, July 2005, Orlando.
[24] Chunyu Huy, Hwangnam Kimz and Jennifer C. Hou, An Analysis of the Binary Exponential Backoff Algorithm in Distributed MAC Protocols, Technical Report No. UIUCDCS-R-2005-2599, July 2005.
[25] Mart L. Molle, A New Binary Logarithmic Arbitration Method for Ethernet, 1994.
[26] Venkatesh Ramaiyan, Anurag Kumar and Eitan Altman, Fixed Point Analysis of Single Cell IEEE 802.11e WLANs: Uniqueness, Multistability and Throughput Differentiation, Tech. Report available at, http://ece.iisc.ernet.in/~anurag/papers/anurag/ramaiyan-etal05fp-general.pdf.gz, 2005.

APPENDIX

A. Proof of Lemma 5.1

We have

$$G(\gamma) \quad := \quad \frac{1+\gamma+\gamma^2\cdots+\gamma^K}{b_0+\gamma b_1+\gamma^2 b_2+\cdots+\gamma^k b_k+\cdots+\gamma^K b_K}$$

and we need to show that the derivative of this function with respect to γ is negative. Taking the derivative we find that we need to show that

$$\sum_{k=0}^{K} b_k\gamma^k\left(\sum_{j=1}^{K} j\gamma^{(j-1)}\right) \le \sum_{k=0}^{K}\gamma^k\left(\sum_{j=1}^{K} jb_j\gamma^{(j-1)}\right)$$

i.e.,

$$\sum_{k=0}^{K}\sum_{j=1}^{K} jb_k\gamma^{(k+j-1)} \le \sum_{k=0}^{K}\sum_{j=1}^{K} jb_j\gamma^{(k+j-1)}$$

or, equivalently, we need to show that

$$\sum_{n=1}^{2K}\gamma^{(n-1)}\sum_{\substack{j=\max\{(n-K),1\}\\k=(n-j)}}^{\min\{n,K\}} j(b_j-b_k)\ge 0$$

Now we consider each term $\sum_{\substack{j=\max\{(n-K),1\}\\k=(n-j)}}^{\min\{n,K\}} j(b_j-b_k)$ and show that it is nonnegative. To this end, define

$$m(n)=|\{(j,k): j+k=n, 1\le j\le K, 0\le k\le K\}|,$$

where $|\cdot|$ denotes set cardinality. When $k=j$, $jb_j-jb_k=0$ and the corresponding term vanishes from the sum. Also, k equals 0 only when $j=n$ and $1\le n\le K$. Hence, simplifying the above expression, we get,

$$\sum_{j=\max\{(n-K),1\}}^{\max\{(n-K),1\}+\lfloor\frac{m}{2}\rfloor-1}\left((n-j)-j\right)(b_{n-j}-b_j)+n(b_n-b_0)1_{\{1\le n\le K\}}$$

which is nonnegative since, in the range of the sum, $(n-j)-j\ge 0$ and $b_{n-j}-b_j\ge 0$. It is also easily seen that the derivative of $G(\cdot)$ is strictly negative for $\gamma>0$ if the b_k are not all equal, this implies that $G(\cdot)$ is strictly decreasing in this case.

B. Proof of Lemma 5.3

Define $G(\gamma):=\frac{u(\gamma)}{v(\gamma)}$. We have

$$\frac{u(\gamma)}{v(\gamma)} \quad = \quad \frac{1+\gamma+\gamma^2\cdots+\gamma^K}{b_0(1+\gamma p+\cdots+\gamma^k p^k+\cdots+\gamma^K p^K)}$$

$$\left(\frac{u}{v}\right)^{'} \quad = \quad \frac{u^{'}v-v^{'}u}{v^2}$$

Since, by Lemma 5.1, $G^{'}(\cdot)\le 0$, $\left(\frac{u}{v}\right)^{'}\le 0$ for all $0\le\gamma\le 1$. Also, with $K\ge 1$, $u,u^{'},v$ and $v^{'}$ are nonnegative for all $0\le\gamma\le 1$. Hence, for all $0\le\gamma\le 1$

$$\left|\left(\frac{u}{v}\right)^{'}\right| \quad \le \quad \frac{v^{'}u}{v^2}$$

Differentiating v, we get,

$$v^{'} \quad = \quad b_0(p+2p^2\gamma+3p^3\gamma^2+\cdots+Kp^K\gamma^{K-1})$$

Multiplying with u, we have,

$$\begin{aligned} v^{'}u &= b_0(p+2p^2\gamma+3p^3\gamma^2+\cdots+Kp^K\gamma^{K-1}) \\ &\quad (1+\gamma+\gamma^2+\cdots+\gamma^K) \\ &= b_0p(1+2p\gamma+3p^2\gamma^2+\cdots+Kp^{K-1}\gamma^{K-1}) \\ &\quad (1+\gamma+\gamma^2+\cdots+\gamma^K) \\ &= b_0p(1+\gamma(1+2p)+\gamma^2(1+2p+3p^2) \\ &\quad +\gamma^3(1+2p+3p^2+4p^3)+\cdots \\ &\quad +\gamma^{K-1}(1+2p+\cdots+Kp^{K-1}) \\ &\quad +\gamma^K(1+2p+\cdots+Kp^{K-1}) \\ &\quad +\gamma^{K+1}(2p+\cdots+Kp^{K-1}) \\ &\quad +\cdots+\gamma^{2K-2}((K-1)p^{K-2}+Kp^{K-1}) \\ &\quad +\gamma^{2K-1}(Kp^{K-1})) \end{aligned}$$

We see that,

$$\begin{aligned} v^{'}u &\leq b_0p(1+\gamma(2+2p)+\gamma^2(3+3p+3p^2) \\ &\quad +\gamma^3(4+4p+4p^2+4p^3)+\cdots \\ &\quad +\gamma^{K-1}(K+Kp+\cdots+Kp^{K-1}) \\ &\quad +\gamma^K(K+Kp+\cdots+Kp^{K-1}) \\ &\quad +\gamma^{K+1}(Kp+\cdots+Kp^{K-1}) \\ &\quad +\cdots+\gamma^{2K-1}(Kp^{K-2}+Kp^{K-1}) \\ &\quad +\gamma^{2K-1}(Kp^{K-1})) \end{aligned}$$

For $p \geq 2$,

$$1+p+p^2+\cdots+p^n < p^{n+1}$$

Hence,

$$\begin{aligned} v^{'}u &\leq b_0p((1+1)+\gamma(2p+2p)+\gamma^2(3p^2+3p^2) \\ &\quad +\gamma^3(4p^3+4p^3)+\cdots \\ &\quad +\gamma^{K-1}(Kp^{K-1}+Kp^{K-1}) \\ &\quad +\gamma^K(Kp^{K-1}+Kp^{K-1}) \\ &\quad +\gamma^{K+1}(Kp^{K-1}+Kp^{K-1}) \\ &\quad +\cdots+\gamma^{2K-1}(Kp^{K-1}+Kp^{K-1}) \\ &\quad +\gamma^{2K-1}(Kp^{K-1}+Kp^{K-1})) \\ &\leq b_02p(1+\gamma(2p)+\gamma^2(3p^2)+\gamma^3(4p^3) \\ &\quad +\cdots+\gamma^{K-1}(Kp^{K-1})+\gamma^K(Kp^{K-1}) \\ &\quad +\gamma^{K+1}(Kp^{K-1})+\cdots+\gamma^{2K-1}(Kp^{K-1}) \\ &\quad +\gamma^{2K-1}(Kp^{K-1})) \end{aligned}$$

But we know that,

$$\begin{aligned} v^2 &= b_0^2(1+p\gamma+p^2\gamma^2+\cdots+p^K\gamma^K)^2 \\ &= b_0^2(1+\gamma(2p)+\gamma^2(3p^2)+\gamma^3(4p^3)+\cdots+ \\ &\quad \gamma^{K-1}(Kp^{K-1})+\gamma^K((K+1)p^K) \\ &\quad +\gamma^{K+1}(Kp^{K+1})+\gamma^{K+2}((K-1)p^{K+2}) \\ &\quad +\cdots+\gamma^{2K-1}(2p^{2K-1})+\gamma^{2K}(p^{2K})) \end{aligned}$$

We see that, for $x \geq 2, y \geq 2$, $(x-1)(y-1) \geq 1 \Rightarrow xy \geq x+y$. Hence, for $K \geq 2, p \geq 2$, $K \leq (K-1)p$. Repeating the above argument for $(K-1)$ and p and so on, we get $K \leq (K-n)p^n$ for $0 \leq n \leq K-1$.

Now, comparing $v^{'}u$ and v^2 term by term in powers of γ and using the fact that $K \leq (K-n)p^n$ for $K \geq 2, p \geq 2$ and $0 \leq n \leq K-1$, we see that,

$$\frac{v^{'}u}{v^2} \leq \frac{2p}{b_0}$$

For the case $K = 1$ and $p \geq 2$, we have $v^{'} = b_0p$ and $v^{'}u = b_0p(1+\gamma)$. Also, $v^2 = b_0^2(1+2p\gamma+\gamma^2)$. Hence,

$$\begin{aligned} \frac{v^{'}u}{v^2} &= \frac{b_0p(1+\gamma)}{b_0^2(1+2p\gamma+\gamma^2)} \\ &= \frac{p}{b_0}\frac{(1+\gamma)}{(1+2p\gamma+\gamma^2)} \\ &\leq \frac{p}{b_0} \leq \frac{2p}{b_0} \end{aligned}$$

C. *Proof of Lemma 6.1*

Rewriting Equation 9, for all i, $1 \leq i \leq n^{(1)}$, we get,

$$\begin{aligned} (1-\gamma_i^{(1)}) &= \prod_{j=1,j\neq i}^{n^{(1)}}(1-\beta_j^{(1)})[\pi(EA) \\ &+ \pi(R)\prod_{k=1}^{n^{(0)}}(1-\beta_k^{(0)})] \end{aligned}$$

Multiplying by $(1-\beta_i^{(1)})$ and using the fact that $\beta_i^{(1)} = G^{(1)}(\gamma_i^{(1)})$, we have,

$$(1-\gamma_i^{(1)})(1-G^{(1)}(\gamma_i^{(1)})) = \pi(EA)q_{EA}+\pi(R)q_R \qquad (23)$$

Observing Equation 23, we see that the right hand side is independent of i. Hence, if the left hand side function, $F^{(1)}(\gamma) := (1-\gamma)(1-G^{(1)}(\gamma))$, is one to one, then $\gamma_i^{(1)} = \gamma_j^{(1)}$ for all $1 \leq i,j, \leq n^{(1)}$. Similarly, we can see from Equation 10 that, for all i, $1 \leq i \leq n^{(0)}$,

$$(1-\gamma_i^{(0)})(1-G^{(0)}(\gamma_i^{(0)})) = q_R \qquad (24)$$

Hence again, $\gamma_i^{(0)} = \gamma_j^{(0)}$ for all $1 \leq i,j, \leq n^{(0)}$, if $F^{(0)}$ is one to one.

D. *Proof of Theorem 6.1*

From Lemma 6.1, we already know that the fixed point is balanced within a class. Now, assume that there exist two vector fixed point solutions, $\boldsymbol{\gamma}$ and $\boldsymbol{\lambda}$, with the first $n^{(1)}$ elements of $\boldsymbol{\gamma}$ are $\gamma^{(1)}$ and the remaining $n^{(0)}$ elements are $\gamma^{(0)}$. Similarly, the first $n^{(1)}$ elements of $\boldsymbol{\lambda}$ are $\lambda^{(1)}$ and the next $n^{(0)}$ elements are $\lambda^{(0)}$.

Let us, in this proof, denote the value of q_R (see Equation 7) for the fixed point $\boldsymbol{\gamma}$ as $q_R(\boldsymbol{\gamma})$ and for the fixed point $\boldsymbol{\lambda}$ as $q_R(\boldsymbol{\lambda})$; similarly, we do for q_{EA} and for other variables.

Lemma D.1: Let $\boldsymbol{\gamma}$ and $\boldsymbol{\lambda}$ be two fixed point solutions and let $F^{(0)}$ be one-to-one. If $\gamma^{(1)} < \lambda^{(1)}$, then $\gamma^{(0)} > \lambda^{(0)}$. Also, $\gamma^{(1)} = \lambda^{(1)}$ *iff* $\gamma^{(0)} = \lambda^{(0)}$.

Proof: Without loss of generality, let $\gamma^{(1)} < \lambda^{(1)}$. Then $G^{(1)}(\gamma^{(1)}) > G^{(1)}(\lambda^{(1)})$ (see Lemma 5.1). Hence,

$$(1-G^{(1)}(\gamma^{(1)}))^{n^{(1)}} < (1-G^{(1)}(\lambda^{(1)}))^{n^{(1)}}$$

If we assume $\gamma^{(0)} < \lambda^{(0)}$, then $q_R(\gamma^{(0)}) > q_R(\lambda^{(0)})$ (see Equation 24). Hence, we require

$$(1-G^{(1)}(\gamma^{(1)}))^{n(1)}(1-G^{(0)}(\gamma^{(0)}))^{n(0)} > (1-G^{(1)}(\lambda^{(1)}))^{n(1)}(1-G^{(0)}(\lambda^{(0)}))^{n(0)}$$

Or,

$$(1-G^{(0)}(\gamma^{(0)}))^{n(0)} > (1-G^{(0)}(\lambda^{(0)}))^{n(0)}$$

which implies $\gamma^{(0)} > \lambda^{(0)}$, which is a contradiction.

If $\gamma^{(0)} = \lambda^{(0)}$, then $q_R(\gamma^{(0)}) = q_R(\lambda^{(0)})$. Hence, $(1 - G^{(1)}(\gamma^{(1)}))^{n^{(1)}} = (1 - G^{(1)}(\lambda^{(1)}))^{n^{(1)}}$, Or, $\gamma^{(1)} = \lambda^{(1)}$. Hence, if $\gamma^{(1)} < \lambda^{(1)}$, then $\gamma^{(0)} > \lambda^{(0)}$. Let $\gamma^{(0)} \neq \lambda^{(0)}$, then $q_R(\gamma^{(0)}) \neq q_R(\lambda^{(0)})$. Hence, $(1 - G^{(1)}(\gamma^{(1)}))^{n^{(1)}} \neq (1 - G^{(1)}(\lambda^{(1)}))^{n^{(1)}}$, Or, $\gamma^{(1)} \neq \lambda^{(1)}$. ■

Now, using Equation 8, write the right hand side of Equation 23 as

$$J(q_{EA}, q_R, l) := \frac{q_{EA}(1 + q_{EA} + \cdots + q_{EA}^{l-1}) + q_R \frac{q_{EA}^l}{1-q_R}}{1 + q_{EA} + q_{EA}^2 + \cdots + q_{EA}^{l-1} + \frac{q_{EA}^l}{1-q_R}}$$

Lemma D.2: If $\gamma^{(1)} < \lambda^{(1)}$, then $J(q_{EA}(\boldsymbol{\gamma}), q_R(\boldsymbol{\gamma}), l) > J(q_{EA}(\boldsymbol{\lambda}), q_R(\boldsymbol{\lambda}), l)$.

Proof:

Consider $J(q_{EA}, q_R, l)$.

$$J(q_{EA}, q_R, l) = \frac{q_{EA}(1 + q_{EA} + \cdots + q_{EA}^{l-1}) + q_R \frac{q_{EA}^l}{1-q_R}}{1 + q_{EA} + \cdots + q_{EA}^{l-1} + \frac{q_{EA}^l}{1-q_R}}$$

Expanding and rewriting the above equation, we get,

$$= \frac{q_{EA} + q_{EA}(q_{EA} - q_R) + \cdots + q_{EA}^{l-1}(q_{EA} - q_R)}{q_{EA} + q_{EA}(q_{EA} - q_R) + \cdots + q_{EA}^{l-1}(q_{EA} - q_R) + (1 - q_R)}$$

which is of the form $\frac{f1}{f1+f2}$. When $\gamma^{(1)} < \lambda^{(1)}$, then $\gamma^{(0)} > \lambda^{(0)}$ (from the previous lemma). Hence,

$$\begin{aligned} q_{EA}(\boldsymbol{\gamma}) &- q_R(\boldsymbol{\gamma}) \\ &= \prod_{i=1}^{n^{(1)}}(1 - G^{(1)}(\gamma^{(1)}))(1 - \prod_{i=1}^{n^{(0)}}(1 - G^{(0)}(\gamma^{(0)}))) \\ &< \prod_{i=1}^{n^{(1)}}(1 - G^{(1)}(\lambda^{(1)}))(1 - \prod_{i=1}^{n^{(0)}}(1 - G^{(0)}(\lambda^{(0)}))) \\ &= q_{EA}(\boldsymbol{\lambda}) - q_R(\boldsymbol{\lambda}) \end{aligned}$$

Also, we can see that,

$$q_{EA}(\boldsymbol{\gamma}) < q_{EA}(\boldsymbol{\lambda})$$
$$q_R(\boldsymbol{\gamma}) < q_R(\boldsymbol{\lambda})$$

Using the above three inequalities, we can see that,

$$J(q_{EA}(\boldsymbol{\gamma}), q_R(\boldsymbol{\gamma}), l) < J(q_{EA}(\boldsymbol{\lambda}), q_R(\boldsymbol{\lambda}), l)$$

■

If $\gamma^{(1)} < \lambda^{(1)}$, then $(1 - \gamma^{(1)})(1 - G^{(1)}(\gamma^{(1)})) > (1 - \lambda^{(1)})(1 - G^{(1)}(\lambda^{(1)}))$. However, from the above lemma and the right hand side of Equation 23, we see that we have a contradiction.

E. Proof of Theorem 7.1

The fixed point equations are, for all $i = 1, \cdots, n$ (and $j = 1, \cdots, c_i$),

$$\gamma_{i,j} = 1 - \prod_{m=1}^{j-1}(1 - \beta_{i,m}) \prod_{\{k=1,k\neq i\}}^{n} \prod_{l=1}^{c_k}(1 - \beta_{k,l})$$

where $\beta_{i,j} = G_{i,j}(\gamma_{i,j})$. Clearly, by Brouwer's fixed point theorem, there exists a fixed point solution for the above system of equations. Rewriting the above equation, we get,

$$(1 - \gamma_{i,j})(1 - \beta_{i,j}) = \prod_{m=1}^{j}(1 - \beta_{i,m}) \prod_{\{k=1,k\neq i\}}^{n} \prod_{l=1}^{c_k}(1 - \beta_{k,l})$$

Notice that, for $2 \leq j \leq c_i$,

$$(1 - \gamma_{i,j})(1 - \beta_{i,j}) = (1 - \gamma_{i,j-1})(1 - \beta_{i,j-1})(1 - \beta_{i,j})$$

or,

$$(1 - \gamma_{i,j}) = (1 - \gamma_{i,j-1})(1 - \beta_{i,j-1}) \quad (25)$$

when $(1 - \beta_{i,j}) > 0$.

Let us assume that there exists two fixed point solutions ($\boldsymbol{\gamma}$ and $\boldsymbol{\lambda}$) for the system. Without loss of generality, assume that for some node i and its AC j, $\gamma_{i,j} < \lambda_{i,j}$. Then, the following lemma shows that $\gamma_{k,l} < \lambda_{k,l}$ for all $k = 1, \cdots, n$ and $l = 1, \cdots, c_k$.

Lemma E.3: Whenever $\boldsymbol{\gamma}$ and $\boldsymbol{\lambda}$ are the fixed point solutions, and if $\gamma_{i,j} < \lambda_{i,j}$ for some $i = 1, \cdots, n$ and $j \in \{1, \cdots, c_i\}$, then $\gamma_{k,l} < \lambda_{k,l}$ for all $k = 1, \cdots, n$ and all $l = 1, \cdots, c_k$.

Proof: Let $\gamma_{i,j} < \lambda_{i,j}$ for some $i \in 1, \cdots, n$ and $j \in \{1, \cdots, c_i\}$. Then, using the fact the $F_{i,j}$ are strictly monotone decreasing, we have

$$(1 - \gamma_{i,j})(1 - G_{i,j}(\gamma_{i,j})) > (1 - \lambda_{i,j})(1 - G_{i,j}(\lambda_{i,j}))$$

Using Equation 25, we see that,

$$(1 - \gamma_{i,j+1}) > (1 - \lambda_{i,j+1})$$

i.e., $\gamma_{i,j+1} < \lambda_{i,j+1}$ when ever $j+1 \in \{1, \cdots, c_i\}$ and, again using Equation 25, we have

$$(1 - \gamma_{i,j-1})(1 - G_{i,j-1}(\gamma_{i,j-1})) > (1 - \lambda_{i,j-1})(1 - G_{i,j-1}(\lambda_{i,j-1}))$$

Or, $\gamma_{i,j-1} < \lambda_{i,j-1}$ when ever $j - 1 \in \{1, \cdots, c_i\}$. Arguing as above, we see that $\gamma_{i,l} < \lambda_{i,l}$ for all $l = 1, \cdots, c_i$.

From the fixed point equations, we observe that for all $k = 1, \cdots, n$,

$$(1 - \gamma_{k,c_k})(1 - G_{k,c_k}(\gamma_{k,c_k})) = \prod_{l=1}^{n}\prod_{m=1}^{c_l}(1 - G_{l,m}(\gamma_{l,m}))$$

$$(1 - \lambda_{k,c_k})(1 - G_{k,c_k}(\lambda_{k,c_k})) = \prod_{l=1}^{n}\prod_{m=1}^{c_l}(1 - G_{l,m}(\lambda_{l,m}))$$

But we know that

$$(1 - \gamma_{i,c_i})(1 - G_{i,c_i}(\gamma_{i,c_i})) > (1 - \lambda_{i,c_i})(1 - G_{i,c_i}(\lambda_{i,c_i}))$$

since $\gamma_{i,c_i} < \lambda_{i,c_i}$. Hence, we have,

$$(1-\gamma_{k,c_k})(1-G_{k,c_k}(\gamma_{k,c_k})) > (1-\lambda_{k,c_k})(1-G_{k,c_k}(\lambda_{k,c_k}))$$

Or, $\gamma_{k,c_k} < \lambda_{k,c_k}$ for all $1 \leq k \leq n$. Arguing as before for node i, we thus have $\gamma_{k,l} < \lambda_{k,l}$ for all $k = 1, \cdots, n$ and $l = 1, \cdots, n$. ■

Hence, if $\boldsymbol{\gamma}$ and $\boldsymbol{\lambda}$ are two fixed point solutions for the system of equations, we see that $\gamma_{i,k} < \lambda_{i,k}$ for all $i = 1, \cdots, n$ (and $k = 1, \cdots, c_i$), which is clearly a contradiction (the proof is similar to that in Section V-B and is not provided). Hence, the system of equations for the multiple access categories per node case (without AIFS) has a unique fixed point solution.

F. Proof of Theorem 7.2

Consider c_i access categories per node i with $c_i^{(1)}$ ACs $(1, \cdots, c_i^{(1)})$ with $AIFS^{(1)}$, and the remaining $c_i^{(0)}$ ACs $(c_i^{(1)}+1, \cdots, c_i)$ with AIFS = $AIFS^{(1)}$ + l slots. The fixed point equations for the system are given in Equations 14 and 15.

As before, by Brouwer's fixed point theorem, there exists a fixed point for the system of equations. Assume that there exist two fixed point solutions for the above system of equations, $\boldsymbol{\gamma}$ and $\boldsymbol{\lambda}$ with $\gamma_{i,j}$ and $\lambda_{i,j}$ as elements.

Let us, in this proof, denote the value of q_R (see Equation 16) for the fixed point $\boldsymbol{\gamma}$ as $q_R(\boldsymbol{\gamma})$ and for the fixed point $\boldsymbol{\lambda}$ as $q_R(\boldsymbol{\lambda})$; similarly, we do for q_{EA} and for other variables.

In a node i, consider two ACs of the same AIFS class, i.e., j and $j-1$ s.t. $C_{i,j} = C_{i,j-1}$. As in the proof of Theorem 7.1, it can be shown from Equation 14 or 15, that

$$(1-\gamma_{i,j}) = (1-\gamma_{i,j-1})(1-G_{i,j-1}(\gamma_{i,j-1}))$$

or,

$$(1-\gamma_{i,j}) = F_{i,j-1}(\gamma_{i,j-1})$$

Hence, using the one-to-one property of $F_{i,j}(\cdot)$ if $\gamma_{i,j} < \lambda_{i,j}$, then $\gamma_{i,k} < \lambda_{i,k}$ for all k such that $C_{i,j} = C_{i,k}$,

Now consider all those nodes with $C_{i,c_i} = 0$, i.e., the least collision priority AC in a node is of AIFS class 0. We then have, using Equations 15 and 16,

$$(1-\gamma_{i,c_i})(1-G_{i,c_i}(\gamma_{i,c_i})) = q_R(\boldsymbol{\gamma})$$
$$(1-\lambda_{i,c_i})(1-G_{i,c_i}(\lambda_{i,c_i})) = q_R(\boldsymbol{\lambda})$$

i.e., $F_{i,c_i}(\gamma_{i,c_i}) = q_R(\boldsymbol{\gamma})$ and $F_{i,c_i}(\lambda_{i,c_i}) = q_R(\boldsymbol{\lambda})$. If $q_R(\boldsymbol{\gamma}) > q_R(\boldsymbol{\lambda})$, then $\gamma_{i,c_i} < \lambda_{i,c_i}$ for all i s.t. $C_{i,c_i} = 0$. If $q_R(\boldsymbol{\gamma}) = q_R(\boldsymbol{\lambda})$, then $\gamma_{i,c_i} = \lambda_{i,c_i}$ for all i s.t. $C_{i,c_i} = 0$. Combining the above two results, we see that for all i, j s.t. $C_{i,j} = 0$, either $\gamma_{i,j} > \lambda_{i,j}$ or $\gamma_{i,j} = \lambda_{i,j}$ or $\gamma_{i,j} < \lambda_{i,j}$.

Without loss of generality, assume that the collision probability of Class 0 ACs is more in $\boldsymbol{\gamma}$ than in $\boldsymbol{\lambda}$ ($\boldsymbol{\gamma^{(0)}} > \boldsymbol{\lambda^{(0)}}$, $\boldsymbol{\gamma^{(0)}}$ and $\boldsymbol{\lambda^{(0)}}$ are the vector of collision probabilities corresponding to AIFS class 0 in the vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\lambda}$ respectively). Hence, $q_R(\boldsymbol{\gamma}) < q_R(\boldsymbol{\lambda})$. Also, $q_{EA}(\boldsymbol{\gamma}) < q_{EA}(\boldsymbol{\lambda})$ (the proof is similar to that provided for AIFS with single AC per node and is not provided), which implies $\boldsymbol{\gamma^{(1)}} < \boldsymbol{\lambda^{(1)}}$.

Now consider the expression $F(\cdot)$ for the least collision priority Class 1 AC, say j, of any node i,

$$(1-\gamma_{i,j})(1-G_{i,j}(\gamma_{i,j})) = \pi(EA,\boldsymbol{\gamma})q_{EA}(\boldsymbol{\gamma}) + \pi(R,\boldsymbol{\gamma})q_{R_{(i,j)}}(\boldsymbol{\gamma})$$
$$(1-\lambda_{i,j})(1-G_{i,j}(\lambda_{i,j})) = \pi(EA,\boldsymbol{\gamma})q_{EA}(\boldsymbol{\lambda}) + \pi(R,\boldsymbol{\gamma})q_{R_{(i,j)}}(\boldsymbol{\lambda})$$

where $q_{R_{(i,j)}} = \prod_{m=1}^{j}(1-\beta_{i,m})\prod_{\{1\leq k\leq n, k\neq i\}}\prod_{l=1}^{c_k}(1-\beta_{k,l})$. Notice that $q_R^{(i,j)}$ is similar to q_R except for terms corresponding to the Class 0 (with lower collision priority) ACs in node i. Hence, if $\boldsymbol{\gamma^{(0)}} > \boldsymbol{\lambda^{(0)}}$, then not only is $q_{EA}(\boldsymbol{\gamma}) < q_{EA}(\boldsymbol{\lambda})$ and $q_R(\boldsymbol{\gamma}) < q_R(\boldsymbol{\lambda})$, but also, $q_{R_{i,j}}(\boldsymbol{\gamma}) < q_{R_{i,j}}(\boldsymbol{\lambda})$. Expanding $(1-\cdot_{i,j})(1-G_{i,j}(\cdot_{i,j}))$, we get,

$$(1-\cdot_{i,j})(1-G_{i,j}(\cdot_{i,j})) = \frac{(1+q_{EA}+q_{EA}^2+\cdots+q_{EA}^{l-1})q_{EA}+\frac{q_{EA}^l}{1-q_R}q_{R_{(i,j)}}}{1+q_{EA}+q_{EA}^2+\cdots+q_{EA}^{l-1}+\frac{q_{EA}^l}{1-q_R}}$$
$$= \frac{q_{EA}+q_{EA}(q_{EA}-q_R)+\cdots+q_{EA}^{l-1}(q_{EA}-q_R)+q_{EA}^l(q_{R_{(i,j)}}-q}{q_{EA}+q_{EA}(q_{EA}-q_R)+\cdots+q_{EA}^{l-1}(q_{EA}-q_R)+(1-q_R)}$$

where $q_{EA}-q_R = q_{EA}(1-\prod_{k=1}^{N}\prod_{\{l=1,C_l^k=0\}}^{n_k}(1-\beta_{k,l}))$ and $q_R^{(i,j)} - q_R = q_R^{(i,j)}(1-\prod_{\{l=1,C_l^i=0\}}^{n_i}(1-\beta_{i,l}))$. Clearly, if $\boldsymbol{\gamma^{(0)}} > \boldsymbol{\lambda^{(0)}}$, then $q_{EA}(\boldsymbol{\gamma}) - q_R(\boldsymbol{\gamma}) < q_{EA}(\boldsymbol{\lambda}) - q_R(\boldsymbol{\lambda})$ and $q_{R_{i,j}}(\boldsymbol{\gamma}) - q_R(\boldsymbol{\gamma}) < q_{R_{i,j}}(\boldsymbol{\lambda}) - q_R(\boldsymbol{\lambda})$. Also, we know that $1-q_R(\boldsymbol{\gamma}) > 1-q_R(\boldsymbol{\lambda})$. From the above observations, we see that, $(1-\gamma_{i,j})(1-G_{i,j}(\gamma_{i,j})) < (1-\lambda_{i,j})(1-G_{i,j}(\lambda_{i,j}))$, which clearly implies that $\gamma_{i,j} > \lambda_{i,j}$. Hence we have $\boldsymbol{\gamma^{(1)}} > \boldsymbol{\lambda^{(1)}}$ which is a contradiction.

Also, we can see that $\boldsymbol{\gamma^{(1)}} = \boldsymbol{\lambda^{(1)}}$ *iff* $\boldsymbol{\gamma^{(0)}} = \boldsymbol{\lambda^{(0)}}$ (the proof is similar to that in Theorem 6.1 and is not provided here).

G. Proof of Lemma 8.1

Consider γ such that $0 \leq \gamma \leq \frac{1}{p}$. Then, $G_\infty(\gamma) = \frac{(1-\gamma p)}{b_0(1-\gamma)}$. Differentiating $(1-\gamma)e^{-G_\infty(\gamma)}$, we have,

$$= e^{-G_\infty(\gamma)}(-1) + (1-\gamma)e^{-G_\infty(\gamma)}(-G_\infty'(\gamma))$$

But $G_\infty'(\gamma) = \frac{1}{b_0}\frac{(1-p)}{(1-\gamma)^2}$. Substituting it in the previous equation, we get,

$$= e^{-G_\infty(\gamma)}\left(-1-(1-\gamma)\frac{1}{b_0}\frac{(1-p)}{(1-\gamma)^2}\right)$$
$$= e^{-G_\infty(\gamma)}\left(-1-\frac{1}{b_0}\frac{(1-p)}{(1-\gamma)}\right)$$

$e^{-G_\infty(\gamma)}$ is always positive (since $G_\infty(\gamma) < 1$). For $0 \leq \gamma \leq \frac{1}{p}$, the absolute value of $\frac{1}{b_0}\frac{(1-p)}{(1-\gamma)}$ is maximum when $\gamma = \frac{1}{p}$, at which the value equals, $\frac{1}{b_0}\frac{(1-p)}{(1-\frac{1}{p})} = -\frac{p}{b_0}$. Hence, the second term is always less than $(-1+\frac{p}{b_0})$. But, if $b_0 \geq 2p+1$, clearly, the second term is negative. Hence, the derivative is always negative and never equal to zero for all $0 \leq \gamma \leq \frac{1}{p}$. Hence, the function $(1-\gamma)e^{-G_\infty(\gamma)}$ is one-to-one in the range $0 \leq \gamma \leq \frac{1}{p}$. For $\frac{1}{p} \leq \gamma \leq 1$, $G_\infty(\gamma) = 0$. Hence, $(1-\gamma)e^{-G_\infty(\gamma)}$ is one-to-one for all γ, $\frac{1}{p} \leq \gamma \leq 1$. Also, the function is decreasing in both the intervals $0 \leq \gamma \leq \frac{1}{p}$ and $\frac{1}{p} \leq \gamma \leq 1$. Hence, $(1-\gamma)e^{-G_\infty(\gamma)}$ is one-to-one for all $0 \leq \gamma \leq 1$. ■

H. Proof of Theorem 8.1

We shall prove Theorem 8.1 by first proving Lemmas H.4 to H.8.

Lemma H.4: In Case 1, with $K = \infty$, $\gamma^{(1)}(\infty, n) \leq \gamma^{(0)}(\infty, n)$ for all n.

Remark: Thus, as expected, the collision probability for the higher priority class is smaller, for each n.

Proof: Since $b_0^{(0)} > b_0^{(1)}$, we see from Equation 18 that, for every $\gamma \in [0,1]$, $G_\infty^{(1)}(\gamma) \geq G_\infty^{(0)}(\gamma)$. Hence, $e^{-G_\infty^{(1)}(\gamma)} \leq e^{-G_\infty^{(0)}(\gamma)}$. Hence, $(1-\gamma)e^{-G_\infty^{(1)}(\gamma)} \leq (1-\gamma)e^{-G_\infty^{(0)}(\gamma)}$. Since the fixed point satisfies Equation 20, it is necessary that $\gamma^{(1)}(\infty, n) \leq \gamma^{(0)}(\infty, n)$ holds (since $(1-\gamma)e^{-G_\infty^{(\cdot)}(\gamma)}$ is one-to-one decreasing). ■

Lemma H.5: In Case 1, with $K = \infty$, $\gamma^{(1)}(\infty, n)$ and $\gamma^{(0)}(\infty, n)$ are strictly increasing functions of n.

Proof: Consider $n1 < n2$. We know that

$$\begin{aligned}(1-\gamma^{(1)}(\infty, n1))e^{-\beta^{(1)}(\infty, n1)} \\ = (1-\gamma^{(0)}(\infty, n1))e^{-\beta^{(0)}(\infty, n1)} \\ (1-\gamma^{(1)}(\infty, n2))e^{-\beta^{(1)}(\infty, n2)} \\ = (1-\gamma^{(0)}(\infty, n2))e^{-\beta^{(0)}(\infty, n2)}\end{aligned}$$

If $\gamma^{(1)}(\infty, n1) = \gamma^{(1)}(\infty, n2)$, then using Lemma 8.1 we see that $\gamma^{(0)}(\infty, n1) = \gamma^{(0)}(\infty, n2)$. Hence $\beta^{(0)}(\infty, n1) = \beta^{(0)}(\infty, n2)$ and $\beta^{(1)}(\infty, n1) = \beta^{(1)}(\infty, n2)$. Since both $\beta^{(0)}(\infty, \cdot)$ and $\beta^{(1)}(\infty, \cdot)$ cannot be zero, and as $n1 < n2$, substituting in Equation 19, we get a contradiction.

Assume that $\gamma^{(1)}(\infty, n1) > \gamma^{(1)}(\infty, n2)$. Then, from Equation 20 and Lemma 8.1, it follows that $\gamma^{(0)}(\infty, n1) > \gamma^{(0)}(\infty, n2)$. Also if collision probabilities decrease with n, it would imply that the attempt rates increase with n, i.e., $\beta^{(1)}(\infty, n1) \leq \beta^{(1)}(\infty, n2)$ and $\beta^{(0)}(\infty, n1) \leq \beta^{(0)}(\infty, n2)$. But from Equations 19, we see that,

$$\begin{aligned}\gamma^{(1)}(\infty, n1) &= 1 - e^{-((n1^{(1)}-1)\beta^{(1)}(\infty, n1) + n1^{(0)}\beta^{(0)}(\infty, n1))} \\ &\leq 1 - e^{-((n2^{(1)}-1)\beta^{(1)}(\infty, n2) + n2^{(0)}\beta^{(0)}(\infty, n2))} \\ &= \gamma^{(1)}(\infty, n2)\end{aligned}$$

Thus we have a contradiction and the result is proved. ■

Lemma H.6: In Case 1, with $K = \infty$, the attempt rates $\beta^{(1)}(\infty, n)$ and $\beta^{(0)}(\infty, n)$ tend to zero as $n \to \infty$.

Proof: If not, the exponent in the collision probability equation (Equation 19) tends to $-\infty$ taking the collision probabilities to 1. However, we know that the attempt rate is zero for all $\gamma \geq \frac{1}{p}$, leading to a contradiction (since we are interested only in the case $p > 1$). ■

Lemma H.7: In Case 1, with $K = \infty$, $\lim_{n\to\infty} \gamma^{(1)}(\infty, n) = \lim_{n\to\infty} \gamma^{(0)}(\infty, n)$

Proof: We have

$$\begin{aligned}0 < \gamma^{(1)}(\infty, n) - \gamma^{(0)}(\infty, n) \\ = e^{-((n^{(1)}-1)\beta^{(1)}(\infty, n) + (n^{(0)}-1)\beta^{(0)}(\infty, n))} \\ (e^{-\beta^{(1)}(\infty, n)} - e^{-\beta^{(0)}(\infty, n)}) \\ \leq (e^{-\beta^{(1)}(\infty, n)} - e^{-\beta^{(0)}(\infty, n)}) \to_{n\to\infty} 0\end{aligned}$$

Hence proved. ■

Lemma H.8: In Case 1, with $K = \infty$, $\gamma^{(1)}(\infty, n) < \gamma^{(0)}(\infty, n) < \frac{1}{p}$ for all n.

Proof: We first observe that $\gamma^{(1)}(\infty, n) < \frac{1}{p}$ for all n. Otherwise, by Lemma H.4 and Lemma H.5, $\frac{1}{p} \leq \gamma^{(1)}(\infty, n) \leq \gamma^{(0)}(\infty, n)$ for all $n > N$ for some N. Hence, $\beta^{(1)}(\infty, n) = \beta^{(0)}(\infty, n) = 0$ for all $n > N$. However, substituting in Equation 19 gives a contradiction.

Now assume that $\gamma^{(0)}(\infty, n) \geq \frac{1}{p}$ for all $n \geq N$ for some N. Since, $\gamma^{(\cdot)}(\infty, n)$ is a strictly increasing function of n, we can, without loss of generality, assume that $\gamma^{(0)}(\infty, n) > \frac{1}{p}$ for all $n \geq N$ for some N. Hence, $\beta^{(0)}(\infty, n)$ is zero for all $n \geq N$. But we know that the collision probability of Class 1 also increases with n and the limit of the collision probability of Class 1 and Class 0 are equal. Hence, $\gamma^{(1)}(\infty, n)$ exceeds $\frac{1}{p}$ for all $n \geq N'$ for some N', which is a contradiction.

Since $\gamma^{(1)}(\infty, n)$ and $\gamma^{(0)}(\infty, n)$ are less than $\frac{1}{p}$, the inequality in Lemma H.4 becomes strict, i.e., $\gamma^{(1)}(\infty, n) < \gamma^{(0)}(\infty, n)$ for all n (when $b_0^{(0)} > b_0^{(1)}$, $G_\infty^{(0)}(\gamma) < G_\infty^{(1)}(\gamma)$ for all $0 \leq \gamma \leq \frac{1}{p}$). ■

Combining the above Lemmas, we see that $\gamma^{(1)}(\infty, n) < \gamma^{(0)}(\infty, n)$ for all n (From Lemma H.8). Using the fact that $\beta^{(\cdot)}(\infty, n) \to 0$ as $n \to \infty$ and $\gamma^{(1)}(\infty, n) < \gamma^{(0)}(\infty, n) < \frac{1}{p}$ for all n, we get $\lim_{n\to\infty} \gamma^{(\cdot)}(\infty, n) = \frac{1}{p}$ as $n \to \infty$ (from Equation 18). Substituting $\gamma^{(\cdot)}(\infty, n) \to \frac{1}{p}$ as $n \to \infty$ in Equations 19, we see that $\lim_{n\to\infty}(n^{(1)}\beta^{(1)}(\infty, n) + n^{(0)}\beta^{(0)}(\infty, n)) \uparrow \ln(\frac{p}{p-1})$, thus completing the proof of Theorem 8.1.

I. Proof of Theorem 8.2

In the following, for notational simplicity, we drop the argument (∞, n). Consider the necessary condition, that a fixed point solution satisfies.

$$(1-\gamma^{(1)})(1-G_\infty^{(1)}(\gamma^{(1)})) = (1-\gamma^{(0)})(1-G_\infty^{(0)}(\gamma^{(0)}))$$

Since we are interested only in the range $0 \leq \gamma \leq \frac{1}{p}$, we can substitute for $G_\infty(\gamma) = \frac{(1-\gamma p)}{b_0(1-\gamma)}$, and further simplify the equation to,

$$(1-\gamma^{(1)}) - \frac{1}{b_0^{(1)}}(1-\gamma^{(1)}p) = (1-\gamma^{(0)}) - \frac{1}{b_0^{(0)}}(1-\gamma^{(0)}p)$$

Rearranging the terms, we have,

$$(\gamma^{(0)} - \gamma^{(1)}) = \frac{1}{b_0^{(1)}}(1-\gamma^{(1)}p) - \frac{1}{b_0^{(0)}}(1-\gamma^{(0)}p)$$

Further

$$\frac{b_0^{(1)}}{(1-\gamma^{(1)}p)}(\gamma^{(0)} - \gamma^{(1)}) = 1 - \frac{b_0^{(1)}}{b_0^{(0)}}\frac{(1-\gamma^{(0)}p)}{(1-\gamma^{(1)}p)}$$

Let us rewrite the left hand side of this equation as follows

$$\begin{aligned}\frac{b_0^{(1)}}{(1-\gamma^{(1)}p)}(\gamma^{(0)} - \gamma^{(1)}) &= \frac{b_0^{(1)}}{p}\frac{(\gamma^{(0)}p - \gamma^{(1)}p)}{(1-\gamma^{(1)}p)} \\ &= \frac{b_0^{(1)}}{p}\frac{(1-\gamma^{(1)}p) - (1-\gamma^{(0)}p)}{(1-\gamma^{(1)}p)} \\ &= \frac{b_0^{(1)}}{p}\left(1 - \frac{(1-\gamma^{(0)}p)}{(1-\gamma^{(1)}p)}\right)\end{aligned}$$

Substituting back this expression for the left hand side into the original equation, we have

$$\frac{b_0^{(1)}}{p}\left(1-\frac{(1-\gamma^{(0)}p)}{(1-\gamma^{(1)}p)}\right) = 1-\frac{b_0^{(1)}}{b_0^{(0)}}\frac{(1-\gamma^{(0)}p)}{(1-\gamma^{(1)}p)}$$

Rearranging terms, we obtain

$$\frac{b_0^{(1)}}{p}-1 = \frac{(1-\gamma^{(0)}p)}{(1-\gamma^{(1)}p)}\left(\frac{b_0^{(1)}}{p}-\frac{b_0^{(1)}}{b_0^{(0)}}\right)$$

Finally the calculation yields

$$\frac{(1-\gamma^{(1)}p)/b_0^{(0)}}{(1-\gamma^{(0)}p)/b_0^{(1)}} = \frac{(b_0^{(0)}-p)}{(b_0^{(1)}-p)} \tag{26}$$

For equal packet length case, the ratio of the throughput of the nodes equals the ratio of their success probabilities in a slot (see, for example, [3] and [1]) which, upon simplification, yields (we reintroduce the dependence on n in the notation)

$$\frac{\beta^{(1)}(\infty,n)}{(1-\beta^{(1)}(\infty,n))}\frac{(1-\beta^{(0)}(\infty,n))}{\beta^{(0)}(\infty,n)}$$

As $n\to\infty$, $\beta^{(1)}(\infty,n)$ and $\beta^{(0)}(\infty,n)$ tends to 0. Also we know that $\beta^{(\cdot)}(\infty,n)$ is of the form $\frac{(1-\gamma^{(\cdot)}(\infty,n)p)}{b_0^{(\cdot)}(1-\gamma^{(\cdot)}(\infty,n))}$ and $\gamma^{(\cdot)}(\infty,n)<\frac{1}{p}$ for all n. Since

$$\lim_{n\to\infty}\gamma^{(1)}(\infty,n)=\lim_{n\to\infty}\gamma^{(0)}(\infty,n)$$

we have $\lim_{n\to\infty}(1-\gamma^{(1)}(\infty,n))=\lim_{n\to\infty}(1-\gamma^{(0)}(\infty,n))$. Hence, using Equation 26, the ratio of the throughputs, as $n\to\infty$, can be seen to converge to $\frac{(b_0^{(0)}-p)}{(b_0^{(1)}-p)}$ as $n\to\infty$.

J. Proof of Theorem 8.3

Lemma J.9: For Case 2, with $K=\infty$, and $F_\infty^{(i)}$ one-to-one

1) $\gamma^{(1)}(\infty,n)<\gamma^{(0)}(\infty,n)$ for all n
2) $\gamma^{(1)}(\infty,n)$ and $\gamma^{(0)}(\infty,n)$ strictly increase with n
3) $\beta^{(1)}(\infty,n)$ and $\beta^{(0)}(\infty,n)$ tend to 0 as $n\to\infty$
4) $\gamma^{(1)}(\infty,n)<\gamma^{(0)}(\infty,n)<\frac{1}{p^{(1)}},\forall n$
5) $\lim_{n\to\infty}\gamma^{(1)}(\infty,n)=\lim_{n\to\infty}\gamma^{(0)}(\infty,n)=\frac{1}{p^{(1)}}$

■

The proof follows in similar lines as in Lemmas H.4 - H.8 and hence is not provided here.

Lemma J.10: In Case 2, with $K=\infty$, $n^{(0)}\beta^{(0)}(\infty,n)\to 0$ as $n\to\infty$.

Proof: Since

$$\lim_{n\to\infty}\gamma^{(1)}(\infty,n)=\lim_{n\to\infty}\gamma^{(0)}(\infty,n)=\frac{1}{p^{(1)}}>\frac{1}{p^{(0)}}$$

we have, $\beta^{(0)}(\infty,n)=0$ for all $n>N$ for some N. Hence, $\lim_{n\to\infty}n^{(0)}\beta^{(0)}(\infty,n)=0$. ■

Remark: Thus, the aggregate attempt rate of the Class 0 goes to zero, while the aggregate attempt rate of the Class 1 governs the system performance.

From Lemma J.9, we see that $\gamma^{(1)}(\infty,n)<\gamma^{(0)}(\infty,n)$ for all n and $\lim_{n\to\infty}\gamma^{(1)}(\infty,n)\uparrow\frac{1}{p^{(1)}},\lim_{n\to\infty}\gamma^{(0)}(\infty,n)\uparrow\frac{1}{p^{(1)}}$. Lemma J.10 shows that $\lim_{n\to\infty}n^{(0)}\beta^{(0)}(\infty,n)=0$. Hence, substituting in the fixed point equations for Case 2, we get $\lim_{n\to\infty}n^{(1)}\beta^{(1)}(\infty,n)\uparrow\ln(\frac{p^{(1)}}{p^{(1)}-1})$ completing the proof of Theorem 8.3.

K. Proof of Theorem 8.4

Lemma K.11: In Case 3, $\gamma^{(1)}(\infty,n)$ and $\gamma^{(0)}(\infty,n)$ are strictly increasing functions of n.

Proof: Rewriting the fixed point equations for AIFS, we have,

$$\begin{aligned}(1-\gamma^{(1)}(\infty,n))e^{-\beta^{(1)}(\infty,n)} &= e^{-n^{(1)}\beta^{(1)}(\infty,n)}(\pi(EA)+\\ &\quad \pi(R)e^{-n^{(0)}\beta^{(0)}(\infty,n)})\\ (1-\gamma^{(0)}(\infty,n))e^{-\beta^{(0)}(\infty,n)} &= e^{-n^{(1)}\beta^{(1)}(\infty,n)-n^{(0)}\beta^{(0)}(\infty,n)}\end{aligned} \tag{27}$$

Consider $n1<n2$.

Assume that $\gamma^{(0)}(\infty,n1)>\gamma^{(0)}(\infty,n2)$ (hence, $\beta^{(0)}(\infty,n1)\le\beta^{(0)}(\infty,n2)$). As $\gamma^{(0)}(\infty,n)$ decreases with n, $(1-\gamma)e^{-G(\gamma)}$ increases. Hence, $q_R=e^{-n^{(1)}\beta^{(1)}(\infty,n)-n^{(0)}\beta^{(0)}(\infty,n)}$ increases with n. Since q_R increases with n and $\beta^{(0)}(\infty,n)$ is non-decreasing with n, we require $q_{EA}=e^{-n^{(1)}\beta^{(1)}(\infty,n)}$ strictly increase with n. Hence, $\beta^{(1)}(\infty,n)$ strictly decreases with n (or $\gamma^{(1)}(\infty,n)$ strictly increase with n). From Lemma D.2, we see that as q_{EA} and q_R both increase with n, the R.H.S. of the first of Equations 27 also increases with n. From the monotonicity of $(1-\gamma)e^{-G(\gamma)}$, we have $\gamma^{(1)}(\infty,n)$ decreasing with n which yields a contradiction.

Assume that $\gamma^{(1)}(\infty,n1)>\gamma^{(1)}(\infty,n2)$ (hence $\beta^{(1)}(\infty,n)$ increases with n). Hence, $q_{EA}=e^{-n^{(1)}\beta^{(1)}(\infty,n)}$ decreases with n. From the second of Equations 27, we see that if q_{EA} decreases, then $\gamma^{(0)}(\infty,n)$ must strictly increase with n (otherwise, the R.H.S. will decrease with n, and from the monotonicity of the L.H.S., we get a contradiction). Since $\gamma^{(0)}(\infty,n)$ increases with n, q_R decreases with n. Using the fact that q_{EA} and q_R decreases with n and from Lemma D.2, we see that the R.H.S. of the first equation also decreases with n, which implies that $\gamma^{(1)}(\infty,n)$ increases with n, which is a contradiction.

Assume that $\gamma^{(1)}(\infty,n1)=\gamma^{(1)}(\infty,n2)$ (clearly, $\beta^{(1)}(\infty,n)>0$ for all n). Then q_{EA} decreases with n. So from the R.H.S. of the second of Equations 27, we need that $\gamma^{(0)}(\infty,n)$ strictly increase with n. So, q_R also decreases with n. Hence, the R.H.S. of the first equation decreases from the Lemma D.2 and hence we obtain a contradiction.

Similarly, if $\gamma^{(0)}(\infty,n1)=\gamma^{(0)}(\infty,n2)$, q_R is constant. Since $e^{-n^{(0)}\beta^{(0)}(\infty,n)}$ is non-increasing, we require that q_{EA} be non-decreasing ($q_R=e^{-n^{(0)}\beta^{(0)}(\infty,n)}q_{EA}$). Hence, we require $\beta^{(1)}(\infty,n)$ strictly decreasing with n. Hence, $\gamma^{(1)}(\infty,n)$ strictly increases with n. Hence, the L.H.S. of the first equation decreases with n. However, since q_R is a constant and q_{EA} is non-decreasing, we have the R.H.S. of the first equation non-decreasing, which is a contradiction.

Hence, $\gamma^{(\cdot)}(\infty,n)$ strictly increases with n. ■

From the above lemma, we can see that $\beta^{(\cdot)}(\infty,n)$ goes to zero as $n\to\infty$.

Lemma K.12: In Case 3, with $K=\infty$, $\gamma^{(1)}(\infty,n)<\gamma^{(0)}(\infty,n)<\frac{1}{p}$ for all n.

Proof: From Equations 27 we can easily see that $(1-\gamma^{(1)}(\infty,n))e^{-\beta^{(1)}(\infty,n)}\ge(1-\gamma^{(0)}(\infty,n))e^{-\beta^{(0)}(\infty,n)}$. Since we assumed that the function $G_\infty(\cdot)$ is the same for both the classes, and since we know that $(1-\gamma)e^{-G_\infty(\gamma)}$ is a strict monotone decreasing function, we have, $\gamma^{(1)}(\infty,n) \leq \gamma^{(0)}(\infty,n)$ for all n.

As in Lemma H.8, it can be seen that $\gamma^{(1)}(\infty,n) < \frac{1}{p}$ for all n. Now suppose that, that $\gamma^{(0)}(\infty,n) > \frac{1}{p}$ for all $n \geq N$ for some N ($\gamma^{(\cdot)}(\infty,n)$ are strictly increasing functions of n). With $\beta^{(0)}(\infty,n) = 0$ for all $n \geq N$, the only factor that governs the collision probability of Class 1 and 0 is $(n^{(1)}-1)\beta^{(1)}$ and $n^{(1)}\beta^{(1)}$. However, we know that $\beta^{(1)}(\infty,n)$ goes to zero, or $\gamma^{(1)}(\infty,n) \to \gamma^{(0)}(\infty,n)$, which requires $\gamma^{(1)}(\infty,n) > \frac{1}{p}$ for some $n > N^{'}$, leading to a contradiction. Hence, $\gamma^{(1)}(\infty,n) \leq \gamma^{(0)}(\infty,n) < \frac{1}{p}$ for all n. Also, when $\gamma^{(\cdot)} < \frac{1}{p}$, the inequality between the collision probabilities becomes strict, i.e., $\gamma^{(1)}(\infty,n) < \gamma^{(0)}(\infty,n)$ (We already know that $\gamma^{(1)}(\infty,n) \leq \gamma^{(0)}(\infty,n)$. The result follows from the Equations 27 and the fact that $G_\infty(\gamma)$ is a strictly decreasing function of γ when $0 \leq \gamma \leq \frac{1}{p}$). ■

Lemma K.13: In Case 3, with $K=\infty$, $n^{(0)}\beta^{(0)}(\infty,n) \to 0$ as $n \to \infty$.

Proof: Since $\beta^{(0)}(\infty,n) \geq 0$

$$1-e^{-(n^{(1)}-1)\beta^{(1)}(\infty,n)} \leq 1-e^{-(n^{(1)}-1)\beta^{(1)}(\infty,n)-n^{(0)}\beta^{(0)}(\infty,n)}$$

If $n^{(0)}\beta^{(0)}(\infty,n)$ converges to a positive value, then this inequality becomes strict in the limit. Hence, $\lim_{n\to\infty}\gamma^{(1)}(\infty,n) < \lim_{n\to\infty}\gamma^{(0)}(\infty,n)$, which is a contradiction, since both $\gamma^{(1)}(\infty,n)$ and $\gamma^{(0)}(\infty,n)$ tend to $\frac{1}{p}$ as $n\to\infty$ (this follows since $\gamma^{(\cdot)}(\infty,n) < \frac{1}{p}$ for all n and $\beta^{(\cdot)}(\infty,n)$ tend to 0 as $n\to\infty$). Hence, $n^{(0)}\beta^{(0)}(\infty,n)$ goes to zero. ■

Using Lemmas K.11 and K.12, we see that $\gamma^{(1)}(\infty,n) < \gamma^{(0)}(\infty,n)$ for all n and $\lim_{n\to\infty}\gamma^{(\cdot)}(\infty,n) \uparrow \frac{1}{p}$. From the previous Lemma, we see that $n^{(0)}\beta^{(0)}(\infty,n) \to 0$. From the Fixed point equations for AIFS and Lemma K.13, we obtain $\lim_{n\to\infty} n^{(1)}\beta^{(1)}(\infty,n) \uparrow \ln(\frac{p}{p-1})$, completing the proof of Theorem 8.4.

L. *Proof of Lemma 8.2*

Consider the case of finite K with $n^{(1)}$ Class 1 nodes and $n^{(0)}$ Class 0 nodes. The success probability for a Class 1 node is given by (we drop K and n in the notation)

$$G^{(1)}(\gamma^{(1)})(\pi(EA)(1-G^{(1)}(\gamma^{(1)}))^{(n^{(1)}-1)} + \pi(R)(1-G^{(1)}(\gamma^{(1)}))^{(n^{(1)}-1)}(1-G^{(0)}(\gamma^{(0)}))^{n^{(0)}})$$

and the success probability for a Class 0 node equals

$$G^{(0)}(\gamma^{(0)})\pi(R)(1-G^{(1)}(\gamma^{(1)}))^{n^{(1)}}(1-G^{(0)}(\gamma^{(0)}))^{(n^{(0)}-1)}$$

The ratio of throughput of a Class 1 node to a Class 0 node is then given by,

$$\frac{G^{(1)}(\gamma^{(1)})(\pi(EA)+\pi(R)(1-G^{(0)}(\gamma^{(0)}))^{n^{(0)}})}{G^{(0)}(\gamma^{(0)})\pi(R)(1-G^{(1)}(\gamma^{(1)}))(1-G^{(0)}(\gamma^{(0)}))^{(n^{(0)}-1)}}$$

$$= \frac{\frac{G^{(1)}(\gamma^{(1)})}{(1-G^{(1)}(\gamma^{(1)}))}(\pi(EA)+\pi(R)(1-G^{(0)}(\gamma^{(0)}))^{n^{(0)}})}{\frac{G^{(0)}(\gamma^{(0)})}{(1-G^{(0)}(\gamma^{(0)}))}\pi(R)(1-G^{(0)}(\gamma^{(0)}))^{n^{(0)}}}$$

$$= \frac{\frac{G^{(1)}(\gamma^{(1)})}{(1-G^{(1)}(\gamma^{(1)}))}}{\frac{G^{(0)}(\gamma^{(0)})}{(1-G^{(0)}(\ gamma^{(0)}))}}\left(\frac{\pi(EA)}{\pi(R)(1-G^{(0)}(\gamma^{(0)}))^{n^{(0)}}}+1\right)$$

Consider the term inside the bracket,

$$\frac{\pi(EA)}{\pi(R)(1-G^{(0)}(\gamma^{(0)}))^{n^{(0)}}}+1$$

Let $l=1$. From Equation 8, we see that $\frac{\pi(EA)}{\pi(R)} = \frac{1-q_R}{q_{EA}}$. Substituting, we get,

$$\frac{1-q_R}{q_{EA}(1-G^{(0)}(\gamma^{(0)}))^{n^{(0)}}}+1$$

We know that $q_{EA}(1-G^{(0)}(\gamma^{(0)}))^{n^{(0)}} = q_R$. Hence, the above expression simplifies to

$$\frac{1-q_R}{q_R}+1 = \frac{1}{q_R}$$

The throughput ratio thus simplifies to

$$\frac{\frac{G^{(1)}(\gamma^{(1)})}{(1-G^{(1)}(\gamma^{(1)}))}}{\frac{G^{(0)}(\gamma^{(0)})}{(1-G^{(0)}(\gamma^{(0)}))}}\frac{1}{q_R}$$